An application of biregularity to quaternionic Lagrange interpolation

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Abstract. We revisit the concept of totally analytic variable of one quaternionic variable introduced by Delanghe [1] and its application to Lagrange interpolation by Güerlebeck and Sprössig [2]. We consider left-regular functions in the kernel of the Cauchy-Riemann operator

$$\mathscr{D} = 2\left(\frac{\partial}{\partial \bar z_1} + j\frac{\partial}{\partial \bar z_2}\right) = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3}.$$

For every imaginary unit $p \in \mathbf{S}^2$, let $\mathbf{C}_p = \langle 1, p\rangle \simeq \mathbf{C}$ and let $J_p = p_1J_1 + p_2J_2 + p_3J_3$ be the corresponding complex structure on $\mathbf{H}$. We identify totally regular variables with real–affine holomorphic functions from $(\mathbf{H}, J_p)$ to $(\mathbf{C}_p, L_p)$, where L_p is the complex structure defined by left multiplication by p. We then show that every J_p–biholomorphic map, which is always a biregular function, gives rise to a Lagrange interpolation formula at any set of distinct points in $\mathbf{H}$.

Keywords: Quaternionic regular function, biregular function, Lagrange interpolation
PACS: 02.30.-f, 02.30.Fn, 02.60.Ed

PRELIMINARIES

We identify the space $\mathbf{C}^2$ with the set $\mathbf{H}$ of quaternions by means of the mapping that associates the pair $(z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$ with the quaternion $q = z_1 + z_2 j = x_0 + ix_1 + jx_2 + kx_3 \in \mathbf{H}$. A quaternionic function $f = f_1 + f_2 j \in C^1(\Omega)$ is *(left) regular* (or *hyperholomorphic*) on Ω if

$$\mathscr{D}f = 2\left(\frac{\partial f}{\partial \bar z_1} + j\frac{\partial f}{\partial \bar z_2}\right) = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} - k\frac{\partial f}{\partial x_3} = 0 \quad \text{on } \Omega.$$

We will denote by $\mathscr{R}(\Omega)$ the space of regular functions on Ω (cf. e.g. [8] and [7] for properties of these functions). The space $\mathscr{R}(\Omega)$ contains the identity mapping and every holomorphic mapping (f_1, f_2) on Ω defines a regular function $f = f_1 + f_2 j$. The original definition of regularity given by Fueter (cf. [8] or [3]) differs from that adopted here by a real coordinate reflection. Let γ be defined by $\gamma(z_1, z_2) = (z_1, \bar z_2)$. Then f is regular on Ω if and only if $f \circ \gamma$ is Fueter–regular on $\gamma(\Omega) = \gamma^{-1}(\Omega)$.

A regular function $f \in C^1(\Omega)$ is called *biregular* if f is invertible and f^{-1} is regular.

Holomorphic functions with respect to a complex structure J_p

Let $J_p = p_1J_1 + p_2J_2 + p_3J_3$ be the orthogonal complex structure on $\mathbf{H}$ defined by a unit imaginary quaternion $p = p_1 i + p_2 j + p_3 k$ in the sphere $\mathbf{S}^2 = \{p \in \mathbf{H} \mid p^2 = -1\}$. Let $\mathbf{C}_p = \langle 1, p\rangle$ be the complex plane spanned by 1 and p and let L_p be the complex structure defined on $T^*\mathbf{C}_p \simeq \mathbf{C}_p$ by left multiplication by p. If $f = f^0 + if^1 : \Omega \to \mathbf{C}$ is a J_p*–holomorphic* function, i.e. $df^0 = J_p^*(df^1)$ or, equivalently, $df + iJ_p^*(df) = 0$, then f defines a regular function $\tilde f = f^0 + pf^1$ on Ω. We can identify $\tilde f$ with a holomorphic function

$$\tilde f : (\Omega, J_p) \to (\mathbf{C}_p, L_p).$$

We have $L_p = J_{\gamma(p)}$, where $\gamma(p) = p_1 i + p_2 j - p_3 k$. More generally, we can consider the space of holomorphic maps from (Ω, J_p) to $(\mathbf{H}, L_p)$

$$Hol_p(\Omega, \mathbf{H}) = \{f : \Omega \to \mathbf{H} \text{ of class } C^1 \mid \overline{\partial}_p f = 0 \text{ on } \Omega\} = Ker(\overline{\partial}_p)$$

where $\overline{\partial}_p$ is the Cauchy–Riemann operator with respect to the structure J_p

$$\overline{\partial}_p = \frac{1}{2}\left(d + pJ_p^* \circ d\right).$$

For any positive orthonormal basis $\{1,p,q,pq\}$ of $\mathbf{H}$ ($p,q \in \mathbf{S}^2$), let $f = f_1 + f_2 q$ be the decomposition of f with respect to the orthogonal sum

$$\mathbf{H} = \mathbf{C}_p \oplus (\mathbf{C}_p)q.$$

Let $f_1 = f^0 + pf^1$, $f_2 = f^2 + pf^3$, with f^0, f^1, f^2, f^3 the real components of f w.r.t. the basis $\{1,p,q,pq\}$. Then the equations of regularity can be rewritten in complex form as

$$\overline{\partial}_p f_1 = J_q^*(\partial_p \overline{f}_2),$$

where $\overline{f}_2 = f^2 - pf^3$ and $\partial_p = \frac{1}{2}\left(d - pJ_p^* \circ d\right)$. Therefore every $f \in Hol_p(\Omega, \mathbf{H})$ is a regular function on Ω.

The energy quadric

In [4] and [6] was introduced the *energy quadric* of a regular function f. It is a family (depending on the point $z \in \Omega$) of positive semi-definite quadrics which contains a lot of information about the (Dirichlet) energy of f and the holomorphicity properties of the function. In particular, this concept can be used to show that there are regular functions that are not J_p–holomorphic for any p, and that an affine biregular function is always J_p–biholomorphic for some p: there exists p such that $f \in Hol_p(\mathbf{H}, \mathbf{H})$ and $f^{-1} \in Hol_{\gamma(p)}(\mathbf{H}, \mathbf{H})$.

TOTALLY REGULAR FUNCTIONS

Definition 1 *A regular function $f \in \mathscr{R}(\Omega)$ is called* totally regular *if the powers f^k are regular on Ω for every integer $k \geq 0$ and f^k is regular on $\Omega' = \{x \in \Omega \mid f(x) \neq 0\}$ for every integer $k < 0$.*

Theorem 1 *Let $f \in \mathscr{R}(\Omega)$ with image $Im(f)$ contained in a (real) plane H. Then there exists $p \in \mathbf{S}^2$ such that $f \in Hol_p(\Omega, \mathbf{H})$. If f is non–constant, the complex structure J_p is uniquely determined.*

If $f = \sum_{\alpha=0}^{4} x_\alpha a_\alpha + b \in \mathscr{R}(\mathbf{H})$, $a_\alpha, b \in \mathbf{H}$, is (real) affine and f has Jacobian matrix of maximum rank 2, the same conclusion of Theorem 1 follows.

Corollary 2 *If $f \in \mathscr{R}(\Omega)$ and $Im(f)$ is contained in $\mathbf{C}_p$ for some $p \in \mathbf{S}^2$, then f is a J_p–holomorphic function, and therefore it is totally regular.*

Remark 1 *The decomposition $f = f_1 + f_2 q$ of a function $f \in Hol_p(\Omega)$ w.r.t. any orthonormal basis $\{p,q,pq\}$ defines totally regular components f_1, $f_2 \in Hol_p(\Omega, \mathbf{C}_p)$.*

We now prove the converse of Corollary 2 for affine functions. Using the energy quadric of a function, we are able to show that the regularity of f and f^2 is sufficient to get that $f \in Hol_p(\mathbf{H}, \mathbf{C}_p)$ and to obtain the total regularity of f.

Theorem 3 *If $f \in \mathscr{R}(\mathbf{H})$ is affine and f^2 is regular, then f has maximum rank 2 and there exists $p \in \mathbf{S}^2$ such that $f \in Hol_p(\mathbf{H}, \mathbf{C}_p)$.*

Corollary 4 *If $f \in \mathscr{R}(\mathbf{H})$ is affine and f^2 is regular, then f is totally regular.*

The condition on the rank of f given in Theorem 3 was proved, in the context of Fueter–regularity, in [2]§1.2 (cf. also [3]§10). The preceding results tell that the set of affine totally regular functions coincides with the set

$$\{f \text{ affine} \mid f \in \cup_{p \in \mathbf{S}^2} Hol_p(\mathbf{H}, \mathbf{C}_p)\}.$$

Note that every subspace $Hol_p(\mathbf{H}, \mathbf{C}_p)$ is a commutative algebra w.r.t. the pointwise product.

TOTALLY REGULAR VARIABLES AND BIREGULARITY

Now our aim is to define, for any $p \in \mathbf{S}^2$, a totally regular function $v_p \in Hol_p(\mathbf{H}, \mathbf{C}_p)$, which generalizes the concept of *Fueter variables* and *totally analytic variables* (cf. e.g. [3]§6.1).

Definition 2 *Let $p \in \mathbf{S}^2$ and $\gamma(p) = p_1 i + p_2 j - p_3 k$. Let $\vec{x}$ denote the vector part (x_1, x_2, x_3) of $x \in \mathbf{H}$. We set*

$$v_p(x) = x_0 + (\overrightarrow{\gamma(p)} \cdot \vec{x})p.$$

In particular, we get the variables $v_i = x_0 + x_1 i = z_1 \in Hol_i(\mathbf{H}, \mathbf{C}_i)$, $v_j = x_0 + x_2 j \in Hol_j(\mathbf{H}, \mathbf{C}_j)$, $v_k = x_0 - x_3 k \in Hol_k(\mathbf{H}, \mathbf{C}_k)$. We can also consider the totally regular variables $v'_p = v_p p \in Hol_p(\mathbf{H}, \mathbf{C}_p)$, which satisfy the additive property

$$\frac{1}{|p+q|}(v'_p + v'_q) = v'_{\frac{p+q}{|p+q|}} \in Hol_{p+q}(\mathbf{H}, \mathbf{C}_{p+q}).$$

For every $a \in \mathbf{H}$, $a \neq 0$, let $rot_a(q) = aqa^{-1}$ be the three–dimensional rotation of $\mathbf{H}$ defined by a. In [5] was studied the effect of rotations on regularity and holomorphicity of functions. As an application of those results, we get that v_p can be seen as one component of a biregular function.

Theorem 5 *a) For every $p \in \mathbf{S}^2$, the function v_p is regular on $\mathbf{H}$ and belongs to the space $Hol_p(\mathbf{H}, \mathbf{C}_p)$. Therefore v_p is totally regular.*

b) For any $p, q \in \mathbf{S}^2$, $q \perp p$, let $a \in \mathbf{H}$ be such that $rot_{\gamma(a)}(i) = p$, $rot_{\gamma(a)}(j) = q$. There exists an affine biregular function $f_a = v_p + w_a q$, with totally regular components $v_p, w_a \in Hol_p(\mathbf{H}, \mathbf{C}_p)$. The function $f_a \in Hol_p(\mathbf{H}, \mathbf{H})$ is J_p–biholomorphic, with inverse of the same type as f_a:

$$f_a^{-1} = f_{a'} = v_{\gamma(p)} + w_{a'} q' \in Hol_{\gamma(p)}(\mathbf{H}, \mathbf{H}) \quad (a' = \gamma(a)^{-1}, \gamma(p) = rot_a^{-1}(i),\ q' = rot_a^{-1}(j)).$$

Remark 2 *The biregular function f_a is defined by the simple formula $f_a = rot_{\gamma(a)a}$.*

QUATERNIONIC LAGRANGE INTERPOLATION

In [2], as an application of totally analytic variables, a Lagrange's Interpolation Theorem was proved. Given k distinct points $b_1, \ldots, b_k \in \mathbf{H}$ and k values $u_1, \ldots, u_k \in \mathbf{H}$, one wants to construct a *Lagrange polynomial* in the module of regular functions, i.e. a polynomial $L \in \mathscr{R}(\mathbf{H})$ such that $L(b_j) = u_j$ for every $j = 1, \ldots, k$.

Theorem 6 *Given a J_p-biholomorphic mapping $f = f_1 + f_2 q \in Hol_p(\mathbf{H}, \mathbf{H})$ ($q \perp p$), there exist (infinitely many) $\alpha, \beta \in \mathbf{C}_p$ such that $g = \alpha f_1 + \beta f_2 \in Hol_p(\mathbf{H}, \mathbf{C}_p)$ is totally regular and satisfies the conditions*

$$g(b_i) \neq g(b_j) \quad \forall\, i \neq j\ (i, j = 1, \ldots, k).$$

The numbers α, β can also be found in the real field.

Then every J_p-biholomorphic mapping f gives rise to a Lagrange interpolation function (a polynomial if f is a polynomial function), given by the formula

$$L = \sum_{s=1}^{k} l_s u_s, \quad \text{where} \quad l_s(x) = \prod_{t \neq s} (g(x) - g(b_t))(g(b_s) - g(b_t))^{-1} \in Hol_p(\mathbf{H}, \mathbf{C}_p).$$

The functions l_s^m are regular on $\mathbf{H}$ for every integer $m > 0$ and $L \in \mathscr{R}(\mathbf{H})$. The powers of L are regular if also the values u_s belong to the subalgebra $\mathbf{C}_p$.

Example 1 *If we take the function f_a of Theorem 5 as J_p-biholomorphic mapping, and $\alpha, \beta \in \mathbf{R}$, then $g = \alpha f_1 + \beta$ is the linear function*

$$rot_{\gamma(a)} \circ (\alpha z_1 + \beta z_2) \circ rot_a.$$

ACKNOWLEDGMENTS

The work was partially supported by MIUR (PRIN Project “Proprietà geometriche delle varietà reali e complesse") and GNSAGA of INdAM.

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