A new approach to slice regularity on real algebras

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Fueter construction

- Given a holomorphic function (a "stem function")

$$F(z) = u(\alpha, \beta) + i\, v(\alpha, \beta) \quad (u, v \text{ real valued})$$

in the upper complex half-plane, real–valued on $\mathbb{R}$, the formula

$$f(q) := u(q_0, |Im(q)|) + \frac{Im(q)}{|Im(q)|} v(q_0, |Im(q)|)$$

$(q = q_0 + q_1 i + q_2 j + q_3 k = q_0 + Im(q) \in \mathbb{H})$
defines a *radially holomorphic* function on $\mathbb{H}$
$\Rightarrow$ Δf is *Fueter–regular* (the *Fueter transform* of F).

- Fueter's construction is naturally related to the concept of *intrinsic function* of a complex variable (Rinehart, Cullen 1960's)
- Higher dimensions: Sce, Qian, Sommen

Fueter construction

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in the upper complex half-plane, real–valued on $\mathbb{R}$, the formula

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($q = q_0 + q_1 i + q_2 j + q_3 k = q_0 + Im(q) \in \mathbb{H}$) defines a slice regular (or Cullen–regular) function on $\mathbb{H}$ (Gentili–Struppa 2006)
$\Rightarrow$ Δf is *Fueter–regular*.

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The quadratic cone of a real algebra

A: a finite–dimensional, alternative real algebra with identity
($\dim(A) > 1$, $\mathbb{R} \subset A$)

Alternative:
the *associator* $(x, y, z) := (xy)z - x(yz)$ is alternating

$Im(A) := \{x \in A \mid x^2 \in \mathbb{R}, x \notin \mathbb{R} \setminus \{0\}\}$
purely imaginary elements of A
(not a subspace of A in general: cf. $\mathbb{R}_3 = Cl_{0,3}(\mathbb{R})$)

$\mathbb{S}_A := \{J \in A \mid J^2 = -1\} \subseteq Im(A)$
"sphere" of square roots of -1

The quadratic cone of a real algebra

- A has an antiinvolution $x \mapsto x^c$, a linear map such that:

$$\begin{cases} (x^c)^c = x \quad \forall x \in A \\ (xy)^c = y^c x^c \quad \forall x, y \in A \\ x^c = x \quad \text{for every real } x \\ J^c = -J \quad \text{for every imaginary unit } J \in \mathbb{S}_A \end{cases}$$

- $t(x) := x + x^c \in A$ (trace of x)
- $n(x) := xx^c \in A$ (norm of x)
- Nondegeneracy condition: $t(x) = n(x) = 0 \Rightarrow x = 0$.

Definition

$Q_A := \{x \in A \mid t(x) \in \mathbb{R},\ n(x) = n(x^c) \in \mathbb{R},\ 4n(x) \geq t(x)^2\} \supseteq \mathbb{R}$
the quadratic cone of A

The quadratic cone of a real algebra

Examples

- $\mathbb{H}$ and $\mathbb{O}$ with the usual conjugation ($Q_{\mathbb{H}} = \mathbb{H}$ and $Q_{\mathbb{O}} = \mathbb{O}$)
- The real Clifford algebra $Cl_{0,n} = \mathbb{R}_n$ with *Clifford conjugation* defined by

$$x^c = (x_{(0)} + x_{(1)} + x_{(2)} + x_{(3)} + x_{(4)} + \cdots)^c$$
$$= x_{(0)} - x_{(1)} - x_{(2)} + x_{(3)} + x_{(4)} - \cdots$$

- In $\mathbb{R}_3$, $Q_A = \{x \in \mathbb{R}_3 \mid x_{123} = 0, x_1 x_{23} - x_2 x_{13} + x_3 x_{12} = 0\}$
  $\mathbb{S}_A = \{x \in Q_A \mid x_0 = 0, \sum_i x_i^2 + \sum_{j,k} x_{jk}^2 = 1\}$

Remark

Q_A is not a subalgebra or a subspace of A in general

The quadratic cone of a real algebra

Proposition

1. *$t(J) = 0,\ n(J) = 1$ for every $J \in \mathbb{S}_A \quad \Rightarrow \quad \mathbb{S}_A \subseteq Q_A$*
2. *every $x \in Q_A$ satisfies a real quadratic equation*

$$x^2 - t(x)x + n(x) = 0$$

3. *every nonzero $x \in Q_A$ is invertible: $x^{-1} = \frac{x^c}{n(x)}$*
4. *$Q_A = A \iff A$ is a quadratic, alternative real algebra without divisors of zero*
   (i.e. A is $\mathbb{C}, \mathbb{H}$ or $\mathbb{O}$ by Frobenius–Zorn's Theorem)
5. *$x \in Q_A, \alpha \in \mathbb{R} \Rightarrow \alpha + x \in Q_A$*

Proposition

For every $x \in Q_A$, there exist unique elements $x_0 \in \mathbb{R}$, $y \in Im(A) \cap Q_A$ with $t(y) = 0$, such that

$$x = x_0 + y$$

Set $Re(x) := x_0 = \frac{x+x^c}{2}$, $Im(x) := y = \frac{x-x^c}{2}$.

Remarks

1. Condition $4n(x) \geq t(x)^2$ assures that $y = \frac{x-x^c}{2}$ belongs to $Im(A)$
2. $n(x) = (Re(x))^2 - (Im(x))^2$

Proposition

Let $\mathbb{C}_J := \langle 1, J \rangle \simeq \mathbb{C}$. If $Q_A \neq \mathbb{R}$, then

- $Q_A = \bigcup_{J \in \mathbb{S}_A} \mathbb{C}_J$
- *If $I, J \in \mathbb{S}_A$, $I \neq \pm J$, then $\mathbb{C}_I \cap \mathbb{C}_J = \mathbb{R}$.*

A–stem functions

Let $A_{\mathbb{C}} = A \otimes_{\mathbb{R}} \mathbb{C}$ be the complexified algebra:

$$A_{\mathbb{C}} = \{w = x + iy \mid x, y \in A, i^2 = -1\}$$

with

- product: $(x + iy)(x' + iy') = xx' - yy' + i(xy' + yx')$
- antiinvolution: $w^c = (x + iy)^c = x^c + iy^c$
- complex conjugation: $\overline{w} = \overline{x + iy} = x - iy$

Definition

Let $D \subseteq \mathbb{C}$. Any *complex intrinsic* $F : D \to A_{\mathbb{C}}$, i.e. satisfying

$$F(\overline{z}) = \overline{F(z)}$$

for every $z \in D$ such that $\overline{z} \in D$ is called an A–stem function on D .

A–stem functions

Remarks

1. No restriction to assume $D = \overline{D}$ (extend to $D \cup \overline{D}$)
2. The *A*–valued components F_1, F_2 of $F = F_1 + iF_2$ must satisfy

$$F_1(\overline{z}) = F_1(z),\ F_2(\overline{z}) = -F_2(z) \quad \text{(an } \textit{even–odd pair} \text{ w.r.t. } Im(z))$$

3. By means of a basis $\mathcal{B} = \{u_k\}_{k=1,\dots,d}$ of A, F can be identified with a complex curve in $\mathbb{C}^d$, $d = \dim(A)$:
   $F(z) = \sum_{k=1}^d F^k(z)u_k, \quad \widetilde{F}_{\mathcal{B}} = (F^1, \dots, F^d) : D \to \mathbb{C}^d$
4. Here no holomorphicity assumption on F

A–valued slice functions

Definition

Any stem–function $F : D \to A_{\mathbb{C}}$ induces a (left) slice function $f = \mathcal{I}(F) : \Omega_D \to A$ with domain in the quadratic cone

$$\Omega_D := \{x = \alpha + \beta J \in \mathbb{C}_J \mid \alpha, \beta \in \mathbb{R},\ \alpha + i\beta \in D,\ J \in \mathbb{S}_A\} \subseteq Q_A$$

$$f(x) = f(\alpha + \beta J) := F_1(z) + JF_2(z) \quad (z = \alpha + i\beta)$$

(F_1, F_2) even–odd pair w.r.t. $\beta \Rightarrow f$ is well defined:

$$f(\alpha + (-\beta)(-J)) = F_1(\bar{z}) + (-J)F_2(\bar{z}) = F_1(z) + JF_2(z)$$

A–valued slice functions

Definition (Space of left slice functions)

$\mathcal{S}(\Omega_D) := \{f : \Omega_D \to A \mid f = \mathcal{I}(F), F : D \to A_{\mathbb{C}}$ stem function$\}$

Examples

1. If $A = \mathbb{H}, \mathbb{O}$, for any element $a \in A$
   $F(z) := z^n a = Re(z^n)a + i\,(Im(z^n)a)$ induces $f(x) = x^n a$
2. By linearity, we get *standard polynomials* $p(x) = \sum_{i=0}^{n} x^i a_i$ with right quaternionic or octonionic coefficients and convergent *power series* $\sum_i x^i a_i$
3. Also $G(z) := Re(z^n)a$ and $H(z) := i\,(Im(z^n)a)$ are complex intrinsic on $\mathbb{C}$. They induce
   $g(x) = Re(x^n)a, \quad h(x) = f(x) - g(x) = (x^n - Re(x^n))a$
4. $G(z) - H(z) = \bar{z}^n a \Rightarrow g(x) - h(x) = \bar{x}^n a$

Smoothness of slice functions

Proposition

Let $f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D)$.

1. *If $F \in C^0(D)$, then $f \in C^0(\Omega_D)$.*
2. *If $F \in C^\infty(D)$ or $C^\omega(D)$, then $f \in C^\infty(\Omega_D)$ or $C^\omega(\Omega_D)$ respectively.*

If F is C^∞–smooth (or C^ω) and $z = \alpha + i\beta$, then

$$F_1(\alpha, \beta) = F_1'(\alpha, \beta^2),\ F_2(\alpha, \beta) = \beta F_2'(\alpha, \beta^2)$$

with $F_1', F_2' \in C^\infty$ or C^ω (Whitney) $\Rightarrow$

$$f(x) = F_1'(Re(x), n(Im(x))) + Im(x) F_2'(Re(x), n(Im(x)))$$

$(x = Re(x) + Im(x) \in Q_A)$

Slice regularity

- Left multiplication by $i \Rightarrow$ complex structure on $A_{\mathbb{C}}$
- $F = F_1 + iF_2 : D \to A_{\mathbb{C}}$, $F \in C^1$, is holomorphic iff Cauchy–Riemann equations hold:

$$\frac{\partial F_1}{\partial \alpha} = \frac{\partial F_2}{\partial \beta}, \quad \frac{\partial F_1}{\partial \beta} = -\frac{\partial F_2}{\partial \alpha} \quad (z = \alpha + i\beta \in D)$$

Definition

A function $f \in \mathcal{S}(\Omega_D)$ is (left) slice regular if its stem function F is holomorphic:
$\mathcal{SR}(\Omega_D) := \{f \in \mathcal{S}(\Omega_D) \mid f = \mathcal{I}(F), F$ holomorphic stem function$\}$

Slice regularity and Cullen regularity

Proposition

If $f \in \mathcal{SR}(\Omega_D)$, then the restriction

$$f_J := f_{|\mathbb{C}_J \cap \Omega_D} \; : \; \mathbb{C}_J \cap \Omega_D \to A$$

is holomorphic for every $J \in \mathbb{S}_A$ (w.r.t. left multiplication by J)

- $f_J(\alpha + \beta J) = F_1(\alpha + i\beta) + JF_2(\alpha + i\beta) \;\Rightarrow$

$$\overline{\partial} f_J = 0 \; \forall J \in \mathbb{S}_A \;\Leftrightarrow\; \text{CR–equations for } F_1, F_2$$

- More precisely:
  $\overline{\partial} f_J = 0$, $\overline{\partial} f_K = 0$ for *two* different units $\Rightarrow\; f \in \mathcal{SR}(\Omega_D)$
- On $\mathbb{H}$ or $\mathbb{O}$, if $D \cap \mathbb{R} \neq \emptyset$, $f \in \mathcal{SR}(\Omega_D) \;\Leftrightarrow\; f$ is Cullen regular (Gentili-Struppa 2006)
- On $\mathbb{R}_n$, slice–regularity generalizes slice–monogenic functions (Colombo–Sabadini–Struppa 2009)

Slice regularity

Proposition

Every $f \in \mathcal{S}(\Omega_D)$ is uniquely determined by its values on two distinct half planes $\mathbb{C}_J^+, \mathbb{C}_K^+$, with $J - K$ invertible.

- The stem function F s.t. $\mathcal{I}(F) = f$ can be recovered from the restrictions f_J, f_K
- Representation formula (also for $K = -J$): $\forall I \in \mathbb{S}_A$

$$f(\alpha+\beta I) = (I-K)\left((J-K)^{-1} f(\alpha+\beta J)\right) - (I-J)\left((J-K)^{-1} f(\alpha+\beta K)\right)$$

(cf. extension formulas for Cullen regular functions on $\mathbb{H}$ Colombo–Gentili–Sabadini–Struppa 2009)

Product of slice functions

Definition

Let $f = \mathcal{I}(F), g = \mathcal{I}(G) \in \mathcal{S}(\Omega_D)$. The product of f and g is the slice function

$$f \cdot g = \mathcal{I}(FG)$$

induced by the pointwise product in $A_{\mathbb{C}}$ (FG is complex intrinsic).

- If $f, g \in \mathcal{SR}(\Omega_D)$, then $f \cdot g \in \mathcal{SR}(\Omega_D)$.
- In general $(f \cdot g)(x) \neq f(x)g(x)$
- If F_1, F_2 are real–valued, then $(f \cdot g)(x) = f(x)g(x)$ for every $x \in \Omega_D$

Definition

$f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D)$ is real if F_1, F_2 are real–valued

"Star product" of power series

Proposition

Let $f(x) = \sum_i x^i a_i$ and $g(x) = \sum_j x^j b_j$ be polynomials or convergent power series ($a_i, b_j \in A$).
Then the product $f \cdot g$ coincides with the star product

$$(f * g)(x) := \sum_k x^k \left(\sum_{i+j=k} a_i b_j \right)$$

i.e. $\mathcal{I}(FG) = \mathcal{I}(F) * \mathcal{I}(G)$.

Example

Let $A = \mathbb{H}$, $I, J \in \mathbb{S}_{\mathbb{H}}$, $F = z - I$, $G = z - J$.
Then $FG = z^2 - z(I + J) + IJ$,
$(f \cdot g)(x) = (x - I) \cdot (x - J) = \mathcal{I}(FG) = x^2 - x(I + J) + IJ = \mathcal{I}(F) * \mathcal{I}(G)$

The normal function associated to a slice function

Definition

Let $f \in \mathcal{S}(\Omega_D)$.

- $f^c := \mathcal{I}(F^c) \in \mathcal{S}(\Omega_D)$, where $F^c(z) := F_1(z)^c + iF_2(z)^c$
- $CN(F) := FF^c = n(F_1) - n(F_2) + i\,t(F_1 F_2{}^c)$
- $N(f) := f \cdot f^c = \mathcal{I}(CN(F)) \in \mathcal{S}(\Omega_D)$ the normal function of f

Remarks

- If $f \in \mathcal{SR}(\Omega_D)$, then f^c and $N(f) \in \mathcal{SR}(\Omega_D)$
- If $A = \mathbb{H}$ or $\mathbb{O}$, then $CN(F)$ is complex valued and $N(f)$ is real

Admissible slice functions

Definition

$N_A := \{x \in A \mid x = 0 \text{ or } n(x) = n(x^c) > 0\}$ (the normal part of A)
$f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D)$ is admissible if $\exists$ subspace $V \subseteq N_A$ s.t.

$$F(z) \in V \otimes \mathbb{C} \quad \text{for every } z \in D$$

Remarks

- If $A = \mathbb{H}$ or $\mathbb{O}$, then *every* slice function is admissible
- If f is admissible, then $CN(F)$ is complex valued and $N(f)$ is real
- If f is real, $N(f) = f^2$ and f is admissible

Example

In $\mathbb{R}_3$, $N_A = \{x \in \mathbb{R}_3 \mid x_0x_{123} + x_1x_{23} - x_2x_{13} + x_3x_{12} = 0\}$ $\Rightarrow$ every polynomial $\sum_i x^i a_i$, ($a_i \in \mathbb{R}^4$ *paravectors*) is admissible slice regular.

Theorem (Structure theorem for $V(f) = f^{-1}(0)$)

Let $f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D)$ *admissible,* $x = \alpha + \beta J \in \Omega_D$, $z = \alpha + i\beta \in D$, $\mathbb{S}_x = \alpha + \mathbb{S}_A \beta$.

1. $\mathbb{S}_x \cap V(f) = \emptyset \quad (\Leftrightarrow \ CN(F)(z) \neq 0\ )$
2. $\mathbb{S}_x \subseteq V(f) \quad (\Leftrightarrow \ F(z) = 0\ )$ — real or spherical zero
3. $\mathbb{S}_x \cap V(f)$ *is a point* $\quad (\Leftrightarrow \ CN(F)(z) = 0,\ F(z) \neq 0\ )$ — isolated zero

In particular, a real slice function has no isolated zeros, and for every f

$$V(N(f)) = \bigcup_{x \in V(f)} \mathbb{S}_x .$$

If $f \in \mathcal{SR}(\Omega_D)$, $f \not\equiv 0$, *then* $\mathbb{C}_J \cap \cup_{x \in V(f)} \mathbb{S}_x$ *is closed and discrete in* $\mathbb{C}_J \cap \Omega_D$ *for every* $J \in \mathbb{S}_A$.

$\mathbb{H}$: Pogorui–Shapiro 2004 (polynomials), Gentili–Stoppato 2008 (power series)

Definition

For any $y \in Q_A$, $\Delta_y(x) := N(x-y) = x^2 - x\,t(y) + n(y)$ is the characteristic polynomial of y

Δ_y is real, $\Delta_y = \Delta_{y'} \Leftrightarrow \mathbb{S}_y = \mathbb{S}_{y'}$, $\Delta_y \equiv 0$ on $\mathbb{S}_y$.

Theorem (Remainder Theorem)

Let $f \in \mathcal{SR}(\Omega_D)$ be slice regular, admissible, $f(y) = 0$.

1. *If y is a real zero, then* $f(x) = (x-y)\,g(x)$
   $g \in \mathcal{SR}(\Omega_D)$ *admissible*
2. *If $y \in \Omega_D \setminus \mathbb{R}$, then* $f(x) = \Delta_y(x)\,h(x) + xa + b$
   $h \in \mathcal{SR}(\Omega_D)$ *admissible,* $a, b \in N_A$
   - y *is a spherical zero of* $f \Leftrightarrow a = b = 0$
   - y *is an isolated, non–real zero of* $f \Leftrightarrow a \neq 0 \quad (y = -ba^{-1})$

(If f is real, then g, h, a, b are real $\Rightarrow a = b = 0$)

($\mathbb{H}$: Beck 1979, $\mathbb{O}$: Serodio 2007)

Multiplicity of zeros

Remarks

1. If $A = \mathbb{H}, \mathbb{O} \Rightarrow$ (1) holds for *every* $y \in V(f)$: $f(x) = (x - y) \cdot g(x)$ (Ghiloni–P. 2009)
2. $\mathbb{S}_y \cap V(f)$ is non–empty $\Leftrightarrow$ $\Delta_y \mid N(f)$

Definition

$y \in \Omega_D$ is a zero of f of multiplicity s if $\Delta_y^s \mid N(f)$ and $\Delta_y^{s+1} \nmid N(f)$

Remark

If p is an admissible polynomial of degree d, $N(p)$ has real coefficients, degree $2d$ and $m_{N(p)}(y_j) = 2m_p(y_j)$.

Theorem (FTA with multiplicities)

Let $p(x) = \sum_{i=0}^{d} x^i a_i$ of degree $d > 0$. Assume that the subspace $\langle a_0, \ldots, a_d \rangle \subseteq N_A$.
Then $V(p) = \{y \in Q_A \mid p(y) = 0\} \neq \emptyset$. For any choice of zeros $y_1, \ldots, y_k \in V(p)$ in different spheres $\mathbb{S}_{y_j}$ such that

$$V(p) \subseteq \bigcup_j \mathbb{S}_{y_j} = V(N(p)) \quad \Rightarrow \quad \sum_j m_p(y_j) = d.$$

$\mathbb{H}, \mathbb{O}$: Niven 1941, Jou 1950, Eilenberg–Steenrod 1952, Pogorui–Shapiro 2004, Gentili–Struppa 2008, Gentili–Struppa–Vlacci 2008, Ghiloni–P. 2009

Examples

1. In $\mathbb{R}_n$ every polynomial $\sum_i x^i a_i$, (a_i paravectors) has roots in Q_A.
2. In $\mathbb{R}_3$, $p(x) = xe_{23} + e_1$ vanishes only at $y = e_{123} \notin Q_A$ (p is *not* admissible: $e_1, e_{23} \in Q_A$ but $e_1 + e_{23} \notin N_A$)

Normal function of a product

Theorem

Let A be associative or $A = \mathbb{O}$. Then

$$N(f \cdot g) = N(f)N(g)$$

for every $f, g \in \mathcal{SR}(\Omega_D)$.

If $A = \mathbb{H}, \mathbb{O} \;\Rightarrow\; A_{\mathbb{C}}$ is a complex alternative algebra with an antiinvolution $\;\Rightarrow\; A_{\mathbb{C}}$ is an algebra with composition: the *complex norm* $cn(x) := xx^c$ is multiplicative (for $\mathbb{O}$ it follows from Artin's Theorem)

$$cn(x\,y) = cn(x)cn(y) \quad \forall\, x, y \in A_{\mathbb{C}}$$

$\Rightarrow\; CN(FG) = CN(F)CN(G)$ for every stem functions F, G.
(Ghiloni–P. 2009: star product of power series on $\mathbb{O}$
$\Rightarrow$ "geometry" of $V(f \cdot g)$)

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