

Zero sets of polynomials and slice regular functions on Clifford algebras

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Abstract

We present the main results of the theory of slice regular functions on a real Clifford algebra \mathbb{R}_n . Our theory includes the theory of slice regular functions of a quaternionic variable and the theory of slice monogenic functions of a Clifford variable. In particular, we show that a fundamental theorem of algebra with multiplicities holds for an ample class of polynomials with coefficients in \mathbb{R}_n . We also give some hints and results about the appearing (almost) complex manifold structure of the quadratic cone of \mathbb{R}_n , which is the natural domain of definition for slice regular functions.

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Math. Subj. Class: 30C15, 30G35, 32A30, 17D05

1 Introduction

The present paper has two aims. The first one is to review some of the most relevant results about slice regularity on real Clifford algebras which can be obtained with the novel approach proposed (in a more general setting) in [9] and [11]. The second aim is to announce some results about the (almost) complex manifold structure which appears when considering the “quadratic cone” of a Clifford algebra (see Section 2 for definitions).

The concept of slice regularity for functions of one quaternionic, octonionic or Clifford variable has been recently introduced by Gentili and Struppa in [5, 6, 8] and by Colombo, Sabadini and Struppa in [1].

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In [9], a theory of slice regular functions on a real alternative algebra A with a fixed antiinvolution was developed. The domains on which slice regular functions can be defined are open subsets of what we call the *quadratic cone* of the algebra. This cone is the whole algebra only in the case in which A is a real division algebra (i.e. the complex numbers, the quaternions or the octonions).

If A is the algebra of quaternions, we get the theory of *slice regular* functions of a quaternionic variable introduced by Gentili and Struppa [5, 6]. If A is the Clifford algebra $Cl_{0,n} = \mathbb{R}_n$, the quadratic cone is a real algebraic (proper for $n \geq 3$) subset \mathcal{Q}_n of \mathbb{R}_n , containing the subspace of paravectors. By restricting the Clifford variables to the paravectors, we get the theory of *slice monogenic* functions introduced by Colombo, Sabadini and Struppa in [1].

In Sections 3 and 4, we recall the definition of slice regular functions on \mathbb{R}_n and the algebra structure induced on these functions by the pointwise product in the complexified algebra $\mathbb{R}_n \otimes \mathbb{C}$. This product for slice functions generalizes the usual product of polynomials and power series.

In Section 5, we recall some properties of the zero set of slice functions. We restrict our attention to *admissible* slice regular functions, which preserve many relevant properties of classical holomorphic functions. We generalize a structure theorem for the zero set proved by Pogorui and Shapiro [14] for quaternionic polynomials and by Gentili and Stoppato [4] for quaternionic power series. We also define a notion of multiplicity for the zeros of an admissible slice regular function.

Polynomials with right Clifford coefficients are slice regular functions on the quadratic cone. We obtain a fundamental theorem of algebra with multiplicities for slice regular admissible polynomials. A version of this theorem was proved, for quaternionic polynomials, by Eilenberg and Niven [3, 13] (case $n = 2$). Gordon and Motzkin [12] proved, for polynomials on a (associative) division ring, that the number of conjugacy classes containing zeros of p cannot exceed the degree m of p . This estimate was improved on the quaternions by Pogorui and Shapiro [14]. Gentili and Struppa [7] showed that, using the right definition of multiplicity, the number of zeros of p equals the degree of the polynomial. In [10], this strong form was generalized to the octonions. Recently, Colombo, Sabadini and Struppa [1, 2] and Yang and Qian [15] proved some results on the structure of the set of zeros of a polynomial with paravector coefficients in a Clifford algebra.

We give several examples that illustrate the relevance of the quadratic cone and of the condition of admissibility for the algebraic and topological properties of the zero set of a Clifford polynomial.

In the last section, we try to shed some light on an interesting aspect of the “smooth” quadratic cone $\mathcal{SQ}_n = \mathcal{Q}_n \setminus \mathbb{R}$ of the Clifford algebra \mathbb{R}_n . In the case $n = 2$ (the quaternions), $\mathcal{SQ}_2 = \mathbb{H} \setminus \mathbb{R}$ has a natural almost complex structure \mathcal{J} related to the quaternionic product. It follows that $(\mathcal{SQ}_2, \mathcal{J})$ is a 2-dim complex manifold. We give an interpretation of slice regularity in terms of holomorphicity w.r.t. this complex structure. We also show that this almost complex structure has a generalization to \mathcal{SQ}_n . In the general case, the results are not definitive. If \mathbb{R}_n satisfies a condition, which is fulfilled for $n = 2$ and 3,

we can show that \mathcal{SQ}_n has a natural almost complex structure \mathcal{J} . In particular, \mathcal{SQ}_n is a smooth manifold of even dimension over \mathbb{R} .

2 The quadratic cone of \mathbb{R}_n

Let \mathbb{R}_n denote the real Clifford algebra $Cl_{0,n} = \mathbb{R}_{0,n}$. An element x of \mathbb{R}_n can be represented in the form $x = \sum_K x_K e_K$, with $K = (i_1, \dots, i_k)$ an increasing multiindex of length k , $0 \leq k \leq n$, $e_K = e_{i_1} \cdots e_{i_k}$, $e_\emptyset = 1$, $x_K \in \mathbb{R}$, $x_\emptyset = x_0$, e_1, \dots, e_n basis elements.

Definition 1. Let $\text{Im}(\mathbb{R}_n) := \{x \in \mathbb{R}_n \mid x^2 \in \mathbb{R}, x \notin \mathbb{R} \setminus \{0\}\}$. The elements of $\text{Im}(\mathbb{R}_n)$ are called *purely imaginary* elements of \mathbb{R}_n .

Definition 2. For every element x of \mathbb{R}_n , the *trace* of x is $t(x) := x + x^c \in \mathbb{R}_n$ and the (squared) *norm* of x is $n(x) := xx^c \in \mathbb{R}_n$. We call *quadratic cone* of \mathbb{R}_n the subset

$$\mathcal{Q}_n := \{x \in \mathbb{R}_n \mid t(x) \in \mathbb{R}, n(x) \in \mathbb{R}\}.$$

We also set $\mathbb{S}_n := \{J \in \mathcal{Q}_n \mid J^2 = -1\}$ (*square roots of -1*). We call *smooth quadratic cone* of the Clifford algebra the set $\mathcal{SQ}_n := \mathcal{Q}_n \setminus \mathbb{R}$.

Proposition 1. *The following statements hold.*

1. Every $x \in \mathcal{Q}_n$ satisfies the real quadratic equation $x^2 - xt(x) + n(x) = 0$.
2. Every nonzero $x \in \mathcal{Q}_n$ is invertible: $x^{-1} = n(x)^{-1}x^c$.
3. $\mathbb{S}_n = \{x \in \mathbb{R}_n \mid t(x) = 0, n(x) = 1\}$.
4. \mathcal{Q}_n contains the subspace of paravectors $\mathbb{R}^{n+1} := \{x \in \mathbb{R}_n \mid [x]_k = 0 \text{ for every } k > 1\}$.
5. \mathcal{Q}_n is the real algebraic subset (proper for $n > 2$) of \mathbb{R}_n defined by $x_K = 0$, $x \cdot (xe_K) = 0$ for every $e_K \neq 1$ such that $e_K^2 = 1$.

Examples 1.

(1) $n = 3$. A direct computation shows that the quadratic cone is the 6-dimensional real algebraic set

$$\mathcal{Q}_3 = \{x \in \mathbb{R}_3 \mid x_{123} = 0, x_2x_{13} - x_1x_{23} - x_3x_{12} = 0\}.$$

(2) $n = 4$. The quadratic cone \mathcal{Q}_4 is the 8-dimensional real algebraic set defined by

$$\begin{aligned} x_{14}x_{23} - x_{13}x_{24} + x_{12}x_{34} &= x_3x_{24} - x_2x_{34} - x_{23}x_4 = x_3x_{14} - x_1x_{34} - x_{13}x_4 = 0, \\ x_2x_{14} - x_1x_{24} - x_{12}x_4 &= x_2x_{13} - x_1x_{23} - x_{12}x_3 = x_{123} = x_{124} = x_{134} = x_{234} = x_{1234} = 0. \end{aligned}$$

Remark 1. The definitions given above can be generalized to a finite-dimensional, alternative real algebra A with a fixed antiinvolution $x \mapsto x^c$ (cf. [9]). In this general case, the quadratic cone is defined as $\mathcal{Q}_A := \mathbb{R} \cup \mathcal{SQ}_A$, where

$$\mathcal{SQ}_A := \{x \in A \mid t(x) \in \mathbb{R}, n(x) \in \mathbb{R}, 4n(x) > t(x)^2\}$$

is called *smooth quadratic cone*.

For example, we can consider the quaternions \mathbb{H} and the octonions \mathbb{O} with the usual conjugation: in these cases $\mathcal{Q}_{\mathbb{H}} = \mathbb{H}$ and $\mathcal{Q}_{\mathbb{O}} = \mathbb{O}$, $\mathbb{S}_{\mathbb{H}}$ is a two-dimensional sphere and $\mathbb{S}_{\mathbb{O}}$ is a six-dimensional sphere. Other examples are the Clifford algebras $Cl_{p,q} = \mathbb{R}_{p,q}$ with Clifford conjugation as antiinvolution. For example, in $\mathbb{R}_{1,2}$ the quadratic cone $\mathcal{Q}_{1,2}$ is a 6-dim real semi-algebraic set.

Proposition 2.

1. If $x \in \mathcal{Q}_n$, then there exist unique $x_0 \in \mathbb{R}$, $y \in \text{Im}(\mathbb{R}_n) \cap \mathcal{Q}_n$, with $t(y) = 0$, such that $x = x_0 + y$. We will denote y by $\text{Im}(x)$.

For $J \in \mathbb{S}_n$, let $\mathbb{C}_J := \langle 1, J \rangle \simeq \mathbb{C}$ be the subalgebra generated by J . Then $\mathcal{Q}_n = \bigcup_{J \in \mathbb{S}_n} \mathbb{C}_J$ and $\mathbb{C}_I \cap \mathbb{C}_J = \mathbb{R}$ for every $I, J \in \mathbb{S}_n$, $I \neq \pm J$.

2. $\mathcal{SQ}_n := \mathcal{Q}_n \setminus \mathbb{R} \simeq \mathbb{C}^+ \times \mathbb{S}_n$ with diffeomorphism $x \mapsto \left(x_0 + |y|\sqrt{-1}, \frac{y}{|y|} \right)$.

3 Slice regular functions

Let $\mathbb{C}_n = \mathbb{R}_n \otimes_{\mathbb{R}} \mathbb{C}$ be the complexified Clifford algebra.

Definition 3. Let $D \subseteq \mathbb{C}$ be an open subset. If a function $F : D \rightarrow \mathbb{C}_n$ is *complex intrinsic*, i.e. $F(\bar{z}) = \overline{F(z)}$ for every $z \in D$ such that $\bar{z} \in D$, then F is called a *stem function* on D . Let

$$\Omega_D := \{x = \alpha + \beta J \in \mathbb{C}_J \mid \alpha + i\beta \in D, J \in \mathbb{S}_n\}$$

be a *circular set* in the quadratic cone \mathcal{Q}_n . Any stem function $F : D \rightarrow \mathbb{C}_n$ induces a (*left*) *slice function* $f =: \mathcal{I}(F) : \Omega_D \rightarrow \mathbb{R}_n$: if $x = \alpha + \beta J \in D_J := \Omega_D \cap \mathbb{C}_J$ and $F = F_1 + iF_2$, we set

$$f(x) := F_1(z) + JF_2(z) \quad (z = \alpha + i\beta).$$

There is an analogous definition for *right* slice functions when J is placed on the right of $F_2(z)$. We will denote the real vector space of (left) slice functions on Ω_D by $\mathcal{S}(\Omega_D)$.

Left multiplication by i defines a complex structure on \mathbb{C}_n . With respect to this structure, a C^1 function $F = F_1 + iF_2 : D \rightarrow \mathbb{C}_n$ is holomorphic if and only if its components F_1, F_2 satisfy the Cauchy–Riemann equations.

Definition 4. A slice function is (*left*) *slice regular* if its stem function F is holomorphic. We set $\mathcal{SR}(\Omega_D) := \{f \in \mathcal{S}(\Omega_D) \mid f = \mathcal{I}(F), F \text{ holomorphic}\}$

Examples 2.

(1) *Standard polynomials* $p(x) = \sum_{j=0}^m x^j a_j$ with right Clifford coefficients are slice regular functions on \mathcal{Q}_n .

(2) *Convergent power series* $\sum_k x^k a_k$ are slice regular functions on the intersection of \mathcal{Q}_n with a ball centered in the origin.

(3) Let $J_0 \in \mathbb{S}_2 \subseteq \mathbb{H}$ be fixed. The function defined on $\mathcal{S}\mathcal{Q}_2 = \mathbb{H} \setminus \mathbb{R}$ by $f(x) = 1 + JJ_0$ for $x \in \mathbb{C}_J^+ = \{x = \alpha + \beta J \in \mathbb{H} \mid \beta \geq 0\}$ is slice regular on $\mathbb{H} \setminus \mathbb{R}$.

Proposition 3. *Let $f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D)$. Then f is slice regular on Ω_D if and only if the restriction $f_J := f|_{\mathbb{C}_J \cap \Omega_D} : D_J = \mathbb{C}_J \cap \Omega_D \rightarrow \mathbb{R}_n$ is holomorphic for every $J \in \mathbb{S}_n$ with respect to the complex structures defined by left multiplication by J .*

Proposition 3 implies that if $\mathbb{R}_2 = \mathbb{H}$ is the algebra of quaternions, and D intersects the real axis, then f is slice regular on Ω_D if and only if it is *Cullen regular* in the sense introduced by Gentili and Struppa in [5, 6]. If $n > 2$, slice regularity generalizes the concept of *slice monogenic functions* introduced by Colombo, Sabadini and Struppa in [1]. If $f = \mathcal{I}(F) \in \mathcal{SR}(\Omega_D)$, $F \in C^1(D)$ and D intersects the real axis, then the restriction of f to the subspace of paravectors is a slice monogenic function.

4 Product of slice functions

In general, the pointwise product of two slice functions is not a slice function. However, pointwise product of stem functions induces a natural product on slice functions.

Definition 5. Let $f = \mathcal{I}(F), g = \mathcal{I}(G) \in \mathcal{S}(\Omega_D)$. The *product* of f and g is the slice function $f \cdot g := \mathcal{I}(FG) \in \mathcal{S}(\Omega_D)$.

Let $f^c := \mathcal{I}(F^c) = \mathcal{I}(F_1^c + iF_2^c)$. The slice function $N(f) := f \cdot f^c$ is called the *normal function* of f .

This product is distributive and associative. If the components F_1, F_2 of the *first* stem function F are real-valued, then $(f \cdot g)(x) = f(x)g(x)$ for every $x \in \Omega_D$. In this case, $f = \mathcal{I}(F)$ is called *real*. Real slice functions are characterized by the following property: for every $J \in \mathbb{S}_n$, $f(\mathbb{C}_J \cap \Omega_D)$ is contained in \mathbb{C}_J .

Remark 2. Let $f(x) = \sum_j x^j a_j$ and $g(x) = \sum_k x^k b_k$ be polynomials or convergent power series with coefficients $a_j, b_k \in \mathbb{R}_n$. Then the product $f \cdot g$ coincides with the star product $(f * g)(x) := \sum_n x^n (\sum_{j+k=n} a_j b_k)$.

If f is slice regular, then also f^c and $N(f)$ are slice regular. If $n = 2$, then the normal function $N(f)$ is always real. For a general algebra \mathbb{R}_n , this is not true for every slice function. This is the motivation for the following definition.

Definition 6. Let

$$\mathcal{N}_n := \{0\} \cup \{x \in \mathbb{R}_n \mid n(x) = n(x^c) \in \mathbb{R} \setminus \{0\}\} \supseteq \mathcal{Q}_n$$

be the *normal cone* of \mathbb{R}_n . A slice function $f = \mathcal{I}(F)$ is *admissible* if the *spherical value* $v_s f(x) = \frac{1}{2}(f(x) + f(x^c)) \in \mathcal{N}_n$ for every $x \in \Omega_D$ and the subspace $\langle v_s f(x), \partial_s f(x) \rangle \subseteq \mathcal{N}_n$ for every $x \in \Omega_D \setminus \mathbb{R}$. Here $\partial_s f(x) = \frac{1}{2} \text{Im}(x)^{-1}(f(x) - f(x^c))$ denotes the *spherical derivative* of f in $x \notin \mathbb{R}$.

We will denote by $\mathcal{AR}(\Omega_D)$ the space of slice regular admissible function on Ω_D . If f is real, then it is admissible. If f is admissible, then $N(f)$ is real. The normal cone \mathcal{N}_n of \mathbb{R}_n contains the subspace of paravectors. Therefore every polynomial $p(x) = \sum_n x^n a_n$ with paravector coefficients is admissible.

5 Zeros of admissible slice functions

Let $\mathbb{S}_x := \{y \in \mathcal{Q}_n \mid y = \alpha + \beta I, I \in \mathbb{S}_n\}$ be the “*sphere*” containing x . The zero set $V(f) = \{x \in \Omega_D \mid f(x) = 0\}$ of an admissible slice function f has a particular structure. For every fixed $x \in \mathcal{Q}_n$, \mathbb{S}_x is entirely contained in $V(f)$ or it contains at most one zero of f . Moreover, if f is not real, there can be isolated, non-real zeros.

Theorem 4. *Let $f \in \mathcal{S}(\Omega_D)$ be an admissible slice function. Let $x \in \Omega_D$. Then one of the following holds:*

1. $\mathbb{S}_x \cap V(f) = \emptyset$.
2. $\mathbb{S}_x \subseteq V(f)$. Then x is called a *real* (if $x \in \mathbb{R}$) or *spherical* (if $x \notin \mathbb{R}$) zero of f .
3. $\mathbb{S}_x \cap V(f)$ consists of a single, non-real point (a \mathbb{S}_x -isolated non-real zero).

Moreover, $V(N(f)) = \bigcup_{x \in V(f)} \mathbb{S}_x$ and if $f \in \mathcal{AR}(\Omega_D)$ is slice regular and $N(f) \neq 0$, then $\mathbb{C}_J \cap \bigcup_{x \in V(f)} \mathbb{S}_x$ is closed and discrete in $D_J = \mathbb{C}_J \cap \Omega_D \forall J \in \mathbb{S}_n$.

In the quaternionic case, a structure theorem for the zero set of slice regular functions was proved by Pogorui and Shapiro [14] for polynomials and by Gentili and Stoppato [4] for power series. Similar results for polynomials with paravector coefficients in \mathbb{R}_n have been obtained by Colombo, Sabadini and Struppa [1, 2] and by Yang and Qian [15].

Theorem 5. *Let $f \in \mathcal{AR}(\Omega_D)$ and $y \in \Omega_D$. If \mathbb{S}_y contains at least one zero of f , then $N(x - y) \mid N(f)$ in $\mathcal{AR}(\Omega_D)$.*

Definition 7. Let $f \in \mathcal{AR}(\Omega_D)$ with $N(f) \neq 0$. Given $y \in V(f)$, we say that y is a *zero of f of multiplicity s* if $N(x - y)^s \mid N(f)$ and $N(x - y)^{s+1} \nmid N(f)$.

If y is a spherical zero, then $s \geq 2$. If $p(x) = \sum_{j=0}^m x^j a_j$ is an admissible polynomial of degree m with coefficients $a_j \in \mathbb{R}_n$, then $N(p)$ has degree $2m$ and real coefficients. A sufficient condition for the admissibility of p is that the real vector subspace $\langle a_0, \dots, a_m \rangle$ is contained in \mathcal{N}_n .

Theorem 6 (Fundamental Theorem of Algebra with multiplicities). *Let $p(x) = \sum_{j=0}^m x^j a_j$ be an admissible polynomial of degree $m > 0$ with coefficients in \mathbb{R}_n . Then $V(p) = \{y \in \mathcal{Q}_n \mid p(y) = 0\}$ is non-empty. Moreover, there are distinct “spheres” $\mathbb{S}_{x_1}, \dots, \mathbb{S}_{x_t}$ such that*

$$V(p) \subseteq \bigcup_{k=1}^t \mathbb{S}_{x_k} = V(N(p)), \quad V(p) \cap \mathbb{S}_{x_j} \neq \emptyset \quad \text{for every } j,$$

and, for any choice of zeros $y_1 \in \mathbb{S}_{x_1}, \dots, y_t \in \mathbb{S}_{x_t}$ of p , $\sum_{k=1}^t m_p(y_k) = m$.

Remark 3. If r denotes the number of real zeros of the polynomial p , i the number of isolated non-real zeros of p and s the number of “spheres” \mathbb{S}_y ($y \notin \mathbb{R}$) containing spherical zeros of p , we have that $r + i + 2s \leq \deg(p)$.

Examples 3. (1) Every polynomial $\sum_{j=0}^m x^j a_j$, with paravector coefficients a_j has m roots in \mathcal{Q}_n (counted with their multiplicities). If the coefficients are real, then it has at least one root in the paravector space \mathbb{R}^{n+1} , since every “sphere” \mathbb{S}_y intersect \mathbb{R}^{n+1} (cf. [15, Theorem 3.1]).

(2) In \mathbb{R}_3 , the polynomial $p(x) = xe_{23} + e_1$ vanishes only at $y = e_{123} \notin \mathcal{Q}_3$ (p is not admissible: $e_1, e_{23} \in \mathcal{N}_3$, but $e_1 + e_{23} \notin \mathcal{N}_3$).

(3) An admissible polynomial of degree m , even in the case of non-spherical zeros, can have more than m roots in the whole algebra. For example, $p(x) = x^2 - 1$ has four roots in \mathbb{R}_3 , two in the quadratic cone ($x = \pm 1$) and two outside it ($x = \pm e_{123}$).

(4) In \mathbb{R}_3 , the admissible polynomial $p(x) = x^3 - 1$ has zero set $V(p) = \{1\} \cup \mathbb{S}_y$ ($y = -\frac{1}{2} + \frac{\sqrt{3}}{2}J$, $J \in \mathbb{S}_3$) in \mathcal{Q}_3 , while in $\mathbb{R}_3 \setminus \mathcal{Q}_3$ the polynomial p vanishes on two 2-spheres.

(5) In the algebras \mathbb{H}, \mathbb{R}_3 , the solutions of the equation $x^2 = -1$ are exactly the elements of $\mathbb{S}_2 \subseteq \mathcal{Q}_2$ and $\mathbb{S}_3 \subseteq \mathcal{Q}_3$, but in \mathbb{R}_4 the equation $x^2 = -1$ has infinite other roots outside the quadratic cone.

6 The almost complex structure of \mathcal{SQ}_n

The smooth quadratic cone $\mathcal{SQ}_2 = \mathbb{H} \setminus \mathbb{R}$ has a natural almost complex structure related to the quaternionic product: given $x \in \mathcal{SQ}_2$, let $J_x := \text{Im}(x)/|\text{Im}(x)| \in \mathbb{S}_2$ and $\mathcal{J}(x)$ the endomorphism of the tangent space $T_x(\mathcal{SQ}_2)$ defined by left multiplication by J_x . The diffeomorphism $\mathcal{SQ}_2 \simeq \mathbb{C}^+ \times \mathbb{S}_2$ is compatible with \mathcal{J} . Therefore $(\mathcal{SQ}_2, \mathcal{J})$ is a 2-dim complex manifold, isomorphic to $\mathbb{C}^+ \times \mathbb{CP}^1$. Now two questions naturally arise.

1. Can we interpret slice regularity in terms of this almost complex structure?
2. Does it generalize to the cone \mathcal{SQ}_n in \mathbb{R}_n ?

The first question has a positive answer.

Proposition 7. *If $\Omega_D \cap \mathbb{R} = \emptyset$, then $\Omega_D \subseteq \mathcal{SQ}_2$ and $\mathcal{SR}(\Omega_D) \simeq \text{Hol}(\Omega_D, \mathbb{H} \otimes \mathbb{C})$ by the map $f = \mathcal{I}(F) \mapsto F' = F \circ \pi_1$ where $\pi_1 : \mathcal{SQ}_2 \simeq \mathbb{C}^+ \times \mathbb{S}_2 \rightarrow \mathbb{C}^+$.*

If $\Omega_D \cap \mathbb{R} \neq \emptyset$, then $\mathcal{SR}(\Omega_D)$ is isomorphic to the space

$$\{F' \in \text{Hol}(\Omega_D \setminus \mathbb{R}, \mathbb{H} \otimes \mathbb{C}) \mid F' \text{ has a } C^0 \text{ extension to } \Omega_D \text{ s.t. } F'|_{\Omega_D \cap \mathbb{R}} \text{ is } \mathbb{H}\text{-valued}\}.$$

Examples 4. (1) $F'(x) = \sum_{j=0}^m z^j a_j$ induces the polynomial $f(x) = \sum_{j=0}^m x^j a_j$ (the extension of F' to \mathbb{H} is \mathbb{H} -valued on \mathbb{R}).

(2) Let $J_0 \in \mathbb{S}_2$ be fixed. The *constant* function $F'(x) = 1 + iJ_0$ induces the slice regular function $f(x) = 1 + J_x J_0$ ($x \in \mathbb{C}_J^+$) (its continuous extension to \mathbb{H} is *not* \mathbb{H} -valued on \mathbb{R}).

Remark 4. The function $f \in \mathcal{SR}(\mathbb{H} \setminus \mathbb{R})$ in Ex.4.2 has normal function $N(f) \equiv 0$ and zero set $V(f) = \mathbb{C}_{J_0}^+$. In general, the non-spherical zero set of a slice regular function with $N(f) \equiv 0$ is a “wing” \mathcal{W}_J diffeomorphic to D (we assume $D \subset \mathbb{C}^+$) defined by a holomorphic curve $J(z) : D \rightarrow (\mathbb{S}_2, \mathcal{J})$, of the form

$$\mathcal{W}_J := \{\text{Re}(z) + \text{Im}(z)J(z) \mid z \in D\}.$$

For the second question posed above, the answer is not definitive. We are able to prove the following result.

Theorem 8. *Assume that for every $y, z \in \mathbb{S}_n$, $y \neq \pm z$, there exists $x \in \mathcal{N}_n$, $x \neq 0$, such that $xyx^{-1} = z$. Then:*

1. \mathcal{SQ}_n and \mathbb{S}_n are smooth manifolds of even dimension over \mathbb{R} .
2. \mathcal{SQ}_n and \mathbb{S}_n have an almost complex structure \mathcal{J} defined by left multiplication by J_x .

It can be shown that the assumption of the preceding theorem is true for $n = 2$ and $n = 3$.

Examples 5. (1) $n = 3$. The normal cone is

$$\mathcal{N}_3 = \{x \in \mathbb{R}_3 \mid x_0 x_{123} + x_2 x_{13} - x_1 x_{23} - x_3 x_{12} = 0\}$$

(diffeomorphic to the Simons minimal cone in \mathbb{R}^8). The almost complex structure \mathcal{J} on the smooth quadratic cone $\mathcal{SQ}_3 = \{x \in \mathbb{R}_3 \setminus \mathbb{R} \mid x_{123} = 0, x_2 x_{13} - x_1 x_{23} - x_3 x_{12} = 0\}$ is integrable. More precisely, $(\mathcal{SQ}_3, \mathcal{J})$ is a 3-dim complex manifold, isomorphic to $\mathbb{C}^+ \times \mathbb{CP}^1 \times \mathbb{CP}^1$.

(2) $n = 4$. The normal cone \mathcal{N}_4 is the 11-dimensional real algebraic set with equations

$$\begin{aligned} x_1 x_{1234} + x_{124} x_{13} - x_{12} x_{134} - x_{123} x_{14} &= 0, & x_2 x_{1234} + x_{124} x_{23} - x_{12} x_{234} - x_{123} x_{24} &= 0, \\ x_3 x_{1234} + x_{134} x_{23} - x_{13} x_{234} - x_{123} x_{34} &= 0, & x_4 x_{1234} - x_{14} x_{234} + x_{134} x_{24} - x_{124} x_{34} &= 0, \\ x_{134} x_2 - x_1 x_{234} - x_{124} x_3 + x_{123} x_4 &= 0, & x_0 x_{1234} - x_{14} x_{23} + x_{13} x_{24} - x_{12} x_{34} &= 0, \\ x_0 x_{234} + x_3 x_{24} - x_2 x_{34} - x_{23} x_4 &= 0, & x_0 x_{134} + x_3 x_{14} - x_1 x_{34} - x_{13} x_4 &= 0, \\ x_0 x_{124} + x_2 x_{14} - x_1 x_{24} - x_{12} x_4 &= 0, & x_0 x_{123} + x_2 x_{13} - x_1 x_{23} - x_{12} x_3 &= 0, \end{aligned}$$

while the quadratic cone \mathcal{Q}_4 is a 8-dim real algebraic set, singular at every real point (cf. Ex.1.2 for its defining equations)

Remark 5. If the assumption of the theorem is true for \mathbb{R}_n , the complex dimension of \mathcal{SQ}_n is equal to n for $n = 2, 3, 4, 5$, but starting from $n = 6$ it is larger than n . The exact formula is

$$\dim_{\mathbb{C}}(\mathcal{SQ}_n) = 1 + \sum_{\substack{k \equiv 1 \pmod{4} \\ 1 \leq k \leq n-1}} \binom{n-1}{k} \in \{2, 3, 4, 5, 7, 13, \dots\}.$$

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