ON REGULAR HARMONICS OF ONE QUATERNIONIC VARIABLE

A. PEROTTI

ABSTRACT. We prove some results about the Fueter-regular homogeneous polynomials, which appear as components in the power series of any quaternionic regular function. We obtain a differential condition that characterizes the homogeneous polynomials whose trace on the unit sphere extends as a regular polynomial. We apply this result to define an injective linear operator from the space of complex spherical harmonics to the module of regular homogeneous polynomials of a fixed degree k.

1. Introduction

Let B denote the unit ball in $\mathbb{C}^2 \simeq \mathbb{H}$ and $S = \partial B$ the group of unit quaternions. In §3.1 we obtain a differential condition that characterizes the homogeneous polynomials whose restriction to S coincides with the restriction of a regular polynomial. This result generalizes a similar characterization for holomorphic extensions of polynomials proved by Kytmanov (cf. [2] and [3]).

In §3.2 we show how to define an injective linear operator $R : \mathcal{H}_k(S) \to U_k^\psi$ from the space $\mathcal{H}_k(S)$ of complex-valued spherical harmonics of degree k to the $\mathbb{H}$-module U_k^ψ of ψ-regular homogeneous polynomials of the same degree (cf. §2.2 and §3.2 for precise definitions). In particular, we show how to construct bases of the module of regular homogeneous polynomials of a fixed degree starting from any choice of $\mathbb{C}$-bases of the spaces of complex harmonic homogeneous polynomials.

This work was partially supported by MIUR (Project "Proprietà geometriche delle varietà reali e complesse") and GNSAGA of INdAM.

2. Notations and definitions

2.1. Let $\Omega = \{z \in \mathbb{C}^2 : \rho(z) < 0\}$ be a bounded domain in $\mathbb{C}^2$ with smooth boundary. Let ν denote the outer unit normal to $\partial\Omega$ and $\tau = i\nu$. For every $F \in C^1(\overline{\Omega})$, let $\overline{\partial}_n F = \frac{1}{2}\left(\frac{\partial F}{\partial \nu} + i\frac{\partial F}{\partial \tau}\right)$ be the normal component of $\overline{\partial} F$ (see Kytmanov [2]§§3.3 and 14.2). It can be expressed by means of the Hodge $*$-operator and the Lebesgue surface measure as $\overline{\partial}_n f d\sigma = *\overline{\partial} f_{|\partial\Omega}$. In a neighbourhood of $\partial\Omega$ we have the decomposition of $\overline{\partial} F$ in the tangential and the normal parts:

1991 *Mathematics Subject Classification.* 32A30, 30G35, 32V10, 32W05.

Key words and phrases. Quaternionic regular functions, spherical harmonics.

$\overline{\partial}F = \overline{\partial}_b F + \overline{\partial}_n F \frac{\overline{\partial}\rho}{|\overline{\partial}\rho|}$. We denote by L the tangential Cauchy-Riemann operator $L = \frac{1}{|\partial\rho|}\left(\frac{\partial\rho}{\partial\bar{z}_2}\frac{\partial}{\partial\bar{z}_1} - \frac{\partial\rho}{\partial\bar{z}_1}\frac{\partial}{\partial\bar{z}_2}\right)$.

Let $\mathbb{H}$ be the algebra of quaternions $q = x_0 + ix_1 + jx_2 + kx_3$, where x_0, x_1, x_2, x_3 are real numbers and i, j, k denote the basic quaternions. We identify the space $\mathbb{C}^2$ with the set $\mathbb{H}$ by means of the mapping that associates the quaternion $q = z_1 + z_2 j$ to $(z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$. We refer to Sudbery [8] for the basic facts of quaternionic analysis. We will denote by $\mathcal{D}$ the left Cauchy-Riemann-Fueter operator

$$\mathcal{D} = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}.$$

A quaternionic C^1 function $f = f_1 + f_2 j$, is *(left-)regular* on a domain $\Omega \subseteq \mathbb{H}$ if $\mathcal{D}f = 0$ on Ω. We prefer to work with another class of regular functions, which is more explicitely connected with the hyperkähler structure of $\mathbb{H}$. It is defined by the Cauchy-Riemann-Fueter operator associated to the structural vector $\psi = \{1, i, j, -k\}$:

$$\mathcal{D}' = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3} = 2\left(\frac{\partial}{\partial\bar{z}_1} + j\frac{\partial}{\partial\bar{z}_2}\right).$$

A quaternionic C^1 function $f = f_1 + f_2 j$, is called *(left-)ψ-regular* on a domain Ω, if $\mathcal{D}'f = 0$ on Ω. This condition is equivalent to the following system of complex differential equations:

$$\frac{\partial f_1}{\partial\bar{z}_1} = \frac{\partial\overline{f_2}}{\partial z_2}, \qquad \frac{\partial f_1}{\partial\bar{z}_2} = -\frac{\partial\overline{f_2}}{\partial z_1}.$$

The identity mapping is ψ-regular, and any holomorphic mapping (f_1, f_2) on Ω defines a ψ-regular function $f = f_1 + f_2 j$. This is no more true if we replace ψ-regularity with regularity. Moreover, the complex components of a ψ-regular function are either both holomorphic or both non-holomorphic (cf. Vasilevski [9], Mitelman et al [4] and Perotti [5]). Let γ be the transformation of $\mathbb{C}^2$ defined by $\gamma(z_1, z_2) = (z_1, \bar{z}_2)$. Then a C^1 function f is regular on the domain Ω if, and only if, $f \circ \gamma$ is ψ-regular on $\gamma^{-1}(\Omega)$.

2.2. The two-dimensional Bochner-Martinelli form $U(\zeta, z)$ is the first complex component of the Cauchy-Fueter kernel $G'(p-q)$ associated to ψ-regular functions (cf. Fueter [1], Vasilevski [9], Mitelman et al [4]). Let $q = z_1 + z_2 j$, $p = \zeta_1 + \zeta_2 j$, $\sigma(q) = dx[0] - idx[1] + jdx[2] + kdx[3]$, where $dx[k]$ denotes the product of dx_0, dx_1, dx_2, dx_3 with dx_k deleted. Then $G'(p-q)\sigma(p) = U(\zeta, z) + \omega(\zeta, z)j$, where $\omega(\zeta, z)$ is the complex $(1,2)$-form

$$\omega(\zeta, z) = -\frac{1}{4\pi^2}|\zeta - z|^{-4}((\bar{\zeta}_1 - \bar{z}_1)d\zeta_1 + (\bar{\zeta}_2 - \bar{z}_2)d\zeta_2) \wedge \overline{d\zeta}.$$

Here $\overline{d\zeta} = \overline{d\zeta_1} \wedge \overline{d\zeta_2}$ and we choose the orientation of $\mathbb{C}^2$ given by the volume form $\frac{1}{4}dz_1 \wedge dz_2 \wedge \overline{dz_1} \wedge \overline{dz_2}$. Given $g(\zeta, z) = \frac{1}{4\pi^2}|\zeta - z|^{-2}$, we can also write $U(\zeta, z) = -2 * \partial_\zeta g(\zeta, z)$ and $\omega(\zeta, z) = -\partial_\zeta(g(\zeta, z)\overline{d\zeta})$.

3. Regular polynomials

3.1. In this section we will obtain a differential condition that characterizes the homogeneous polynomials whose restrictions to the unit sphere extend regularly or ψ-regularly. We will use a computation made by Kytmanov in [3] (cf. also [2] Corollary 23.4), where the analogous result for holomorphic extensions is proved.

Let Ω be the unit ball B in $\mathbb{C}^2$, $S=\partial B$ the unit sphere. In this case the operators $\overline{\partial}_n$ and L have the following forms:

$$\overline{\partial}_n=\bar z_1\frac{\partial}{\partial\bar z_1}+\bar z_2\frac{\partial}{\partial\bar z_2},\qquad L=z_2\frac{\partial}{\partial\bar z_1}-z_1\frac{\partial}{\partial\bar z_2}$$

and they preserve harmonicity. Let $\Delta=\frac{\partial^2}{\partial z_1\partial\bar z_1}+\frac{\partial^2}{\partial z_2\partial\bar z_2}$ be the Laplacian in $\mathbb{C}^2$ and D_k the differential operator

$$D_k=\sum_{0\le l\le k/2-1}\frac{(k-2l-1)!(2l-1)!!}{k!(l+1)!}2^l\Delta^{l+1}.$$

Theorem 1. *Let $f=f_1+f_2j$ be a $\mathbb{H}$-valued, homogeneous polynomial of degree k. Then its restriction to S extends as a ψ-regular function into B if, and only if,*

$$(\overline{\partial}_n-D_k)f_1+\overline{L(f_2)}=0\quad\text{on } S.$$

Proof. In the first part we can proceed as in [3]. The harmonic extension $\tilde f_1$ of $f_{1|S}$ into B is given by Gauss's formula: $\tilde f_1=\sum_{s\ge0}g_{k-2s}$, where g_{k-2s} is the homogeneous harmonic polynomial of degree $k-2s$ defined by

$$g_{k-2s}=\frac{k-2s+1}{s!(k-s+1)!}\sum_{j\ge0}\frac{(-1)^j(k-j-2s)!}{j!}|z|^{2j}\Delta^{j+s}f_1.\qquad(*)$$

Then $\overline{\partial}_n\tilde f_1=\overline{\partial}_nf_1-D_kf_1$ on S (cf. [2] §23). Let $\tilde f_2$ be the harmonic extension of f_2 into B and $\tilde f=\tilde f_1+\tilde f_2j$. Then $(\overline{\partial}_n-D_k)f_1+\overline{L(f_2)}=0$ on S is equivalent to $\overline{\partial}_n\tilde f_1+\overline{L(f_2)}=0$ on S. We now show that this implies the ψ-regularity of $\tilde f$. Let F^+ and F^- be the ψ-regular functions defined respectively on B and on $\mathbb{C}^2\setminus\overline{B}$ by the Cauchy-Fueter integral of $\tilde f$:

$$F^\pm(z)=\int_S U(\zeta,z)\tilde f(\zeta)+\int_S\omega(\zeta,z)j\tilde f(\zeta).$$

From the equalities $U(\zeta,z)=-2{*}\partial_\zeta g(\zeta,z)$, $\omega(\zeta,z)=-\partial_\zeta(g(\zeta,z)d\bar\zeta)$, we get that

$$F^-(z)=-2\int_S(\tilde f_1(\zeta)+f_2(\zeta)j){*}\partial_\zeta g(\zeta,z)-\int_S\partial_\zeta(g(\zeta,z)\overline{d\zeta})(\overline{\tilde f_1}j-\overline{\tilde f_2})$$

for every $z\notin\overline{B}$. From the complex Green formula and Stokes' Theorem and from the equality $\overline{\partial}\tilde f_2\wedge d\zeta_{|S}=2L(f_2)d\sigma$ on S, we get that the first complex component of $F^-(z)$ is

$$-2\int_S \tilde{f}_1 \partial_n g d\sigma + \int_S \overline{\tilde{f}_2} \partial_\zeta g \wedge \overline{d\zeta} = -2\int_S g \overline{\partial}_n \tilde{f}_1 d\sigma - \int_S g \partial_\zeta \overline{\tilde{f}_2} \wedge \overline{d\zeta}$$
$$= -2\int_S g(\overline{\partial}_n \tilde{f}_1 + \overline{L(f_2)})d\sigma$$

and then it vanishes on $\mathbb{C}^2 \setminus \overline{B}$. Therefore, $F^- = F_2 j$, with F_2 a holomorphic function that can be holomorphically continued to the whole space. Let $\tilde{F}^- = \tilde{F}_2 j$ be such extension. Then $F = F^+ - \tilde{F}^-_{|B}$ is a ψ-regular function on B (indeed a polynomial of the same degree k), continuous on $\overline{B}$, such that $F_{|S} = f_{|S}$. The converse is immediate from the equations of ψ-regularity. □

Let N and T be the differential operators

$$N = \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + z_2 \frac{\partial}{\partial z_2}, \qquad T = \bar{z}_2 \frac{\partial}{\partial \bar{z}_1} - z_1 \frac{\partial}{\partial z_2}.$$

T is a tangential operator w.r.t. S, while N is non-tangential, such that $N(\rho) = |\overline{\partial}\rho|^2$, $\mathrm{Re}(N) = |\overline{\partial}\rho| \, \mathrm{Re}(\overline{\partial}_n)$, where $\rho = |z_1|^2 + |z_2|^2 - 1$. Let γ be the reflection introduced at the end of §1.1. The operator D_k is γ-invariant, i.e. $D_k(f \circ \gamma) = D_k(f) \circ \gamma$, since Δ is invariant. It follows a criterion for regularity of homogeneous polynomials.

Corollary 2. *Let $f = f_1 + f_2 j$ be a $\mathbb{H}$-valued, homogeneous polynomial of degree k. Then its restriction to S extends as a regular function into B if, and only if,*

$$(N - D_k) f_1 + \overline{T(f_2)} = 0 \quad \text{on } S.$$

Let $g = \sum_k g^k$ be the homogeneous decomposition of a polynomial g. After replacing $D_k g$ by $\sum_k D_k g^k$, we can extend the preceding results also to non-homogeneous polynomials.

3.2. Let $\mathcal{P}_k$ denote the space of homogeneous complex-valued polynomials of degree k on $\mathbb{C}^2$, and $\mathcal{H}_k$ the space of harmonic polynomials in $\mathcal{P}_k$. The space $\mathcal{H}_k$ is the sum of the pairwise $L^2(S)$-orthogonal spaces $\mathcal{H}_{p,q}$ $(p+q=k)$, whose elements are the harmonic homogeneous polynomials of degree p in z_1, z_2 and q in $\bar{z}_1, \bar{z}_2$ (cf. for example Rudin [7]§12.2). The spaces $\mathcal{H}_k$ and $\mathcal{H}_{p,q}$ can be identified with the spaces of the restrictions of their elements to S (*spherical harmonics*). These spaces will be denoted by $\mathcal{H}_k(S)$ and $\mathcal{H}_{p,q}(S)$ respectively.

Let U_k^ψ be the right $\mathbb{H}$-module of (left)ψ-regular homogeneous polynomials of degree k. The elements of the modules U_k^ψ can be identified with their restrictions to S, which we will call *regular harmonics*.

Theorem 3. *For every $f_1 \in \mathcal{P}_k$, there exists $f_2 \in \mathcal{P}_k$ such that the trace of $f = f_1 + f_2 j$ on S extends as a ψ-regular polynomial of degree at most k on $\mathbb{H}$. If $f_1 \in \mathcal{H}_k$, then $f_2 \in \mathcal{H}_k$ and $f = f_1 + f_2 j \in U_k^\psi$.*

Proof. We can suppose that f_1 has degree p in z and q in $\bar{z}$, $p+q=k$, and then extend by linearity. Let $\tilde{f_1} = \sum_{s\geq 0} g_{p-s,q-s}$ be the harmonic extension of f_1 into B, where $g_{p-s,q-s} \in \mathcal{H}_{p-s,q-s}$ is given by formula (*). Then $\overline{\partial}_n \overline{L(g_{p-s,q-s})} = (p-s+1)\overline{L(g_{p-s,q-s})}$. We set

$$\tilde{f_2} = \sum_{s\geq 0} \frac{1}{p-s+1} \overline{L(g_{p-s,q-s})} \in \bigoplus_{s\geq 0} \mathcal{H}_{k-2s}.$$

Then $\overline{\partial}_n \tilde{f_2} = \overline{L(f_1)}$ on S and we can conclude as in the proof of Theorem 1 that $\tilde{f} = \tilde{f_1} + \tilde{f_2} j$ is a ψ-regular polynomial of degree at most k. Now it suffices to define

$$f_2 = \sum_{s\geq 0} \frac{|z|^{2s}}{p-s+1} \overline{L(g_{p-s,q-s})} \in \mathcal{P}_k$$

to get a homogeneous polynomial $f = f_1 + f_2 j$, of degree k, that has the same restriction to S as $\tilde{f}$. If $f_1 \in \mathcal{H}_k$, then $\tilde{f_1} = f_1$, $\tilde{f_2} = f_2$ and therefore $f \in U_k^\psi$. □

Let $C : U_k^\psi \to \mathcal{H}_k(S)$ be the complex-linear operator that associates to $f = f_1 + f_2 j$ the restriction to S of its first complex component f_1. The function $\tilde{f}$ in the preceding proof gives a right inverse $R : \mathcal{H}_k(S) \to U_k^\psi$ of the operator C. The function $R(f_1)$ is uniquely determined by the orthogonality condition with respect to the functions holomorphic on a neighbourhood of $\overline{B}$:

$$\int_S (R(f_1) - f_1)\overline{h} d\sigma = 0 \quad \forall h \in \mathcal{O}(\overline{B}).$$

Corollary 4. *(i) The restriction operator C defined on U_k^ψ induces isomorphisms of real vector spaces*

$$\frac{U_k^\psi}{\mathcal{H}_{k,0} j} \simeq \mathcal{H}_k(S), \qquad \frac{U_k^\psi}{\mathcal{H}_{k,0} + \mathcal{H}_{k,0} j} \simeq \frac{\mathcal{H}_k(S)}{\mathcal{H}_{k,0}(S)}.$$

(ii) U_k^ψ has dimension $\frac{1}{2}(k+1)(k+2)$ over $\mathbb{H}$.

Proof. The first part follows from $\ker C = \{f = f_1 + f_2 j \in U_k^\psi \ : \ f_1 = 0 \text{ on } S\} = \mathcal{H}_{k,0} j$. Part (ii) can be obtained from any of the above isomorphisms, since $\mathcal{H}_{k,0}$ (as every space $\mathcal{H}_{p,q}, p+q=k$) and $\mathcal{H}_k(S)$ have real dimensions respectively $2(k+1)$ and $2(k+1)^2$. □

As an application of Corollary 2, we have another proof of the known result (cf. Sudbery [8] Theorem 7) that the right $\mathbb{H}$-module U_k of left-regular homogeneous polynomials of degree k has dimension $\frac{1}{2}(k+1)(k+2)$ over $\mathbb{H}$.

3.3. The operator $R : \mathcal{H}_k(S) = \bigoplus_{p+q=k} \mathcal{H}_{p,q}(S) \to U_k^\psi$ can also be used to obtain $\mathbb{H}$-bases for U_k^ψ starting from bases of the complex spaces $\mathcal{H}_{p,q}(S)$. On $\mathcal{H}_{p,q}(S)$, R acts in the following way:

$$R(h) = h + M(h)j, \quad \text{where } M(h) = \frac{1}{p+1}\overline{L(h)} \in \mathcal{H}_{q-1,p+1} \quad (h \in \mathcal{H}_{p,q})$$

Note that $M \equiv 0$ on $\mathcal{H}_{k,0}(S)$. If $q > 0$, $M^2 = -Id$ on $\mathcal{H}_{p,q}(S)$, since $qh = \overline{\partial}_n h = -\overline{L(M(h))}$ on S, and therefore

$$h = -\frac{1}{q}\overline{L(M(h))} = -\frac{1}{q(p+1)}\overline{L}L(h) = -M^2(h).$$

If $k = 2m+1$ is odd, then M is a complex conjugate isomorphism of $\mathcal{H}_{m,m+1}(S)$. Then M induces a quaternionic structure on this space, which has real dimension $4(m+1)$. We can find complex bases of $\mathcal{H}_{m,m+1}(S)$ of the form

$$\{h_1, M(h_1), \dots, h_{m+1}, M(h_{m+1})\}.$$

Theorem 5. *Let $\mathcal{B}_{p,q}$ denote a complex base of the space $\mathcal{H}_{p,q}(S)$ ($p+q = k$). Then:*
(i) if $k = 2m$ is even, a basis of U_k^ψ over $\mathbb{H}$ is given by the set

$$\mathcal{B}_k = \{R(h) : h \in \mathcal{B}_{p,q}, p+q = k, 0 \le q \le p \le k\}.$$

(ii) if $k = 2m+1$ is odd, a basis of U_k^ψ over $\mathbb{H}$ is given by

$$\mathcal{B}_k = \{R(h) : h \in \mathcal{B}_{p,q}, p+q = k, 0 \le q < p \le k\} \cup \{R(h_1), \dots, R(h_{m+1})\},$$

where $h_1, \dots, h_{m+1}$ are chosen such that the set

$$\{h_1, M(h_1), \dots, h_{m+1}, M(h_{m+1})\}$$

forms a complex basis of $\mathcal{H}_{m,m+1}(S)$.

If the bases $\mathcal{B}_{p,q}$ are orthogonal in $L^2(S)$ and $h_1, \dots, h_{m+1} \in \mathcal{H}_{m,m+1}(S)$ are mutually orthogonal, then $\mathcal{B}_k$ is orthogonal, with norms

$$\|R(h)\|_{L^2(S,\mathbb{H})} = \left(\frac{p+q+1}{p+1}\right)^{1/2} \|h\|_{L^2(S)} \quad (h \in \mathcal{B}_{p,q})$$

w.r.t. the scalar product of $L^2(S, \mathbb{H})$.

Proof. From dimension count, it suffices to prove that the sets $\mathcal{B}_k$ are linearly independent. When $q \le p$, $q' \le p'$, $p+q = p'+q' = k$, the spaces $\mathcal{H}_{p,q}$ and $\mathcal{H}_{q'-1,p'+1}$ are distinct. Since $R(h) = h + M(h)j \in \mathcal{H}_{p,q} \oplus \mathcal{H}_{q-1,p+1}j$, this implies the independence over $\mathbb{H}$ of the images $\{R(h) : h \in \mathcal{B}_{p,q}\}$. It remains to consider the case when $k = 2m+1$ is odd. If $h \in \mathcal{H}_{m,m+1}(S)$, the complex components h and $M(h)$ of $R(h)$ belong to the same space. The independence of $\{R(h_1), \dots, R(h_{m+1})\}$ over $\mathbb{H}$ follows from the particular form of the complex basis chosen in $\mathcal{H}_{m,m+1}(S)$.

The scalar product of $L(h)$ and $L(h')$ in $\mathcal{H}_{p,q}(S)$ is

$$(L(h), L(h')) = (h, L^* L(h')) = -(h, \overline{L} L(h')) = q(p+1)(h, h'),$$

since the adjoint L^* is equal to $-\overline{L}$ (cf. [7]§18.2.2) and $\overline{L}L = q(p+1)M^2 = -q(p+1)Id$. Therefore, if h, h' are orthogonal, $M(h)$ and $M(h')$ are orthogonal in $\mathcal{H}_{q-1,p+1}$ and then also $R(h)$ and $R(h')$. Finally, the norm of $R(h)$, $h \in \mathcal{H}_{p,q}(S)$, is

$$\|R(h)\|^2 = \|h\|^2 + \|M(h)\|^2 = \|h\|^2 + \frac{1}{(p+1)^2}\|L(h)\|^2 = \frac{p+q+1}{p+1}\|h\|^2$$

and this concludes the proof. □

From Theorem 3 it is immediate to obtain also bases of the right $\mathbb{H}$-module U_k of left-regular homogeneous polynomials of degree k.

Examples. (i) The case $k = 2$. Starting from the orthogonal bases $\mathcal{B}_{2,0} = \{z_1^2, 2z_1 z_2, z_2^2\}$ of $\mathcal{H}_{2,0}$ and $\mathcal{B}_{1,1} = \{z_1\bar{z}_2, |z_1|^2 - |z_2|^2, z_2\bar{z}_1\}$ of $\mathcal{H}_{1,1}$ we get the orthogonal basis of regular harmonics

$$\mathcal{B}_2 = \{z_1^2, 2z_1z_2, z_2^2, z_1\bar{z}_2 - \frac{1}{2}\bar{z}_1^2 j, |z_1|^2 - |z_2|^2 + \bar{z}_1\bar{z}_2 j, z_2\bar{z}_1 + \frac{1}{2}\bar{z}_2^2 j\}$$

of the six-dimensional right $\mathbb{H}$-module U_2^{ψ}.

(ii) The case $k = 3$. From the orthogonal bases

$\mathcal{B}_{3,0} = \{z_1^3, 3z_1^2 z_2, 3z_1 z_2^2, z_2^3\}$, $\quad\mathcal{B}_{2,1} = \{z_1^2\bar{z}_2, 2z_1|z_2|^2 - z_1|z_1|^2, 2z_2|z_1|^2 - z_2|z_2|^2, z_2^2\bar{z}_1\}$,

$\mathcal{B}_{1,2} = \{h_1 = z_1\bar{z}_2^2, M(h_1) = -z_2\bar{z}_1^2, h_2 = -2\bar{z}_2|z_1|^2 + \bar{z}_2|z_2|^2, M(h_2) = -2\bar{z}_1|z_2|^2 + \bar{z}_1|z_1|^2\}$,

we get the orthogonal basis of regular harmonics

$$\mathcal{B}_3 = \{z_1^3, 3z_1^2z_2, 3z_1z_2^2, z_2^3, z_1^2\bar{z}_2 - \frac{1}{3}\bar{z}_1^3 j, 2z_1|z_2|^2 - z_1|z_1|^2 - \bar{z}_1^2\bar{z}_2 j, 2z_2|z_1|^2 - z_2|z_2|^2 + \bar{z}_1\bar{z}_2^2 j,$$
$$z_2^2\bar{z}_1 + \frac{1}{3}\bar{z}_2^3 j, z_1\bar{z}_2^2 - z_2\bar{z}_1^2 j, -2\bar{z}_2|z_1|^2 + \bar{z}_2|z_2|^2 + (\bar{z}_1|z_1|^2 - 2\bar{z}_1|z_2|^2 +)j\}.$$

of the ten-dimensional right $\mathbb{H}$-module U_3^{ψ}.

In general, for any k, an orthogonal basis of $\mathcal{H}_{p,q}$ $(p + q = k)$ is given by the polynomials $\{P_{q,l}^k\}_{l=0,\dots,k}$ defined by formula (6.14) in Sudbery [8]. The basis of U_k obtained from these bases by means of Theorem 3 and applying the reflection γ is essentially the same given in Proposition 8 of Sudbery [8].

Another spanning set of the space $\mathcal{H}_{p,q}$ is given by the functions

$$g_\alpha^{p,q}(z_1, z_2) = (z_1 + \alpha z_2)^p(\bar{z}_2 - \alpha\bar{z}_1)^q \quad (\alpha \in \mathbb{C})$$

(cf. Rudin [7]§12.5.1). Since $M(g_\alpha^{p,q}) = \frac{(-1)^q q\bar{\alpha}^{p+q}}{p+1} g_{-1/\bar{\alpha}}^{q-1,p+1}$ for $\alpha \neq 0$ and $M(g_0^{p,q}) = -\frac{q}{p+1} z_2^{q-1}\bar{z}_1^{p+1}$, where we set $g_\alpha^{p,q} \equiv 0$ if $p < 0$, from Theorem 3 we get that U_k^{ψ} is spanned over $\mathbb{H}$ by the polynomials

$$R(g_\alpha^{p,q}) = \begin{cases} g_\alpha^{p,q} + \frac{(-1)^q q\bar{\alpha}^{p+q}}{p+1} g_{-1/\bar{\alpha}}^{q-1,p+1} j & \text{for } \alpha \neq 0 \\ z_1^p\bar{z}_2^q - \frac{q}{p+1} z_2^{q-1}\bar{z}_1^{p+1} j & \text{for } \alpha = 0 \end{cases} \qquad (\alpha \in \mathbb{C},\ p + q = k)$$

Any choice of $k+1$ distinct numbers $\alpha_0, \alpha_1, \dots, \alpha_k$ gives rise to a basis of U_k^{ψ}.

The results obtained in this paper enabled the writing of a *Mathematica* package [6], named `RegularHarmonics`, which implements efficient computations with regular and ψ-regular functions and with harmonic and holomorphic functions of two complex variables.

References

[1] R. Fueter, Über einen Hartogs'schen Satz in der Theorie der analytischen Funktionen von n komplexen Variablen, *Comment. Math. Helv.*, **14** (1942), 394–400.

[2] A.M. Kytmanov, *The Bochner-Martinelli integral and its applications*, Birkhäuser Verlag, Basel, Boston, Berlin (1995).

[3] A.M. Kytmanov, Some differential criteria for the holomorphy of functions in $\mathbb{C}^n$, In: *Some problems of multidimensional complex analysis*, Akad. Nauk SSSR Sibirsk. Otdel., Inst. Fiz., Krasnoyarsk (1980), 51–64, 263–264.

[4] I.M. Mitelman and M.V. Shapiro, Differentiation of the Martinelli-Bochner integrals and the notion of hyperderivability, *Math. Nachr.*, **172** (1995), 211–238.

[5] A. Perotti, A differential criterium for regularity of quaternionic functions, *C. R. Acad. Sci. Paris, Ser. I Math.*, **337** no. 2 (2003), 89–92.

[6] A. Perotti, RegularHarmonics - a package for Mathematica 4.2 for computations with regular quaternionic functions
(available at `http://www.science.unitn.it/~perotti/regular harmonics.htm`).

[7] W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$, Springer-Verlag, New York, Heidelberg, Berlin (1980).

[8] A. Sudbery, Quaternionic analysis, *Mat. Proc. Camb. Phil. Soc.*, **85** (1979), 199–225.

[9] N.L. Vasilevski and M.V. Shapiro, Some questions of hypercomplex analysis, In: *Complex analysis and applications '87 (Varna, 1987)*, Publ. House Bulgar. Acad. Sci., Sofia (1989), 523–531.

Department of Mathematics, University of Trento, Via Sommarive, 14, I-38050 Povo Trento ITALY

E-mail address: `perotti@science.unitn.it`