Holomorphic functions and regular quaternionic functions on the hyperkähler space $\mathbb{H}$

A. Perotti

Department of Mathematics
University of Trento, Italy

V ISAAC Congress
Catania 2005

Outline

1. Regular functions
   - Fueter-regular and ψ-regular functions
   - q-holomorphic functions (Joyce)
2. Holomorphic maps
   - J_p-holomorphic maps
   - Quaternionic maps (Sommese)
3. Question: Does ψ-regular imply holomorphic?
   - Energy and regularity
   - A criterion for holomorphicity
   - Answer: There exist ψ-regular functions that are not holomorphic

Notations and definitions

- $\mathbb{H} \simeq \mathbb{C}^2$

$$\mathbb{C}^2 \ni (z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$$

$$\longleftrightarrow q = z_1 + z_2 j = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H}$$

- Ω bounded domain in $\mathbb{H} \simeq \mathbb{C}^2$.
  A quaternionic function $f = f_1 + f_2 j \in C^1(\Omega)$ is (left) regular on Ω if

$$\mathcal{D}f = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} + k\frac{\partial f}{\partial x_3} = 0 \quad \text{on } \Omega \quad \text{(Fueter)}$$

- "structural vector" $\psi = (1, i, j, -k) \Rightarrow$ f is (left) ψ-regular on Ω if

$$\mathcal{D}'f = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} - k\frac{\partial f}{\partial x_3} = 0 \quad \text{on } \Omega.$$

Some properties of regular functions

1. f is ψ-regular $\Leftrightarrow$ $\dfrac{\partial f_1}{\partial \bar{z}_1} = \dfrac{\partial \overline{f_2}}{\partial z_2}, \quad \dfrac{\partial f_1}{\partial \bar{z}_2} = -\dfrac{\partial \overline{f_2}}{\partial z_1}$
2. Every holomorphic map (f_1, f_2) on Ω defines a ψ-regular function $f = f_1 + f_2 j$.
3. The complex components are both holomorphic or both non-holomorphic.
4. Every regular or ψ-regular function is harmonic.
5. If Ω is pseudoconvex, every complex harmonic function is the complex component of a ψ-regular function on Ω.

$$*\overline{\partial} f_1 = -\tfrac{1}{2}\partial(\overline{f_2} d\bar{z}_1 \wedge d\bar{z}_2)$$

6. The space $\mathcal{R}(\Omega)$ of ψ-regular functions on Ω is a *right* $\mathbb{H}$-module with integral representation formulas.

q-holomorphic functions on the hypercomplex manifold $\mathbb{H}$

- Hypercomplex structure on $\mathbb{H} \simeq \mathbb{C}^2$:
  J_1, J_2 complex structures on $T\mathbb{H} \simeq \mathbb{H}$ defined by left multiplication by i and $j \Rightarrow J_1 J_2 + J_2 J_1 = 0$.
- J_1^*, J_2^* dual structures on $T^*\mathbb{H}$. In complex coordinates

$$\Rightarrow \begin{cases} J_1^* dz_1 = i\, dz_1, & J_1^* dz_2 = i\, dz_2 \\ J_2^* dz_1 = -d\bar{z}_2, & J_2^* dz_2 = d\bar{z}_1 \\ J_3^* dz_1 = i\, d\bar{z}_2, & J_3^* dz_2 = -i\, d\bar{z}_1 \end{cases}$$

  where we make the choice $J_3^* = J_1^* J_2^* \Rightarrow J_3 = -J_1 J_2$.

q-holomorphic functions on the hypercomplex manifold $\mathbb{H}$

f is ψ-regular $\Leftrightarrow$ f is q-holomorphic (Joyce):

$$df + iJ_1^*(df) + jJ_2^*(df) + kJ_3^*(df) = 0$$

Joyce defined on them a (commutative) product.
In complex components $f = f_1 + f_2 j$, we can rewrite the equations of ψ-regularity as

$$\overline{\partial} f_1 = J_2^*(\partial \overline{f}_2)$$

Holomorphic functions w.r.t. a complex structure J_p

Let $J_p = p_1 J_1 + p_2 J_2 + p_3 J_3$ be the complex structure on $\mathbb{H}$ defined by a unit imaginary quaternion $p = p_1 i + p_2 j + p_3 k$ in the sphere S^2.
(i.e. compatible with the standard hyperkähler structure of $\mathbb{H}$.)
Every J_p-holomorphic function $f = f^0 + if^1 : \Omega \to \mathbb{C}$ i.e.

$$df^0 = J_p^*(df^1) \quad \Leftrightarrow \quad df + iJ_p^*(df) = 0$$

defines a ψ-regular function $\tilde{f} = f^0 + pf^1$ on Ω.
We can identify $\tilde{f}$ with a holomorphic function

$$\tilde{f} : (\Omega, J_p) \to (\mathbb{C}_p, L_p)$$

where $\mathbb{C}_p = \langle 1, p \rangle$ is a copy of $\mathbb{C}$ in $\mathbb{H}$ and L_p is the complex structure defined on $T^*\mathbb{C}_p \simeq \mathbb{C}_p$ by left multiplication by p.

Holomorphic maps w.r.t. a complex structure J_p

Space of holomorphic maps from (Ω, J_p) to $(\mathbb{H}, L_p)$

$$Hol_p(\Omega, \mathbb{H}) = \{f : \Omega \to \mathbb{H} \mid \overline{\partial}_p f = 0 \text{ on } \Omega\} = Ker\overline{\partial}_p$$

(J_p-holomorphic maps on Ω) where $\overline{\partial}_p$ is the Cauchy-Riemann operator w.r.t. J_p:

$$\overline{\partial}_p = \frac{1}{2}\left(d + pJ_p^* \circ d\right).$$

For any positive orthonormal basis $\{1, p, q, pq\}$ of $\mathbb{H}$ ($p, q \in S^2$), the equations of ψ-regularity can be rewritten in complex form as

$$\overline{\partial}_p f_1 = J_q^*(\partial_p \overline{f}_2)$$

where $f = (f^0 + pf^1) + (f^2 + pf^3)q = f_1 + f_2 q$

$\Rightarrow$ every $f \in Hol_p(\Omega, \mathbb{H})$ is a ψ-regular function on Ω.

Some properties of J_p-holomorphic maps

- The *identity* map is in $Hol_i(\Omega, \mathbb{H}) \cap Hol_j(\Omega, \mathbb{H})$ but not in $Hol_k(\Omega, \mathbb{H})$.
- $Hol_{-p}(\Omega, \mathbb{H}) = Hol_p(\Omega, \mathbb{H})$
- If $f \in Hol_p(\Omega, \mathbb{H}) \cap Hol_{p'}(\Omega, \mathbb{H})$, with $p \neq \pm p'$, $\Rightarrow f \in Hol_{p''}(\Omega, \mathbb{H})$ for every $p'' = \frac{\alpha p + \beta p'}{\|\alpha p + \beta p'\|}$.
- ψ-regularity distinguishes between holomorphic and anti-holomorphic maps: if f is an *anti-holomorphic* map from (Ω, J_p) to $(\mathbb{H}, L_p)$, then f can be ψ-regular or not.
  - $f = \bar{z}_1 + \bar{z}_2 j \in Hol_j(\Omega, \mathbb{H}) \cap Hol_k(\Omega, \mathbb{H})$ is a ψ-regular function induced by the anti-holomorphic map
$$(\bar{z}_1, \bar{z}_2) : (\Omega, J_1) \to (\mathbb{H}, L_i)$$
  - $(\bar{z}_1, 0) : (\Omega, J_1) \to (\mathbb{H}, L_i)$ induces the function $g = \bar{z}_1 \notin \mathcal{R}(\Omega)$.

Quaternionic maps on the quaternionic manifold Ω

Example

A quaternionic map between hypercomplex manifolds

$$f : (X, J_1, J_2) \to (Y, K_1, K_2)$$

is a map that is holomorphic from (X, J_1) to (Y, K_1) *and* from (X, J_2) to (Y, K_2) (Sommese).
In particular, a quaternionic map

$$f : (\Omega, J_1, J_2) \to (\mathbb{H}, J_1, J_2)$$

is an element of $Hol_i(\Omega, \mathbb{H}) \cap Hol_j(\Omega, \mathbb{H}) \Rightarrow$ a ψ-regular function on Ω.
Sommese showed that these quaternionic maps are affine.
(transition functions for 4-dim *quaternionic manifolds*)

Question: Does ψ-regular imply holomorphic?

$$\mathcal{R}(\Omega) \supseteq \bigcup_{p \in S^2} Hol_p(\Omega, \mathbb{H}) \quad \text{properly?}$$

Q.: Can ψ-regular maps always be made holomorphic
by rotating the complex structure
or do they constitute a new class of harmonic maps?

Chen and Li (JDG 2000): analogous question for the larger class of q-maps between hyperkähler manifolds.
In their definition, the complex structures of the source and target manifold can rotate *independently*.
($\Rightarrow$ also anti-holomorphic maps are q-maps)

Energy functional

The energy (w.r.t. the euclidean metric g) of a map $f : \Omega \to \mathbb{C}^2 \simeq \mathbb{H}$, of class $C^1(\overline{\Omega})$, is the integral

$$\mathcal{E}(f) = \frac{1}{2}\int_\Omega \|df\|^2 dV = \frac{1}{2}\int_\Omega \langle g, f^*g\rangle dV = \frac{1}{2}\int_\Omega tr(J_\mathbb{C}(f)\overline{J_\mathbb{C}(f)}^T)dV$$

where $J_\mathbb{C}(f)$ is the Jacobian matrix of f with respect to the coordinates $\bar{z}_1, z_1, \bar{z}_2, z_2$.

Theorem

(Lichnerowicz) Holomorphic maps between Kähler manifolds minimize the energy functional in their homotopy classes.

(for maps smooth on $\overline{\Omega}$ the homotopy class contains the maps u with $u_{|\partial\Omega} = f_{|\partial\Omega}$ which are homotopic to f relative to $\partial\Omega$.)

Energy functional and ψ-regularity

From the theorem, functions $f \in Hol_p(\Omega, \mathbb{H})$ minimize the energy functional in their homotopy classes (relative to $\partial\Omega$). More generally:

Proposition

If f is ψ-regular on Ω, then it minimizes energy in its homotopy class (relative to $\partial\Omega$).

Sketch of proof (Lichnerowicz, Chen and Li).
Let $i_1 = i, i_2 = j, i_3 = k$ and

$$\mathcal{K}(f) = \int_\Omega \sum_{\alpha=1}^{3} \langle J_\alpha, f^* L_{i_\alpha} \rangle dV, \quad \mathcal{I}(f) = \frac{1}{2} \int_\Omega \| df + \sum_{\alpha=1}^{3} L_{i_\alpha} \circ df \circ J_\alpha \|^2 dV$$

Then $\mathcal{K}(f)$ is a homotopy invariant of f, $\mathcal{I}(f) = 0 \iff f \in \mathcal{R}(\Omega)$ and

$$\mathcal{E}(f) + \mathcal{K}(f) = \frac{1}{4} \mathcal{I}(f) \geq 0$$

A criterion for holomorphicity

Let $f : \Omega \to \mathbb{H}$ be a function of class $C^1(\overline{\Omega})$.

Theorem

Let $A = (a_{\alpha\beta})$ *be the* 3×3 *matrix with entries* $a_{\alpha\beta} = -\int_\Omega \langle J_\alpha, f^* L_{i_\beta} \rangle dV$.

1. f *is* ψ*-regular* $\Longleftrightarrow \mathcal{E}(f) = trA$.
2. *If* $f \in \mathcal{R}(\Omega)$, *then* A *is real, symmetric and* $trA \geq \lambda_1 = \max\{\text{eigenvalues of } A\} \Longrightarrow \det(A - (trA)I_3) \leq 0$.
3. *If* $f \in \mathcal{R}(\Omega)$, *then* f *belongs to some space* $Hol_p(\Omega, \mathbb{H})$ $\Longleftrightarrow \mathcal{E}(f) = trA = \lambda_1 \Longleftrightarrow \det(A - (trA)I_3) = 0$.
4. *If* $\mathcal{E}(f) = trA = \lambda_1$, $X_p = (p_1, p_2, p_3)$ *is a unit eigenvector of* A *relative to the largest eigenvalue* $\lambda_1 \Longleftrightarrow f \in Hol_p(\Omega, \mathbb{H})$.

Answer

A.: On every domain Ω, there exist ψ-regular functions that are not holomorphic.

Linear examples

- Let $f = z_1 + \bar{z}_1 + \bar{z}_2 j$. Then f is ψ-regular (on any Ω) but not J_p-holomorphic, for any p, since $rkJ_{\mathbb{C}}(f)$ is odd.
- Let $g = z_1 + z_2 + \bar{z}_1 + (z_1 + z_2 + \bar{z}_2)j$. Then g is ψ-regular, but not holomorphic even if $rkJ_{\mathbb{C}}(g) = 4$.
  On the unit ball B in $\mathbb{C}^2$, g has energy $\mathcal{E}(g) = 6$ and the matrix A of the theorem is

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \mathcal{E}(g) = trA > 2 = \lambda_1$$

More examples

Linear examples

The linear, ψ-regular functions constitute a $\mathbb{H}$-module of dimension 3 over $\mathbb{H}$, generated e.g. by $\{id = z_1 + z_2 j, z_2 + z_1 j, \bar{z}_1 + \bar{z}_2 j\}$. An element

$$f = (z_1 + z_2 j)q_1 + (z_2 + z_1 j)q_2 + (\bar{z}_1 + \bar{z}_2 j)q_3$$

is holomorphic $\iff$ the coefficients $q_1 = a_1 + a_2 j$, $q_2 = b_1 + b_2 j$, $q_3 = c_1 + c_2 j$ satisfy the 6^{th}-degree real homogeneous equation

$$\det(A - (trA)I_3) = 0$$

obtained after integration on B. So "almost all" (linear) ψ-regular functions are not-holomorphic.

More examples

Linear examples

A positive example (with $p \neq i, j, k$): Let $h = \bar{z}_1 + (z_1 + \bar{z}_2)j$. On the unit ball h has energy 3 and the matrix A is

$$A = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

$\Longrightarrow \mathcal{E}(h) = trA$ is equal to the (simple) largest eigenvalue, with unit eigenvector $X = \frac{1}{\sqrt{5}}(1, 0, 2) \Longrightarrow h$ is J_p-holomorphic with $p = \frac{1}{\sqrt{5}}(i + 2k)$, i.e. it satisfies the equation

$$df + \tfrac{1}{5}(i + 2k)(J_1^* + 2J_3^*)(df) = 0.$$

More examples

Example

A quadratic example: Let $f = |z_1|^2 - |z_2|^2 + \bar{z}_1 \bar{z}_2 j$. f has energy 2 on B and the matrix A is

$$A = \begin{bmatrix} -2/3 & 0 & 0 \\ 0 & 4/3 & 0 \\ 0 & 0 & 4/3 \end{bmatrix}$$

$\Longrightarrow$ f is ψ-regular but not holomorphic w.r.t. any complex structure J_p.

Other applications

- If $f \in Hol_p(\Omega, \mathbb{H}) \cap Hol_{p'}(\Omega, \mathbb{H})$ for two $\mathbb{R}$-independent p, p' $\Rightarrow X_p, X_{p'}$ are independent eigenvectors relative to λ_1

$$\Rightarrow \text{ the eigenvalues are } \lambda_1 = \lambda_2 = -\lambda_3.$$

- If $f \in Hol_p(\Omega, \mathbb{H}) \cap Hol_{p'}(\Omega, \mathbb{H}) \cap Hol_{p''}(\Omega, \mathbb{H})$ for three $\mathbb{R}$-independent p, p', p''

$$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0 \Rightarrow A = 0$$

and then f has energy 0 $\Rightarrow$ f is a (locally) *constant map*.

Application: $Hol_p(\Omega,\mathbb{H}) \cap Hol_{p'}(\Omega,\mathbb{H})$ $(p \neq \pm p')$ contains only affine maps (cf. Sommese)

Let Ω be connected. We can assume $p = i$, $p' = j$. Let $f \in Hol_i(\Omega,\mathbb{H}) \cap Hol_j(\Omega,\mathbb{H})$ and $a = \left(\frac{\partial f_1}{\partial z_1}, \frac{\partial f_2}{\partial z_1}\right)$, $b = \left(\overline{\frac{\partial f_2}{\partial z_2}}, -\overline{\frac{\partial f_1}{\partial z_2}}\right)$.

Since $f \in Hol_i(\Omega,\mathbb{H})$, the matrix A is obtained after integration on Ω of

$$\begin{bmatrix} |a|^2+|b|^2 & 0 & 0 \\ 0 & 2Re\langle a,b\rangle & -2Im\langle a,b\rangle \\ 0 & -2Im\langle a,b\rangle & -2Re\langle a,b\rangle \end{bmatrix}$$

$f \in Hol_j(\Omega,\mathbb{H}) \implies \int_\Omega Im\langle a,b\rangle dV = 0$ and $\int_\Omega |a-b|^2 dV = 0 \implies a = b$ on Ω. Then a is holomorphic *and* anti-holomorphic (w.r.t. J_1) $\implies$ a is constant on $\Omega \implies f$ is an affine map with linear part of the form

$$(a_1 z_1 - \bar{a}_2 z_2) + (a_2 z_1 + \bar{a}_1 z_2) j$$

i.e. the *right multiplication* of $q = z_1 + z_2 j$ by the quaternion $a_1 + a_2 j$.

Classification of ψ-regular functions

Let Ω be connected. Given a function $f \in \mathcal{R}(\Omega)$, we set

$$\mathcal{J}(f) = \{p \in S^2 \mid f \in Hol_p(\Omega, \mathbb{H})\}.$$

The space $\mathcal{R}(\Omega)$ of ψ-regular functions is the disjoint union of subsets of functions of the following four types:

1. f is J_p-holomorphic for three $\mathbb{R}$-independent structures $\Longrightarrow$ f is a constant and $\mathcal{J}(f) = S^2$.
2. f is J_p-holomorphic for exactly two $\mathbb{R}$-independent structures $\Longrightarrow$ f is a ψ-regular, invertible affine map and $\mathcal{J}(f)$ is an equator $S^1 \subset S^2$.
3. f is J_p-holomorphic for exactly one structure J_p (up to sign of p) $\Longrightarrow$ $\mathcal{J}(f)$ is a two-point set S^0.
4. f is ψ-regular but not J_p-holomorphic w.r.t. any complex structure $\Longrightarrow$ $\mathcal{J}(f) = \emptyset$.

Sketch of proof of the criterion

If $f \in \mathcal{R}(\Omega) \Rightarrow \mathcal{E}(f) = -\mathcal{K}(f) = trA$.
Let $\mathcal{I}_p(f) = \frac{1}{2}\int_\Omega \|df + L_p \circ df \circ J_p\|^2 dV$. Then

$$\mathcal{E}(f) + \int_\Omega \langle J_p, f^* L_p \rangle dV = \frac{1}{4}\mathcal{I}_p(f).$$

If $X_p = (p_1, p_2, p_3)$, then

$$XAX^T = \sum_{\alpha,\beta} p_\alpha p_\beta a_{\alpha\beta} = -\int_\Omega \langle \sum_\alpha p_\alpha J_\alpha, f^* \sum_\beta p_\beta L_{i_\beta} \rangle dV =$$

$$-\int_\Omega \langle J_p, f^* L_p \rangle dV = \mathcal{E}(f) - \frac{1}{4}\mathcal{I}_p(f).$$

Then $trA = \mathcal{E}(f) = XAX^T + \frac{1}{4}\mathcal{I}_p(f) \geq XAX^T$, with equality $\Leftrightarrow \mathcal{I}_p(f) = 0 \Leftrightarrow f$ is a J_p-holomorphic map.

Back

Sketch of proof of the criterion

Let M_α ($\alpha = 1, 2, 3$) be the matrix associated to J_α^* w.r.t. the basis $\{d\bar{z}_1, dz_1, d\bar{z}_2, dz_2\}$. The entries of the matrix A can be computed by the formula

$$a_{\alpha\beta} = -\int_\Omega \langle J_\alpha, f^* L_{i_\beta} \rangle dV = \frac{1}{2} \int_\Omega tr(\overline{B_\alpha}^T C_\beta) dV$$

where $B_\alpha = M_\alpha J_\mathbb{C}(f)^T$ for $\alpha = 1, 2$, $B_\alpha = -M_\alpha J_\mathbb{C}(f)^T$ for $\alpha = 3$ and $C_\beta = J_\mathbb{C}(f)^T M_\beta$ for $\beta = 1, 2, 3$.
The particular form of the Jacobian matrix of a ψ-regular function gives the symmetry property of A.

Back

References

- Chen, J. and Li, J., *Quaternionic maps between hyperkähler manifolds*, J. Differential Geom. **55** (2000) 355–384
- Joyce, D., *Hypercomplex algebraic geometry*, Quart. J. Math. Oxford **49** (1998) 129–162
- Perotti, A., *RegularHarmonics*: a *Mathematica* 4.2 package available at `www.science.unitn.it/~perotti /regular harmonics.htm` (2004)
- Sommese, A.J., *Quaternionic Manifolds*, Math. Ann. **212** (1975) 191–214

$$\tfrac{1}{16}det(A-(trA)I_3)=$$
$$a_1a_2b_2c_1^2\bar b_1-a_1a_2b_1c_1c_2\bar b_1-a_1^2b_2c_1c_2\bar b_1+a_1^2b_1c_2^2\bar b_1-a_1c_1^2\bar a_1\bar b_1^2-$$
$$a_1c_1c_2\bar a_2\bar b_1^2+a_2^2b_2c_1^2\bar b_2-a_2^2b_1c_1c_2\bar b_2-a_1a_2b_2c_1c_2\bar b_2+a_1a_2b_1c_2^2\bar b_2-$$
$$a_2c_1^2\bar a_1\bar b_1\bar b_2-a_1c_1c_2\bar a_1\bar b_1\bar b_2-a_2c_1c_2\bar a_2\bar b_1\bar b_2-a_1c_2^2\bar a_2\bar b_1\bar b_2-a_2c_1c_2\bar a_1\bar b_2^2-$$
$$a_2c_2^2\bar a_2\bar b_2^2+a_1a_2b_1b_2c_1\bar c_1-a_1^2b_2^2c_1\bar c_1-a_1a_2b_1^2c_2\bar c_1+a_1^2b_1b_2c_2\bar c_1-$$
$$2a_1b_1c_1\bar a_1\bar b_1\bar c_1-a_1b_2c_1\bar a_2\bar b_1\bar c_1-a_1b_1c_2\bar a_2\bar b_1\bar c_1-a_2b_1c_1\bar a_1\bar b_2\bar c_1-$$
$$2a_1b_2c_1\bar a_1\bar b_2\bar c_1+a_1b_1c_2\bar a_1\bar b_2\bar c_1-2a_2b_2c_1\bar a_2\bar b_2\bar c_1+a_2b_1c_2\bar a_2\bar b_2\bar c_1-$$
$$a_1b_2c_2\bar a_2\bar b_2\bar c_1+c_1\bar a_1\bar a_2\bar b_1\bar b_2\bar c_1+c_2\bar a_2^2\bar b_1\bar b_2\bar c_1-c_1\bar a_1^2\bar b_2^2\bar c_1-c_2\bar a_1\bar a_2\bar b_2^2\bar c_1-$$
$$a_1b_1^2\bar a_1\bar c_1^2-a_1b_1b_2\bar a_2\bar c_1^2+b_1\bar a_1\bar a_2\bar b_2\bar c_1^2+b_2\bar a_2^2\bar b_2\bar c_1^2+a_2^2b_1b_2c_1\bar c_2-$$
$$a_1a_2b_2^2c_1\bar c_2-a_2^2b_1^2c_2\bar c_2+a_1a_2b_1b_2c_2\bar c_2-a_2b_1c_1\bar a_1\bar b_1\bar c_2+a_1b_2c_1\bar a_1\bar b_1\bar c_2-$$
$$2a_1b_1c_2\bar a_1\bar b_1\bar c_2+a_2b_2c_1\bar a_2\bar b_1\bar c_2-2a_2b_1c_2\bar a_2\bar b_1\bar c_2-a_1b_2c_2\bar a_2\bar b_1\bar c_2-$$
$$c_1\bar a_1\bar a_2\bar b_1^2\bar c_2-c_2\bar a_2^2\bar b_1^2\bar c_2-a_2b_2c_1\bar a_1\bar b_2\bar c_2-a_2b_1c_2\bar a_1\bar b_2\bar c_2-2a_2b_2c_2\bar a_2\bar b_2\bar c_2+$$
$$c_1\bar a_1^2\bar b_1\bar b_2\bar c_2+c_2\bar a_1\bar a_2\bar b_1\bar b_2\bar c_2-a_2b_1^2\bar a_1\bar c_1\bar c_2-a_1b_1b_2\bar a_1\bar c_1\bar c_2-$$
$$a_2b_1b_2\bar a_2\bar c_1\bar c_2-a_1b_2^2\bar a_2\bar c_1\bar c_2-b_1\bar a_1\bar a_2\bar b_1\bar c_1\bar c_2-b_2\bar a_2^2\bar b_1\bar c_1\bar c_2-b_1\bar a_1^2\bar b_2\bar c_1\bar c_2-$$
$$b_2\bar a_1\bar a_2\bar b_2\bar c_1\bar c_2-a_2b_1b_2\bar a_1\bar c_2^2-a_2b_2^2\bar a_2\bar c_2^2+b_1\bar a_1^2\bar b_1\bar c_2^2+b_2\bar a_1\bar a_2\bar b_1\bar c_2^2=0$$

Back