

HOLOMORPHIC FUNCTIONS AND REGULAR QUATERNIONIC FUNCTIONS ON THE HYPERKÄHLER SPACE \mathbb{H}

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Let \mathbb{H} be the space of quaternions, with its standard hypercomplex structure. Let $\mathcal{R}(\Omega)$ be the module of ψ -regular functions on Ω . For every unitary vector p in $\mathbb{S}^2 \subset \mathbb{H}$, $\mathcal{R}(\Omega)$ contains the space of holomorphic functions w.r.t. the complex structure J_p induced by p . We prove the existence, on any bounded domain Ω , of ψ -regular functions that are not J_p -holomorphic for any p . Our starting point is a result of Chen and Li concerning maps between hyperkähler manifolds, where a similar result is obtained for a less restricted class of quaternionic maps. We give a criterion, based on the energy-minimizing property of holomorphic maps, that distinguishes J_p -holomorphic functions among ψ -regular functions.

1. Introduction

Let \mathbb{H} be the space of quaternions, with its standard hypercomplex structure given by the complex structures J_1, J_2 on $T\mathbb{H} \simeq \mathbb{H}$ defined by left multiplication by i and j . Let J_1^*, J_2^* be the dual structures on $T^*\mathbb{H}$.

We consider the module $\mathcal{R}(\Omega) = \{f = f_1 + f_2j \mid \bar{\partial}f_1 = J_2^*(\bar{\partial}f_2) \text{ on } \Omega\}$ of left ψ -regular functions on Ω . These functions are in a simple correspondence with Fueter left regular functions, since they can be obtained from them by means of a real coordinate reflection in \mathbb{H} . They have been studied by many authors (see for instance Sudbery⁸, Shapiro and Vasilevski⁶ and Nōno⁵). The space $\mathcal{R}(\Omega)$ contains the identity mapping and any holomorphic mapping (f_1, f_2) on Ω defines a ψ -regular function $f = f_1 + f_2j$. This is no more true if we replace the class of ψ -regular functions with that of regular functions. The definition of ψ -regularity is also equivalent to that of q -holomorphicity given by Joyce² in the setting of hypercomplex manifolds.

For every unitary vector p in $\mathbb{S}^2 \subset \mathbb{H}$, $\mathcal{R}(\Omega)$ contains the space $Hol_p(\Omega, \mathbb{H}) = \{f : \Omega \rightarrow \mathbb{H} \mid df + pJ_p(df) = 0 \text{ on } \Omega\}$ of holomorphic functions w.r.t. the complex

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structure $J_p = p_1J_1 + p_2J_2 + p_3J_3$ on Ω and to the structure induced on \mathbb{H} by left-multiplication by p (J_p -holomorphic functions on Ω).

We show that on every domain Ω there exist ψ -regular functions that are not J_p -holomorphic for any p . A similar result was obtained by Chen and Li¹ for the larger class of q -maps between hyperkähler manifolds.

This result is a consequence of a criterion (cf. Theorem 4.1) of J_p -holomorphicity, which is obtained using the energy-minimizing property of ψ -regular functions (cf. Proposition 4.1) and ideas of Lichnerowicz⁴ and Chen and Li¹.

In Sec. 4.4 we give some other applications of the criterion. In particular, we show that if Ω is connected, then the intersection $Hol_p(\Omega, \mathbb{H}) \cap Hol_{p'}(\Omega, \mathbb{H})$ ($p \neq \pm p'$) contains only affine maps. This result is in accord with what was proved by Sommese⁷ about *quaternionic maps* (cf. Sec. 3.2 for definitions).

2. Fueter-regular and ψ -regular functions

2.1. Notations and definitions

We identify the space \mathbb{C}^2 with the set \mathbb{H} of quaternions by means of the mapping that associates the pair $(z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$ with the quaternion $q = z_1 + z_2j = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H}$. Let Ω be a bounded domain in $\mathbb{H} \simeq \mathbb{C}^2$. A quaternionic function $f = f_1 + f_2j \in C^1(\Omega)$ is (*left*) *regular* on Ω (in the sense of Fueter) if

$$\mathcal{D}f = \frac{\partial f}{\partial x_0} + i \frac{\partial f}{\partial x_1} + j \frac{\partial f}{\partial x_2} + k \frac{\partial f}{\partial x_3} = 0 \quad \text{on } \Omega.$$

Given the “structural vector” $\psi = (1, i, j, -k)$, f is called (*left*) ψ -regular on Ω if

$$\mathcal{D}'f = \frac{\partial f}{\partial x_0} + i \frac{\partial f}{\partial x_1} + j \frac{\partial f}{\partial x_2} - k \frac{\partial f}{\partial x_3} = 0 \quad \text{on } \Omega.$$

We recall some properties of regular functions, for which we refer to the papers of Sudbery⁸, Shapiro and Vasilevski⁶ and Nōno⁵:

- (1) f is ψ -regular $\Leftrightarrow \frac{\partial f_1}{\partial \bar{z}_1} = \frac{\partial \bar{f}_2}{\partial z_2}, \frac{\partial f_1}{\partial \bar{z}_2} = -\frac{\partial \bar{f}_2}{\partial z_1}$.
- (2) Every holomorphic map (f_1, f_2) on Ω defines a ψ -regular function $f = f_1 + f_2j$.
- (3) The complex components are both holomorphic or both non-holomorphic.
- (4) Every regular or ψ -regular function is harmonic.
- (5) If Ω is pseudoconvex, every complex harmonic function is the complex component of a ψ -regular function on Ω .
- (6) The space $\mathcal{R}(\Omega)$ of ψ -regular functions on Ω is a *right* \mathbb{H} -module with integral representation formulas.

2.2. q -holomorphic functions

A definition equivalent to ψ -regularity has been given by Joyce² in the setting of hypercomplex manifolds. Joyce introduced the module of q -holomorphic functions

on a hypercomplex manifold. On this module he defined a (commutative) product. A hypercomplex structure on the manifold \mathbb{H} is given by the complex structures J_1, J_2 on $T\mathbb{H} \simeq \mathbb{H}$ defined by left multiplication by i and j . Let J_1^*, J_2^* be the dual structures on $T^*\mathbb{H}$. In complex coordinates

$$\begin{cases} J_1^* dz_1 = i dz_1, & J_1^* dz_2 = i dz_2 \\ J_2^* dz_1 = -d\bar{z}_2, & J_2^* dz_2 = d\bar{z}_1 \\ J_3^* dz_1 = i d\bar{z}_2, & J_3^* dz_2 = -i d\bar{z}_1 \end{cases}$$

where we make the choice $J_3^* = J_1^* J_2^* \Rightarrow J_3 = -J_1 J_2$.

A function f is ψ -regular if and only if f is q -holomorphic, i.e.

$$df + iJ_1^*(df) + jJ_2^*(df) + kJ_3^*(df) = 0.$$

In complex components $f = f_1 + f_2 j$, we can rewrite the equations of ψ -regularity as

$$\bar{\partial} f_1 = J_2^*(\partial \bar{f}_2).$$

3. Holomorphic maps

3.1. Holomorphic functions w.r.t. a complex structure J_p

Let $J_p = p_1 J_1 + p_2 J_2 + p_3 J_3$ be the complex structure on \mathbb{H} defined by a unit imaginary quaternion $p = p_1 i + p_2 j + p_3 k$ in the sphere $\mathbb{S}^2 = \{p \in \mathbb{H} \mid p^2 = -1\}$. It is well-known that every complex structure compatible with the standard hyperkähler structure of \mathbb{H} is of this form. If $f = f^0 + i f^1 : \Omega \rightarrow \mathbb{C}$ is a J_p -holomorphic function, i.e. $df^0 = J_p^*(df^1)$ or, equivalently, $df + iJ_p^*(df) = 0$, then f defines a ψ -regular function $\tilde{f} = f^0 + p f^1$ on Ω . We can identify \tilde{f} with a holomorphic function

$$\tilde{f} : (\Omega, J_p) \rightarrow (\mathbb{C}_p, L_p)$$

where $\mathbb{C}_p = \langle 1, p \rangle$ is a copy of \mathbb{C} in \mathbb{H} and L_p is the complex structure defined on $T^*\mathbb{C}_p \simeq \mathbb{C}_p$ by left multiplication by p .

More generally, we can consider the space of holomorphic maps from (Ω, J_p) to (\mathbb{H}, L_p)

$$Hol_p(\Omega, \mathbb{H}) = \{f : \Omega \rightarrow \mathbb{H} \mid \bar{\partial}_p f = 0 \text{ on } \Omega\} = Ker \bar{\partial}_p$$

(the J_p -holomorphic maps on Ω) where $\bar{\partial}_p$ is the Cauchy-Riemann operator w.r.t. the structure J_p

$$\bar{\partial}_p = \frac{1}{2} (d + p J_p^* \circ d).$$

For any positive orthonormal basis $\{1, p, q, pq\}$ of \mathbb{H} ($p, q \in \mathbb{S}^2$), the equations of ψ -regularity can be rewritten in complex form as

$$\bar{\partial}_p f_1 = J_q^*(\partial_p \bar{f}_2),$$

where $f = (f^0 + pf^1) + (f^2 + pf^3)q = f_1 + f_2q$. Then every $f \in Hol_p(\Omega, \mathbb{H})$ is a ψ -regular function on Ω .

Remark 3.1. 1) The *identity* map is in $Hol_i(\Omega, \mathbb{H}) \cap Hol_j(\Omega, \mathbb{H})$, but not in $Hol_k(\Omega, \mathbb{H})$.

2) $Hol_{-p}(\Omega, \mathbb{H}) = Hol_p(\Omega, \mathbb{H})$

3) If $f \in Hol_p(\Omega, \mathbb{H}) \cap Hol_{p'}(\Omega, \mathbb{H})$, with $p \neq \pm p'$, then $f \in Hol_{p''}(\Omega, \mathbb{H})$ for every $p'' = \frac{\alpha p + \beta p'}{\|\alpha p + \beta p'\|}$.

4) ψ -regularity distinguishes between holomorphic and anti-holomorphic maps: if f is an *anti-holomorphic* map from (Ω, J_p) to (\mathbb{H}, L_p) , then f can be ψ -regular or not. For example, $f = \bar{z}_1 + \bar{z}_2j \in Hol_j(\Omega, \mathbb{H}) \cap Hol_k(\Omega, \mathbb{H})$ is a ψ -regular function induced by the anti-holomorphic map

$$(\bar{z}_1, \bar{z}_2) : (\Omega, J_1) \rightarrow (\mathbb{H}, L_i),$$

while $(\bar{z}_1, 0) : (\Omega, J_1) \rightarrow (\mathbb{H}, L_i)$ induces the function $g = \bar{z}_1 \notin \mathcal{R}(\Omega)$.

3.2. Quaternionic maps

A particular class of J_p -holomorphic maps is constituted by the *quaternionic maps* on the quaternionic manifold Ω . Sommese⁷ defined quaternionic maps between hypercomplex manifolds: a quaternionic map is a map

$$f : (X, J_1, J_2) \rightarrow (Y, K_1, K_2)$$

that is holomorphic from (X, J_1) to (Y, K_1) and from (X, J_2) to (Y, K_2) .

In particular, a quaternionic map

$$f : (\Omega, J_1, J_2) \rightarrow (\mathbb{H}, J_1, J_2)$$

is an element of $Hol_i(\Omega, \mathbb{H}) \cap Hol_j(\Omega, \mathbb{H})$ and then a ψ -regular function on Ω . Sommese showed that quaternionic maps are affine. They appear for example as transition functions for 4-dimensional *quaternionic manifolds*.

4. Non-holomorphic ψ -regular maps

A natural question can now be raised: can ψ -regular maps always be made holomorphic by rotating the complex structure or do they constitute a new class of harmonic maps? In other words, does the space $\mathcal{R}(\Omega)$ contain the union $\bigcup_{p \in \mathbb{S}^2} Hol_p(\Omega, \mathbb{H})$ properly?

Chen and Li¹ posed and answered the analogous question for the larger class of q -maps between hyperkähler manifolds. In their definition, the complex structures of the source and target manifold can rotate *independently*. This implies that also anti-holomorphic maps are q -maps.

4.1. Energy and regularity

The *energy* (w.r.t. the euclidean metric g) of a map $f : \Omega \rightarrow \mathbb{C}^2 \simeq \mathbb{H}$, of class $C^1(\bar{\Omega})$, is the integral

$$\mathcal{E}(f) = \frac{1}{2} \int_{\Omega} \|df\|^2 dV = \frac{1}{2} \int_{\Omega} \langle g, f^*g \rangle dV = \frac{1}{2} \int_{\Omega} \text{tr}(J_{\mathbb{C}}(f)\overline{J_{\mathbb{C}}(f)}^T) dV,$$

where $J_{\mathbb{C}}(f)$ is the Jacobian matrix of f with respect to the coordinates $\bar{z}_1, z_1, \bar{z}_2, z_2$.

Lichnerowicz⁴ proved that holomorphic maps between Kähler manifolds minimize the energy functional in their homotopy classes. Holomorphic maps f smooth on $\bar{\Omega}$ minimize energy in the homotopy class constituted by maps u with $u|_{\partial\Omega} = f|_{\partial\Omega}$ which are homotopic to f relative to $\partial\Omega$.

From the theorem, functions $f \in \text{Hol}_p(\Omega, \mathbb{H})$ minimize the energy functional in their homotopy classes (relative to $\partial\Omega$). More generally:

Proposition 4.1. *If f is ψ -regular on Ω , then it minimizes energy in its homotopy class (relative to $\partial\Omega$).*

Proof. We repeat arguments of Lichnerowicz, Chen and Li. Let $i_1 = i, i_2 = j, i_3 = k$ and let

$$\mathcal{K}(f) = \int_{\Omega} \sum_{\alpha=1}^3 \langle J_{\alpha}, f^*L_{i_{\alpha}} \rangle dV, \quad \mathcal{I}(f) = \frac{1}{2} \int_{\Omega} \|df + \sum_{\alpha=1}^3 L_{i_{\alpha}} \circ df \circ J_{\alpha}\|^2 dV.$$

Then $\mathcal{K}(f)$ is a homotopy invariant of f and $\mathcal{I}(f) = 0$ if and only if $f \in \mathcal{R}(\Omega)$. A computation similar to that made by Chen and Li¹ gives

$$\mathcal{E}(f) + \mathcal{K}(f) = \frac{1}{4}\mathcal{I}(f) \geq 0.$$

From this the result follows immediately. \square

4.2. A criterion for holomorphicity

We now come to our main result. Let $f : \Omega \rightarrow \mathbb{H}$ be a function of class $C^1(\bar{\Omega})$.

Theorem 4.1. *Let $A = (a_{\alpha\beta})$ be the 3×3 matrix with entries $a_{\alpha\beta} = -\int_{\Omega} \langle J_{\alpha}, f^*L_{i_{\beta}} \rangle dV$. Then*

- (1) f is ψ -regular if and only if $\mathcal{E}(f) = \text{tr}A$.
- (2) If $f \in \mathcal{R}(\Omega)$, then A is real, symmetric and

$$\text{tr}A \geq \lambda_1 = \max\{\text{eigenvalues of } A\}.$$

It follows that $\det(A - (\text{tr}A)I_3) \leq 0$.

- (3) *If $f \in \mathcal{R}(\Omega)$, then f belongs to some space $\text{Hol}_p(\Omega, \mathbb{H})$ if and only if $\mathcal{E}(f) = \text{tr}A = \lambda_1$ or, equivalently, $\det(A - (\text{tr}A)I_3) = 0$.*
- (4) *If $\mathcal{E}(f) = \text{tr}A = \lambda_1$, $X_p = (p_1, p_2, p_3)$ is a unit eigenvector of A relative to the largest eigenvalue λ_1 if and only if $f \in \text{Hol}_p(\Omega, \mathbb{H})$.*

4.3. The existence of non-holomorphic ψ -regular maps

The criterion can be applied to show that on every domain Ω in \mathbb{H} , there exist ψ -regular functions that are not holomorphic.

Example 4.1. Let $f = z_1 + z_2 + \bar{z}_1 + (z_1 + z_2 + \bar{z}_2)j$. Then f is ψ -regular, but not holomorphic, since on the unit ball B in \mathbb{C}^2 , f has energy $\mathcal{E}(f) = 6$ and the matrix A of the theorem is

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Therefore $\mathcal{E}(f) = \text{tr}A > 2 = \lambda_1$.

In the preceding example, the Jacobian matrix of the function has even rank, a necessary condition for a holomorphic map. In the case when the rank is odd, the non-holomorphicity follows immediately. For example, $g = z_1 + \bar{z}_1 + \bar{z}_2 j$ is ψ -regular (on any Ω) but not J_p -holomorphic, for any p , since $\text{rk}J_{\mathbb{C}}(f)$ is odd.

Example 4.2. The linear, ψ -regular functions constitute a \mathbb{H} -module of dimension 3 over \mathbb{H} , generated e.g. by the set $\{z_1 + z_2 j, z_2 + z_1 j, \bar{z}_1 + \bar{z}_2 j\}$. An element

$$f = (z_1 + z_2 j)q_1 + (z_2 + z_1 j)q_2 + (\bar{z}_1 + \bar{z}_2 j)q_3$$

is holomorphic if and only if the coefficients $q_1 = a_1 + a_2 j$, $q_2 = b_1 + b_2 j$, $q_3 = c_1 + c_2 j$ satisfy the 6th-degree real homogeneous equation

$$\det(A - (\text{tr}A)I_3) = 0$$

obtained after integration on B . The explicit expression of this equation is given in the Appendix. So “almost all” (linear) ψ -regular functions are non-holomorphic.

Example 4.3. A positive example (with $p \neq i, j, k$). Let $h = \bar{z}_1 + (z_1 + \bar{z}_2)j$. On the unit ball h has energy 3 and the matrix A is

$$A = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

then $\mathcal{E}(h) = \text{tr}A$ is equal to the (simple) largest eigenvalue, with unit eigenvector $X = \frac{1}{\sqrt{5}}(1, 0, 2)$. It follows that h is J_p -holomorphic with $p = \frac{1}{\sqrt{5}}(i + 2k)$, i.e. it satisfies the equation

$$df + \frac{1}{5}(i + 2k)(J_1^* + 2J_3^*)(df) = 0.$$

Example 4.4. We give a quadratic example. Let $f = |z_1|^2 - |z_2|^2 + \bar{z}_1 \bar{z}_2 j$. f has energy 2 on B and the matrix A is

$$A = \begin{bmatrix} -2/3 & 0 & 0 \\ 0 & 4/3 & 0 \\ 0 & 0 & 4/3 \end{bmatrix}$$

Then f is ψ -regular but not holomorphic w.r.t. any complex structure J_p .

4.4. Other applications of the criterion

1) If $f \in Hol_p(\Omega, \mathbb{H}) \cap Hol_{p'}(\Omega, \mathbb{H})$ for two \mathbb{R} -independent p, p' , then $X_p, X_{p'}$ are independent eigenvectors relative to λ_1 . Therefore the eigenvalues of the matrix A are $\lambda_1 = \lambda_2 = -\lambda_3$.

If $f \in Hol_p(\Omega, \mathbb{H}) \cap Hol_{p'}(\Omega, \mathbb{H}) \cap Hol_{p''}(\Omega, \mathbb{H})$ for three \mathbb{R} -independent p, p', p'' then $\lambda_1 = \lambda_2 = \lambda_3 = 0 \Rightarrow A = 0$ and therefore f has energy 0 and f is a (locally) constant map.

2) If Ω is connected, then $Hol_p(\Omega, \mathbb{H}) \cap Hol_{p'}(\Omega, \mathbb{H})$ ($p \neq \pm p'$) contains only affine maps (cf. Sommesé⁷).

We can assume $p = i, p' = j$ since in view of property 3) of Remark 3.1 we can suppose p and p' orthogonal quaternions and then we can rotate the space of imaginary quaternions. Let $f \in Hol_i(\Omega, \mathbb{H}) \cap Hol_j(\Omega, \mathbb{H})$ and $a = \left(\frac{\partial f_1}{\partial z_1}, \frac{\partial f_2}{\partial z_1} \right)$, $b = \left(\frac{\partial f_2}{\partial z_2}, -\frac{\partial f_1}{\partial z_2} \right)$. Since $f \in Hol_i(\Omega, \mathbb{H})$, the matrix A is obtained after integration on Ω of the matrix

$$\begin{bmatrix} |a|^2 + |b|^2 & 0 & 0 \\ 0 & 2Re\langle a, b \rangle & -2Im\langle a, b \rangle \\ 0 & -2Im\langle a, b \rangle & -2Re\langle a, b \rangle \end{bmatrix}$$

where $\langle a, b \rangle$ denotes the standard hermitian product of \mathbb{C}^2 .

Since $f \in Hol_j(\Omega, \mathbb{H})$, we have $\int_{\Omega} Im\langle a, b \rangle dV = 0$ and $\int_{\Omega} |a - b|^2 dV = 0$. Therefore $a = b$ on Ω . Then a is holomorphic and anti-holomorphic w.r.t. the standard structure J_1 . This means that a is constant on Ω and f is an affine map with linear part of the form

$$(a_1 z_1 - \bar{a}_2 z_2) + (a_2 z_1 + \bar{a}_1 z_2)j$$

i.e. the right multiplication of $q = z_1 + z_2 j$ by the quaternion $a_1 + a_2 j$.

3) We can give a classification of ψ -regular functions based on the dimension of the set of complex structures w.r.t. which the function is holomorphic. Let Ω be connected. Given a function $f \in \mathcal{R}(\Omega)$, we set

$$\mathcal{J}(f) = \{p \in \mathbb{S}^2 \mid f \in Hol_p(\Omega, \mathbb{H})\}.$$

The space $\mathcal{R}(\Omega)$ of ψ -regular functions is the disjoint union of subsets of functions of the following four types:

- (i) f is J_p -holomorphic for three \mathbb{R} -independent structures
 $\implies f$ is a constant and $\mathcal{J}(f) = \mathbb{S}^2$.
- (ii) f is J_p -holomorphic for exactly two \mathbb{R} -independent structures
 $\implies f$ is a ψ -regular, invertible affine map and $\mathcal{J}(f)$ is an equator $S^1 \subset \mathbb{S}^2$.
- (iii) f is J_p -holomorphic for exactly one structure J_p (up to sign of p) $\implies \mathcal{J}(f)$
is a two-point set S^0 .
- (iv) f is ψ -regular but not J_p -holomorphic w.r.t. any complex structure \implies
 $\mathcal{J}(f) = \emptyset$.

We shall return in a subsequent paper to the application of the criterion to the study of ψ -biregular functions, which are invertible ψ -regular functions with ψ -regular inverse (see Królikowski and Porter³ for the class of *biregular* functions). This class contains as a proper subset the invertible holomorphic maps.

5. Sketch of proof of Theorem 4.1

If $f \in \mathcal{R}(\Omega)$, then $\mathcal{E}(f) = -\mathcal{K}(f) = \text{tr}A$. Let

$$\mathcal{I}_p(f) = \frac{1}{2} \int_{\Omega} \|df + L_p \circ df \circ J_p\|^2 dV.$$

Then we obtain, as in Chen and Li¹

$$\mathcal{E}(f) + \int_{\Omega} \langle J_p, f^* L_p \rangle dV = \frac{1}{4} \mathcal{I}_p(f).$$

If $X_p = (p_1, p_2, p_3)$, then

$$\begin{aligned} XAX^T &= \sum_{\alpha, \beta} p_{\alpha} p_{\beta} a_{\alpha\beta} = - \int_{\Omega} \langle \sum_{\alpha} p_{\alpha} J_{\alpha}, f^* \sum_{\beta} p_{\beta} L_{i_{\beta}} \rangle dV \\ &= - \int_{\Omega} \langle J_p, f^* L_p \rangle dV = \mathcal{E}(f) - \frac{1}{4} \mathcal{I}_p(f). \end{aligned}$$

Then $\text{tr}A = \mathcal{E}(f) = XAX^T + \frac{1}{4} \mathcal{I}_p(f) \geq XAX^T$, with equality if and only if $\mathcal{I}_p(f) = 0$ i.e if and only if f is a J_p -holomorphic map.

Let M_{α} ($\alpha = 1, 2, 3$) be the matrix associated to J_{α}^* w.r.t. the basis $\{d\bar{z}_1, dz_1, d\bar{z}_2, dz_2\}$. The entries of the matrix A can be computed by the formula

$$a_{\alpha\beta} = - \int_{\Omega} \langle J_{\alpha}, f^* L_{i_{\beta}} \rangle dV = \frac{1}{2} \int_{\Omega} \text{tr}(\overline{B_{\alpha}}^T C_{\beta}) dV$$

where $B_{\alpha} = M_{\alpha} J_{\mathbb{C}}(f)^T$ for $\alpha = 1, 2$, $B_{\alpha} = -M_{\alpha} J_{\mathbb{C}}(f)^T$ for $\alpha = 3$ and $C_{\beta} = J_{\mathbb{C}}(f)^T M_{\beta}$ for $\beta = 1, 2, 3$.

A direct computation shows how from the particular form of the Jacobian matrix of a ψ -regular function it follows the symmetry property of A .

Appendix

We give the explicit expression of the 6^{th} -degree real homogeneous equation satisfied by the complex coefficients of a linear J_p -holomorphic ψ -regular function.

$$\begin{aligned} \frac{1}{16} \det(A - (tr A)I_3) = & a_1 a_2 b_2 c_1^2 \bar{b}_1 - a_1 a_2 b_1 c_1 c_2 \bar{b}_1 - a_1^2 b_2 c_1 c_2 \bar{b}_1 + a_1^2 b_1 c_2^2 \bar{b}_1 - \\ & a_1 c_1^2 \bar{a}_1 \bar{b}_1^2 - a_1 c_1 c_2 \bar{a}_2 \bar{b}_1^2 + a_2^2 b_2 c_1^2 \bar{b}_2 - a_2^2 b_1 c_1 c_2 \bar{b}_2 - a_1 a_2 b_2 c_1 c_2 \bar{b}_2 + a_1 a_2 b_1 c_2^2 \bar{b}_2 - \\ & a_2 c_1^2 \bar{a}_1 \bar{b}_1 \bar{b}_2 - a_1 c_1 c_2 \bar{a}_1 \bar{b}_1 \bar{b}_2 - a_2 c_1 c_2 \bar{a}_2 \bar{b}_1 \bar{b}_2 - a_1 c_2^2 \bar{a}_2 \bar{b}_1 \bar{b}_2 - a_2 c_1 c_2 \bar{a}_1 \bar{b}_2^2 - a_2 c_2^2 \bar{a}_2 \bar{b}_2^2 + \\ & a_1 a_2 b_1 b_2 c_1 \bar{c}_1 - a_1^2 b_2^2 c_1 \bar{c}_1 - a_1 a_2 b_1^2 c_2 \bar{c}_1 + a_1^2 b_1 b_2 c_2 \bar{c}_1 - 2a_1 b_1 c_1 \bar{a}_1 \bar{b}_1 \bar{c}_1 - a_1 b_2 c_1 \bar{a}_2 \bar{b}_1 \bar{c}_1 - \\ & a_1 b_1 c_2 \bar{a}_2 \bar{b}_1 \bar{c}_1 - a_2 b_1 c_1 \bar{a}_1 \bar{b}_2 \bar{c}_1 - 2a_1 b_2 c_1 \bar{a}_1 \bar{b}_2 \bar{c}_1 + a_1 b_1 c_2 \bar{a}_1 \bar{b}_2 \bar{c}_1 - 2a_2 b_2 c_1 \bar{a}_2 \bar{b}_2 \bar{c}_1 + \\ & a_2 b_1 c_2 \bar{a}_2 \bar{b}_2 \bar{c}_1 - a_1 b_2 c_2 \bar{a}_2 \bar{b}_2 \bar{c}_1 + c_1 \bar{a}_1 \bar{a}_2 \bar{b}_1 \bar{b}_2 \bar{c}_1 + c_2 \bar{a}_2^2 \bar{b}_1 \bar{b}_2 \bar{c}_1 - c_1 \bar{a}_1^2 \bar{b}_2^2 \bar{c}_1 - c_2 \bar{a}_1 \bar{a}_2 \bar{b}_2^2 \bar{c}_1 - \\ & a_1 b_1^2 \bar{a}_1 \bar{c}_1^2 - a_1 b_1 b_2 \bar{a}_2 \bar{c}_1^2 + b_1 \bar{a}_1 \bar{a}_2 \bar{b}_2 \bar{c}_1^2 + b_2 \bar{a}_2^2 \bar{b}_2 \bar{c}_1^2 + a_2^2 b_1 b_2 c_1 \bar{c}_2 - a_1 a_2 b_2^2 c_1 \bar{c}_2 - a_2^2 b_1^2 c_2 \bar{c}_2 + \\ & a_1 a_2 b_1 b_2 c_2 \bar{c}_2 - a_2 b_1 c_1 \bar{a}_1 \bar{b}_1 \bar{c}_2 + a_1 b_2 c_1 \bar{a}_1 \bar{b}_1 \bar{c}_2 - 2a_1 b_1 c_2 \bar{a}_1 \bar{b}_1 \bar{c}_2 + a_2 b_2 c_1 \bar{a}_2 \bar{b}_1 \bar{c}_2 - \\ & 2a_2 b_1 c_2 \bar{a}_2 \bar{b}_1 \bar{c}_2 - a_1 b_2 c_2 \bar{a}_2 \bar{b}_1 \bar{c}_2 - c_1 \bar{a}_1 \bar{a}_2 \bar{b}_1^2 \bar{c}_2 - c_2 \bar{a}_2^2 \bar{b}_1^2 \bar{c}_2 - a_2 b_2 c_1 \bar{a}_1 \bar{b}_2 \bar{c}_2 - a_2 b_1 c_2 \bar{a}_1 \bar{b}_2 \bar{c}_2 - \\ & 2a_2 b_2 c_2 \bar{a}_2 \bar{b}_2 \bar{c}_2 + c_1 \bar{a}_1^2 \bar{b}_1 \bar{b}_2 \bar{c}_2 + c_2 \bar{a}_1 \bar{a}_2 \bar{b}_1 \bar{b}_2 \bar{c}_2 - a_2 b_1^2 \bar{a}_1 \bar{c}_1 \bar{c}_2 - a_1 b_1 b_2 \bar{a}_1 \bar{c}_1 \bar{c}_2 - \\ & a_2 b_1 b_2 \bar{a}_2 \bar{c}_1 \bar{c}_2 - a_1 b_2^2 \bar{a}_2 \bar{c}_1 \bar{c}_2 - b_1 \bar{a}_1 \bar{a}_2 \bar{b}_1 \bar{c}_1 \bar{c}_2 - b_2 \bar{a}_2^2 \bar{b}_1 \bar{c}_1 \bar{c}_2 - b_1 \bar{a}_1^2 \bar{b}_2 \bar{c}_1 \bar{c}_2 - b_2 \bar{a}_1 \bar{a}_2 \bar{b}_2 \bar{c}_1 \bar{c}_2 - \\ & a_2 b_1 b_2 \bar{a}_1 \bar{c}_2^2 - a_2 b_2^2 \bar{a}_2 \bar{c}_2^2 + b_1 \bar{a}_1^2 \bar{b}_1 \bar{c}_2^2 + b_2 \bar{a}_1 \bar{a}_2 \bar{b}_1 \bar{c}_2^2 = 0 \end{aligned}$$

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