SLICE REGULARITY IN SEVERAL VARIABLES

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Abstract. We introduce a class of slice regular functions of several variables on a real alternative algebra. In the quaternionic case, several variables have been considered recently by Colombo, Sabadini and Struppa [1]. Our approach to the definition of slice functions, which is based on the concept of stem functions, is the same as the one adopted by these authors. However, the condition of regularity is different, and allows to include in our class, in particular, the family of ordered polynomials in several variables. We prove some basic properties and results of slice and slice regular functions and give examples to illustrate this function theory.

1 Introduction

The theory of power series and more generally of *slice regular* functions of *one variable* in a real alternative algebra is now fairly developed. It was introduced firstly for functions of one quaternionic variable by Gentili and Struppa in [3, 4]. A related theory, concerning *slice monogenic* functions on Clifford algebras, was introduced by Colombo, Sabadini and Struppa in [2]. In [5] and [6], a new approach to slice regularity, based on the concept of *stem function*, allowed to extend the theory to any real alternative algebra A of finite dimension.

In the present paper, we propose a possible generalization of the theory to *several variables* in A. Our function theory includes, in particular, the class of (ordered) polynomials in several variables. For $A = \mathbb{H}$, several variables have been studied recently by Colombo, Sabadini and Struppa [1]. The approach via stem functions is similar, but the definition of regularity is different, as we will see in Section 4.3.

After having given the basic definitions, we state some results which show the richness of this function theory. We give a Cauchy integral formula and some of its fundamental consequences, and we show that some results about the removability of singularities, which are true for several complex variables, continue to hold in our setting.

2 The quadratic cone

Let A be a *finite-dimensional, alternative real algebra with identity* with a fixed $\mathbb{R}$-linear *antiinvolution.* Define the *trace* $t(x) := x + x^c \in A$ and the *norm* $n(x) := xx^c \in A$.

Definition 1. *The* quadratic cone *of A is the subset*

$$\mathcal{Q}_A := \mathbb{R} \cup \{x \in A \mid t(x) \in \mathbb{R},\ n(x) \in \mathbb{R},\ 4n(x) > t(x)^2\}.$$

The square roots of -1 in the algebra are the elements of $\mathbb{S}_A := \{J \in \mathcal{Q}_A \mid J^2 = -1\}$.

Examples 1. *1. $\mathbb{H}$ and $\mathbb{O}$ with the usual conjugation ($\mathcal{Q}_\mathbb{H} = \mathbb{H}$ and $\mathcal{Q}_\mathbb{O} = \mathbb{O}$).*

2. *The real Clifford algebra $Cl_{0,n} = \mathbb{R}_n$ with Clifford conjugation. The quadratic cone $\mathcal{Q}_n$ of $\mathbb{R}_n$ is the real algebraic subset defined by $x_K = 0,\ x \cdot (xe_K) = 0\ \forall e_K \neq 1$ such that $e_K^2 = 1$. It contains the subspace of* paravectors.

3. *In $\mathbb{R}_3$, $\mathcal{Q}_A = \{x \in \mathbb{R}_3 \mid x_{123} = 0, x_1x_{23} - x_2x_{13} + x_3x_{12} = 0\}$.*

The algebras with $\mathcal{Q}_A = A$ are *only* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ with the usual conjugation (cf. Frobenius–Zorn's Theorem)

Proposition 1. *Let $Im(A) := \{x \in A \mid x^2 \in \mathbb{R}, x \notin \mathbb{R} \setminus \{0\}\}$. For every $x \in \mathcal{Q}_A$, there exist unique elements $x_0 \in \mathbb{R}$, $y \in Im(A) \cap \mathcal{Q}_A$ with $t(y) = 0$, such that $x = x_0 + y$. For $J \in \mathbb{S}_A$, let $\mathbb{C}_J := \langle 1, J \rangle \simeq \mathbb{C}$ be the subalgebra generated by J. Then $\mathcal{Q}_A = \bigcup_{J \in \mathbb{S}_A} \mathbb{C}_J$ and $\mathbb{C}_I \cap \mathbb{C}_J = \mathbb{R}$ for every $I, J \in \mathbb{S}_A$, $I \neq \pm J$.*

3 Slice regular functions: one variable

We recall some definitions from [5, 6], where the slice regular functions of one variable in A were introduced. Let $A \otimes \mathbb{C} = A \otimes_\mathbb{R} \mathbb{C}$ be the complexified algebra.

Definition 2. *Let $D \subseteq \mathbb{C}$. If a function $F : D \to A \otimes \mathbb{C}$ is complex intrinsic, i.e. $F(\overline{z}) = \overline{F(z)}$ for every $z \in D$ such that $\overline{z} \in D$, then F is called a* stem function *on D. Let $\Omega_D := \{x = \alpha + \beta J \in \mathbb{C}_J \mid \alpha + i\beta \in D,\ J \in \mathbb{S}_A\}$ be a circular set in the quadratic cone $\mathcal{Q}_A$. Any stem function $F = F_1 + iF_2 : D \to A \otimes \mathbb{C}$ induces a* (left) slice function *$f = \mathcal{I}(F) : \Omega_D \to A$. If $x = \alpha + \beta J \in D_J := \Omega_D \cap \mathbb{C}_J$, we set*

$$f(x) := F_1(z) + JF_2(z) \quad (z = \alpha + i\beta).$$

Definition 3 ([5, 6])**.** *A slice function is* slice regular *if its stem function F is holomorphic. $\mathcal{SR}(\Omega_D) := \{f \in \mathcal{S}(\Omega_D) \mid f = \mathcal{I}(F), F$ holomorphic$\}$*

Examples 2. *1.* Polynomials $p(x) = \sum_{j=0}^{m} x^j a_j$ *with right coefficients in* A *are slice regular functions on* $\mathcal{Q}_A$.

2. Convergent power series $\sum_k x^k a_k$ *are slice regular functions on the intersection of* $\mathcal{Q}_A$ *with a ball centered in the origin.*

3. If $A = \mathbb{H}$ *and* $D \cap \mathbb{R} \neq \emptyset$, *then* $f \in \mathcal{SR}(\Omega_D)$ *if and only if it is Cullen regular [3, 4].*

4. If $A = \mathbb{R}_n$, $n > 2$, *and* $D \cap \mathbb{R} \neq \emptyset$, *then* $f \in \mathcal{SR}(\Omega_D)$ *if and only if the restriction of* f *to the paravectors is a* slice monogenic function *[2].*

4 Slice regular functions: several variables

4.1 Stem functions and slice functions

Let D be an open subset of $\mathbb{C}^n$, invariant w.r.t. complex conjugation in every variable $z_1, \ldots, z_n$.

Definition 4. *Given a function* $F : D \to A \otimes \mathbb{R}_n$, *with* $F = \sum_{K \in \mathcal{P}(n)} e_K F_K$, *we say that* F *is* Clifford-intrinsic *if, for each* $K \in \mathcal{P}(n)$, $h \in \{1, \ldots, n\}$ *and* $z = (z_1, \ldots, z_n) \in D$, *the components* $F_K : D \to A$ *satisfy:*

$$F_K(z_1, \ldots, z_{h-1}, \overline{z}_h, z_{h+1}, \ldots, z_n) = \begin{cases} F_K(z) & \text{if } h \notin K \\ -F_K(z) & \text{if } h \in K \end{cases} .$$

Definition 5. *A continuous Clifford–intrinsic function is a* stem function *on* D.

Define the (ordered) product $\prod_{h \in H}^{\rightarrow} x_h$ of $x_{h_1}, \ldots, x_{h_p} \in A$ as $\prod_{h \in H}^{\rightarrow} x_h := (\cdots((x_{h_1} x_{h_2}) x_{h_3}) \cdots) x_{h_p}$. Let Ω_D be the *circular* subset of $\mathcal{Q}_A^n$ associated to the open set $D \subseteq \mathbb{C}^n$:

$$\Omega_D = \{x \in \mathcal{Q}_A^n : x_h = \alpha_h + \beta_h J_h \in \mathbb{C}_J, J_h \in \mathbb{S}_A, (\alpha_1 + i\beta_1, \ldots, \alpha_n + i\beta_n) \in D\}.$$

Definition 6. *Given a stem function* $F : D \to A \otimes \mathbb{R}_n$ *with* $F = \sum_{K \in \mathcal{P}(n)} e_K F_K$, *we define the* (left) slice function $\mathcal{I}(F) : \Omega_D \to A$ *induced by* F *as follows*

$$\mathcal{I}(F)(x) := \sum_{K \in \mathcal{P}(n)} J_K F_K(\alpha_1 + i\beta_1, \ldots, \alpha_n + i\beta_n)$$

for each $x = (x_1, \ldots, x_n) = (\alpha_1 + J_1\beta_1, \ldots, \alpha_n + J_n\beta_n)$, *where* $J_K := \prod_{k \in K}^{\rightarrow} J_k$.

Examples 3. *1. For each $h = 1, \dots, n$, the coordinate function x_h is a slice function on $\mathcal{Q}_A^n$: if $x_h = \alpha_h + J_h\beta_h$, x_h is induced by the stem function $\zeta_h(z) := \alpha_h + e_h\beta_h$.*

2. *For each $\ell \in \mathbb{N}^n$ and $a \in A$, the stem function $\zeta_1^{\ell_1}(z)\cdots\zeta_n^{\ell_n}(z)a := (\alpha_1 + e_1\beta_1)^{\ell_1}\cdots(\alpha_n + e_n\beta_n)^{\ell_n}a$ induces the* monomial slice function $x^\ell a = (\prod^{\rightarrow}_{h\in\{1,\dots,n\}} x_h^{\ell_h})a$.

3. *Let $L \subset \mathbb{N}^n$ and $a_\ell \in A$. Then the* polynomial function *from $\mathcal{Q}_A^n$ to A, sending x into $p(x) = \sum_{\ell\in L} x^\ell a_\ell$, is a slice function.*

4. Convergent power series $\sum_{\ell\in\mathbb{N}^n} x^\ell a_\ell$ *are slice functions on the intersection of $\mathcal{Q}_A^n$ with a product of balls centered in the origin.*

Proposition 2 (Smoothness)**.** *Let $f = \mathcal{I}(F) : \Omega_D \to A$ be a slice function. The following statements hold:*

1. *If $F \in C^0(D, A\otimes\mathbb{R}_n)$, then $f \in C^0(\Omega_D, A)$.*

2. *Let $s_1 = 2^n(s+1) - 1$. If $F \in C^{s_1}(D, A\otimes\mathbb{R}_n)$ for some $s \in \mathbb{N}^* \cup \{\infty\}$, then $f \in C^s(\Omega_D, A)$.*

3. *If $F \in C^\omega(D, A\otimes\mathbb{R}_n)$, then $f \in C^\omega(\Omega_D, A)$.*

Proposition 3 (Identity principle)**.** *Let $f, g : \Omega_D \to A$ be slice functions and let $I \in \mathbb{S}_A$ such that $f = g$ on $\Omega_D \cap (\mathbb{C}_I)^n$. Then $f = g$ on the whole Ω_D.*

4.2 Complex structures on $\mathbb{R}_n$

Let us introduce some complex structures on $\mathbb{R}_n$.

Definition 7. *For each $h = 1, \dots, n$, define the* complex structure $\mathcal{J}_h$ *on $\mathbb{R}_n$ by*

$$\mathcal{J}_h(e_K) := \begin{cases} -e_{K\setminus\{h\}} & \text{if } h \in K \\ e_{K\cup\{h\}} & \text{if } h \notin K \end{cases}.$$

From the definition, it follows immediately that $\mathcal{J}_h^2 = -id_{\mathbb{R}_n}$. In other words, the endomorphisms $\mathcal{J}_h$ are complex structures on $\mathbb{R}_n$. One can easily verify that $\mathcal{J}_1$ is the left multiplication by e_1, $\mathcal{J}_n$ is the right multiplication by e_n and, for every $h = 1, \dots, n$, $\mathcal{J}_h$ coincides with the left multiplication by e_h on the complex plane $\mathbb{C}_{e_h} = \langle 1, e_h\rangle$ of $\mathbb{R}_n$.

Proposition 4. *The complex structures $\mathcal{J}$ are pairwise commuting and therefore they define commuting* Cauchy-Riemann operators *w.r.t.* $\mathcal{J}_h$*:*

$$\partial_h F = \frac{1}{2}\left(\frac{\partial F}{\partial \alpha_h} - \mathcal{J}_h\left(\frac{\partial F}{\partial \beta_h}\right)\right), \quad \bar{\partial}_h F = \frac{1}{2}\left(\frac{\partial F}{\partial \alpha_h} + \mathcal{J}_h\left(\frac{\partial F}{\partial \beta_h}\right)\right).$$

4.3 Slice regularity: several variables

Extend the complex structures $\mathcal{J}_h$ to $A \otimes \mathbb{R}_n$ by setting $\mathcal{J}_h(a \otimes x) = a \otimes \mathcal{J}_h(x)$ for every $a \in A, x \in \mathbb{R}_n$.

Definition 8. *Let $F : D \to A \otimes \mathbb{R}_n$ be a C^1 stem function and let $f = \mathcal{I}(F) : \Omega_D \to A$. F is a* holomorphic stem function *if, for each $h = 1, \dots, n$ and each fixed $z^0 := (z_1^0, \dots, z_n^0) \in D$, the function $F_h^{z_0} : D_h \to (A \otimes \mathbb{R}_n, \mathcal{J}_h)$ defined by $z_h \mapsto F(z_1^0, \dots, z_{h-1}^0, z_h, z_{h+1}^0, \dots, z_n^0)$ is holomorphic on a domain $D_h \ni z_h^0$ of $\mathbb{C}$ or, equivalently, if $\bar{\partial}_h F = 0$ on D for every $h = 1, \dots, n$. If F is holomorphic, then we say that $f = \mathcal{I}(F)$ is a* slice regular function.

Remark 1. *For $A = \mathbb{H}$, several variables have been considered recently by Colombo, Sabadini and Struppa [1]. Slice functions defined via stem functions are the same, but regularity is different (they use the complex structure L_{e_h} in the place of $\mathcal{J}_h$).*

Examples 4.

1. *For each $\ell \in \mathbb{N}^n$ and $a \in A$, the* monomial slice function *$x^\ell a : \mathcal{Q}_A^n \to A$ is regular. Therefore every* (ordered) polynomial function *$p(x) = \sum_{\ell \in L} x^\ell a_\ell$ with right coefficients in A is slice regular.*
2. Convergent power series $\sum_{\ell \in \mathbb{N}^n} x^\ell a_\ell$ *are slice regular functions on the intersection of $\mathcal{Q}_A^n$ with a product of balls centered in the origin.*

Proposition 5. *Let $F = \sum_{K \in P(n)} e_K F_K : D \to A \otimes \mathbb{R}_n$ be a stem function of class C^1. Let $f = \mathcal{I}(F) : \Omega_D \to A$. We denote by $f_I : \Omega_D \cap (\mathbb{C}_I)^n \to A$ the restriction of f on $\Omega_D \cap (\mathbb{C}_I)^n$. The following assertions are equivalent:*

(1) *f is slice regular.*

(2) $\frac{\partial F_K}{\partial \alpha_h} = \frac{\partial F_{K \cup \{h\}}}{\partial \beta_h}$, $\frac{\partial F_K}{\partial \beta_h} = -\frac{\partial F_{K \cup \{h\}}}{\partial \alpha_h}$ *for each K, h with $K \not\ni h$.*

(3) *There exists $I \in \mathbb{S}_A$ such that f_I is holomorphic w.r.t. the complex structures on $(\mathbb{C}_I)^n$ and on A defined by the left multiplication by I.*

(3′) *For every $I \in \mathbb{S}_A$, f_I is holomorphic w.r.t. the complex structures on $(\mathbb{C}_I)^n$ and on A defined by the left multiplication by I.*

4.4 Products and derivatives

Proposition 6. *Let $D = \prod_{h=1}^{n} D_h$. For each $h = 1, \ldots, n$, let $F^h : D_h \to A \otimes \mathbb{C}_{e_h} \subseteq A \otimes \mathbb{R}_n$ be a (one variable) stem function of class C^1. Let $a \in A$ and $F : D \to A \otimes \mathbb{R}_n$ defined by $F(z_1, \ldots, z_n) = \left(\prod^{\rightarrow}_{h \in \{1,\ldots,n\}} F^h(z_h)\right) a$. Then F is a stem function, holomorphic if every F^h is holomorphic.*

In general, the *pointwise* product of two slice functions is *not* a slice function. However, the pointwise product of stem functions (in the algebra $A \otimes \mathbb{R}_n$) is still a stem function.

Definition 9. *Let $f = \mathcal{I}(F)$, $g = \mathcal{I}(G)$ slice functions. The* product *of f and g is the slice function $f \cdot g := \mathcal{I}(FG)$.*

Proposition 7. *If f, g are slice regular* and $F = \sum_{K \in \mathcal{S}} e_K F_K$, $G = \sum_{H \in \mathcal{S}'} e_H G_H$, *with $K \leq H$ for each $K \in \mathcal{S}$, $H \in \mathcal{S}'$, then $f \cdot g$ is slice regular.*

Remark 2. *The* ordering *of the variables is important for regularity: e.g. the function $x_2 \cdot x_1 = \mathcal{I}(\zeta_2 \zeta_1)$ is a slice function but it is not slice regular.*

If $f = \mathcal{I}(F)$ is a slice function, of class C^1 on Ω_D, then the functions $\partial_h F$ and $\bar{\partial}_h F$ are stem functions on D.

Definition 10. *We set*

$$\frac{\partial f}{\partial x_h} := \mathcal{I}\left(\partial_h F\right), \quad \frac{\partial f}{\partial x_h^c} := \mathcal{I}\left(\bar{\partial}_h F\right), \quad h = 1, \ldots, n.$$

These functions are continuous slice functions on Ω_D.

The slice function f is slice regular if and only if $\frac{\partial f}{\partial x_h^c} = 0$ for every $h = 1, \ldots, n$. If f is slice regular, then also the derivatives $\frac{\partial f}{\partial x_h}$ are slice regular (it follows from the commutativity of the structures $\mathcal{J}_h$).

4.5 Cauchy integral formula

We now show that slice regular functions satisfy a Cauchy integral formula. As a consequence, we obtain that on a polydisc the class of slice regular functions coincides with that of convergent ordered power series.

Definition 11. *Let $\Delta_y(x) = x^2 - t(y)x + n(y)$ and $\Gamma_A := \{(x, y) \in \mathcal{Q}_A \times \mathcal{Q}_A \mid \Delta_y(x) \neq 0\}$. We define the* Cauchy kernel of A *as the C^ω–function $C_A : \Gamma_A \to A$, slice regular in x, given by*

$$C_A(x, y) := \left(\Delta_y(x)\right)^{-1}(y^c - x).$$

Fix $I \in \mathbb{S}_A$ and, for each $h = 1, \dots, n$, a bounded open subset E_h of $\mathbb{C}$, whose boundary is piecewise of class C^1. Let $E_h(I) := \Omega_{E_h} \cap \mathbb{C}_I$ and let $\partial E_h(I)$ be the boundary of $E_h(I)$ in $\mathbb{C}_I$. Let $E := E_1 \times E_2 \times \dots \times E_n \subset \mathbb{C}^n$. Denote by $\partial^* E(I)$ the *distinguished boundary* $\partial E_1(I) \times \partial E_2(I) \times \cdots \times \partial E_n(I)$ of $E(I) := E_1(I) \times E_2(I) \times \cdots \times E_n(I)$.

Theorem 1 (Cauchy integral formula)**.** *If $f \in \mathcal{SR}(\Omega_E, A) \cap C^0(\overline{\Omega}_E, A)$, then*

$$f(x) = \frac{1}{(2\pi)^n} \int_{\partial^* E(I)} C_A(x_1, \xi_1) \cdots C_A(x_n, \xi_n) d\xi_1 d\xi_2 \cdots d\xi_n I^{-n} f(\xi_1, \dots, \xi_n)$$

for each $x = (x_1, \dots, x_n) \in \Omega_E$ if A is associative or for each $x = (x_1, \dots, x_n) \in E(I)$ if A is not–associative. In particular, f is real analytic.

Suppose that there exists a norm $\|\cdot\|_A$ on A which induces the euclidean topology on A and such that $\|x\|_A = \sqrt{\alpha^2 + \beta^2}$ for each $x = \alpha + J\beta \in \mathcal{Q}_A$.
Let $r = (r_1, \dots, r_n) \in (\mathbb{R}^+)^n$. Denote by B_r the polydisc $B(0, r_1) \times \cdots \times B(0, r_n)$ of $\mathbb{C}^n$ and by $B_A(r)$ the polydisc $\{(x_1, \dots, x_n) \in A^n \mid \|x_1\|_A < r_1, \dots, \|x_n\|_A < r_n\}$ of A^n. Note that $B_A(r)$ is an open neighborhood of 0 in A^n and $B_A(r) \cap \mathcal{Q}_A^n = \Omega_{B_r}$.

Corollary 1 (Ordered analyticity)**.** *Let $f \in \mathcal{SR}(\Omega_{B_r}, A) \cap C^0(\overline{\Omega}_{B_r}, A)$. Choose $I \in \mathbb{S}_A$ and, for each $\ell = (\ell_1, \dots, \ell_n) \in \mathbb{N}^n$, define $a_\ell \in A$ by setting*

$$a_\ell := (2\pi I)^{-n} \int_{\partial^* B_r(I)} \xi_1^{-\ell_1 - 1} \cdots \xi_n^{-\ell_n - 1} d\xi_1 \cdots d\xi_n \, f(\xi_1, \dots, \xi_n).$$

Then the ordered power series $\sum_{\ell=(\ell_1,\dots,\ell_n)\in\mathbb{N}^n} x_1^{\ell_1} \cdots x_n^{\ell_n} a_\ell$ converges uniformly on compact subsets of $B_A(r)$ and its sum is equal to $f(x)$ for each $x \in \Omega_{B_r}$.

Corollary 2. *On Ω_{B_r}, the set of slice regular functions coincides with that of convergent ordered power series.*

Corollary 3 (Cauchy's inequalities)**.** *Let $f \in \mathcal{SR}(\Omega_{B_r}, A) \cap C^0(\overline{\Omega}_{B_r}, A)$ and let $M > 0$ be a constant such that $\sup_{x \in \partial^* B_r(I)} \|f(x)\|_A \leq M$ for some $I \in \mathbb{S}_A$. Then there exists a constant N_A (depending only on $\|\cdot\|_A$) such that, for each $\ell = (\ell_1, \dots, \ell_n) \in \mathbb{N}^n$, it holds:*

$$\|\partial_\ell f(0)\|_A \leq N_A \cdot M \cdot \ell! \cdot r_1^{-\ell_1} \cdots r_n^{-\ell_n},$$

where $\partial_\ell := \partial^{\ell_1 + \dots + \ell_n} / \partial Re(x_1)^{\ell_1} \cdots \partial Re(x_n)^{\ell_n}$.

4.6 Removability of singularities

Theorem 2 (Hartogs extension phenomenon). *Let $D' \subset D \subset \mathbb{C}^n$ open with compact closure $K := \overline{D'} \subset D$ such that $D \setminus K$ is connected. If f is a slice regular function on $\Omega_{D\setminus K} = \Omega_D \setminus \overline{\Omega}_{D'}$, then it extends uniquely to a slice regular function on the whole set Ω_D.*

Theorem 3. *Let Θ be a circular open subset of A^n, let $Z = \Omega_W$ be a proper closed subset of Θ with W locally analytic in $\mathbb{C}^n$ and let $f \in \mathcal{SR}(\Theta \setminus Z, A)$. Suppose that at least one of the following two condition holds:* (1) *f is locally bounded in Θ,* (2) $\operatorname{codim}(W) \geq 2$. *Then f extends to a slice regular function on the whole Θ.*

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