A new approach to slice regularity on real algebras

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Abstract. We expose the main results of a theory of slice regular functions on a real alternative algebra A, based on a well–known Fueter's construction. Our general theory includes the theory of slice regular functions of a quaternionic or octonionic variable and the theory of slice monogenic functions of a Clifford variable. Our approach permits to extend the range of these function theories and to obtain new results. In particular, we show that a fundamental theorem of algebra with multiplicities holds for an ample class of polynomials with coefficients in A. We give several examples to illustrate some interesting aspects of the theory.

1. Introduction

In this survey paper, we propose a new approach to the concepts of "slice regularity" for functions of one quaternionic, octonionic or Clifford variable which have been recently introduced by Gentili and Struppa in [17, 18] and by Colombo, Sabadini and Struppa in [8].

Actually, the starting point for our approach is not new: it dates back to a paper of Rudolf Fueter [12], in which he proposed a simple method, which is now known as Fueter's Theorem, to generate quaternionic regular functions (cf. [35] and [25] for the theory of Fueter regular functions) by means of complex holomorphic functions. Given a holomorphic "stem function"

$$F(z) = u(\alpha, \beta) + i\, v(\alpha, \beta) \quad (z = \alpha + i\beta \text{ complex}, u, v \text{ real–valued})$$

in the upper complex half–plane, real–valued on $\mathbb{R}$, the formula

$$f(q) := u\left(q_0, |\operatorname{Im}(q)|\right) + \frac{\operatorname{Im}(q)}{|\operatorname{Im}(q)|} v\left(q_0, |\operatorname{Im}(q)|\right)$$

(with $q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}$, $\operatorname{Im}(q) = q_1 i + q_2 j + q_3 k$) defines a function on $\mathbb{H}$, whose Laplacian is Fueter regular. Fueter's construction was later extended to higher dimensions by Sce [32], Qian [30] and Sommen [34] in the setting of octonionic and Clifford analysis.

We extend Fueter's construction in order to develop a theory of slice regular functions on a real alternative algebra A with a fixed antiinvolution. These functions will be obtained by taking A–valued components u, v of the stem function F. The domains on which slice regular functions can be defined are open subsets of what we call the *quadratic cone* of the algebra. This cone is the whole algebra only in the case in which A is a real division algebra (i.e. the complex numbers, the quaternions $\mathbb{H}$ or the octonions $\mathbb{O}$). We refer the reader to the article [22] for complete proofs of the main results stated in the present paper.

If A is the algebra of quaternions, we get the theory of *slice regular* (or *Cullen regular*) functions of a quaternionic variable introduced by Gentili and Struppa [17, 18]. If A is the algebra of octonions, we obtain the corresponding theory of regular functions already considered in [16, 20, 23]. If A is the Clifford algebra $Cl_{0,n} = \mathbb{R}_n$, the quadratic cone is a real algebraic (proper for $n \geq 3$) subset of $\mathbb{R}_n$, containing the subspace of paravectors. By restricting the Clifford variables to the paravectors, we get the theory of *slice monogenic* functions introduced by Colombo, Sabadini and Struppa in [8].

In Section 2, we define the normal cone and the quadratic cone of an algebra A and prove that the quadratic cone is a union of complex planes of A. This property is the starting point for the extension of Fueter's construction. We compute the cones for some relevant algebras. In particular, we show that the quadratic cone can be a semi–algebraic set (e.g. in Clifford algebras with non–definite signature) and depends on the antiinvolution chosen in A. We also find the dimensions of the cones.

In Section 3, we introduce complex intrinsic functions with values in the complexified algebra $A \otimes_{\mathbb{R}} \mathbb{C}$ and use them as stem functions to generate A–valued *(left) slice functions*. This approach does not require the holomorphy of the stem function. Moreover, slice functions can be defined also on domains which do not intersect the *real axis* of A.

In Section 4, we restrict our attention to slice functions with *holomorphic* stem function, what we call *(left) slice regular* functions on A. These functions forms a real vector space that is not closed w.r.t. the pointwise product in A. The pointwise product for stem functions induces a natural product on slice functions (cf. Section 5), that generalizes the usual product of polynomials and power series.

In Section 6, we recall some properties of the zero set of slice functions. We restrict our attention to *admissible* slice regular functions, which preserve many relevant properties of classical holomorphic functions. We generalize a structure

theorem for the zero set proved by Pogorui and Shapiro [29] for quaternionic polynomials and by Gentili and Stoppato [15] for quaternionic power series. A Remainder Theorem (Theorem 11) gives us the possibility to define a notion of multiplicity for the zeros of an admissible slice regular function.

Polynomials with right coefficients in A are slice regular functions on the quadratic cone. We obtain a version of the fundamental theorem of algebra for slice regular admissible polynomials. This theorem was proved for quaternionic polynomials by Eilenberg and Niven [10, 28] and for octonionic polynomials by Jou [26]. See also [11, pp. 308ff] for a topological proof of the theorem valid for a class of real algebras including $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, and [36], [31] and [21] for other proofs. Gordon and Motzkin [24] proved, for polynomials on a (associative) division ring, that the number of conjugacy classes containing zeros of p cannot be greater than the degree m of p. This estimate was improved on $\mathbb{H}$ by Pogorui and Shapiro [29]: if p has s spherical zeros and l non–spherical zeros, then $2s + l \leq m$. Gentili and Struppa [19] showed that, using the right definition of multiplicity, the number of zeros of p equals the degree of the polynomial. In [23], this strong form was generalized to the octonions. Recently, Colombo, Sabadini and Struppa [8, 9] and Yang and Qian [37] proved some results on the structure of the set of zeros of a polynomial with paravector coefficients in a Clifford algebra.

We obtain a strong form of the fundamental theorem of algebra (Theorem 14), which contains and generalizes the above results. We show that the sum of the multiplicities of the zeros of a slice regular admissible polynomial is equal to its degree. The last section contains several examples that illustrate the relevance of the quadratic cone and of the condition of admissibility for the algebraic and topological properties of the zero set of a polynomial. We see that the quadratic cone is sufficiently large to contain the "right" number of (isolated or spherical) zeros, and sufficiently small to exclude "wild" sets of zeros of the polynomial. Outside the quadratic cone, an admissible polynomial can have infinite non–spherical zeros. If the polynomial is not admissible, its zero set loses its regular structure.

2. The quadratic cone of a real alternative algebra

Let A be a finite–dimensional alternative real algebra with a unity, of dimension $d > 1$. We will identify the field of real numbers with the subalgebra of A generated by the unity. In a real algebra, we can consider the *imaginary space* consisting of all non–real elements whose square is real.

Definition 1. Let $\mathrm{Im}(A) := \{x \in A \mid x^2 \in \mathbb{R}, \ x \notin \mathbb{R} \setminus \{0\}\}$. The elements of $\mathrm{Im}(A)$ are called *purely imaginary* elements of A.

In general, the imaginary space $\mathrm{Im}(A)$ is not a vector subspace of A. In what follows, we will assume that on A an *antiinvolution* is fixed. It is a linear map $x \mapsto x^c$ of A into A satisfying the following properties:

$$(x^c)^c = x \quad \forall x \in A, \quad (xy)^c = y^c x^c \quad \forall x, y \in A, \quad x^c = x \quad \text{for every real } x.$$

Definition 2. For every element x of A, the *trace* of x is $t(x) := x + x^c \in A$ and the (squared) *norm* of x is $n(x) := xx^c \in A$.

Definition 3. We call *normal cone* of the algebra A the subset

$$\mathcal{N}_A := \{0\} \cup \{x \in A \mid n(x) = n(x^c) \in \mathbb{R} \setminus \{0\}\},$$

and *quadratic cone* of A the set

$$\mathcal{Q}_A := \mathbb{R} \cup \{x \in A \mid t(x) \in \mathbb{R}, \ n(x) \in \mathbb{R}, \ 4n(x) > t(x)^2\}.$$

We also set $\mathbb{S}_A := \{J \in \mathcal{Q}_A \mid J^2 = -1\} \subseteq \mathrm{Im}(A)$. Elements of $\mathbb{S}_A$ will be called *square roots of* -1 in the algebra A. For every $J \in \mathbb{S}_A$, we will denote by $\mathbb{C}_J := \langle 1, J \rangle \simeq \mathbb{C}$ the subalgebra of A generated by J.

Proposition 1. *Let A be an alternative real algebra with a fixed antiinvolution $x \mapsto x^c$. The following statements hold.*

(1) *Every $x \in \mathcal{Q}_A$ satisfies the real quadratic equation $x^2 - x\,t(x) + n(x) = 0$.*

(2) *$x \in \mathcal{Q}_A$ is equivalent to $x^c \in \mathcal{Q}_A$. Moreover, $\mathcal{Q}_A \subseteq \mathcal{N}_A$.*

(3) *Every nonzero $x \in \mathcal{N}_A$ is invertible: $x^{-1} = n(x)^{-1} x^c$.*

(4) *$J^c = -J$ for every $J \in \mathbb{S}_A$, i.e. $t(J) = 0$, $n(J) = 1$.*

(5) *$\mathcal{Q}_A = A$ if and only if A is isomorphic to one of the division algebras $\mathbb{C}, \mathbb{H}$ or $\mathbb{O}$ with the usual conjugation mapping.*

(6) *For every $x \in \mathcal{Q}_A$, there exist uniquely determined elements $x_0 \in \mathbb{R}$, $y \in \mathrm{Im}(A) \cap \mathcal{Q}_A$, with $t(y) = 0$, such that $x = x_0 + y$.*

(7) *$\mathcal{Q}_A = \bigcup_{J \in \mathbb{S}_A} \mathbb{C}_J$ and $\mathbb{C}_I \cap \mathbb{C}_J = \mathbb{R}$ for every $I, J \in \mathbb{S}_A$, $I \neq \pm J$.*

Using the notation of the above proposition, for every $x \in \mathcal{Q}_A$, we set $\mathrm{Re}(x) := x_0 = \frac{x + x^c}{2}$, $\mathrm{Im}(x) := y = \frac{x - x^c}{2}$.

As shown by the examples below, the quadratic cone $\mathcal{Q}_A$ needs not be a subalgebra or a subspace of A.

Examples 1. (1) Let A be the algebra $\mathbb{H}$ of the quaternions or the algebra $\mathbb{O}$ of the octonions. Let $x^c = \bar{x}$ be the usual conjugation mapping. Then $\mathrm{Im}(A)$ is a subspace, $A = \mathbb{R} \oplus \mathrm{Im}(A)$ and $\mathcal{Q}_\mathbb{H} = N_\mathbb{H} = \mathbb{H}$, $\mathcal{Q}_\mathbb{O} = N_\mathbb{O} = \mathbb{O}$. In these cases, $\mathbb{S}_\mathbb{H}$ is a two-dimensional sphere and $\mathbb{S}_\mathbb{O}$ is a six-dimensional sphere.

(2) Let A be the real Clifford algebra $Cl_{p,q} = \mathbb{R}_{p,q}$, with the *conjugation*

$$x^c = ([x]_0 + [x]_1 + [x]_2 + [x]_3 + [x]_4 + \cdots)^c = [x]_0 - [x]_1 - [x]_2 + [x]_3 + [x]_4 - \cdots,$$

where $[x]_k$ denotes the k–vector component of x in $\mathbb{R}_{p,q}$ (cf. for example [7, §4.1] or [25, §3.2]). Let $n := p + q$. An element x of $\mathbb{R}_{p,q}$ can be represented in the form $x = \sum_K x_K e_K$, with $K = (i_1, \ldots, i_k)$ an increasing multiindex of length k, $0 \leq k \leq n$, $e_K = e_{i_1} \cdots e_{i_k}$, $e_\emptyset = 1$, $x_K \in \mathbb{R}$, $x_\emptyset = x_0$, $e_1, \ldots, e_n$ basis elements.

Let $\mathbb{R}_n := \mathbb{R}_{0,n}$. In this case, the quadratic cone $\mathcal{Q}_n := \mathcal{Q}_{\mathbb{R}_n}$ is the real algebraic (proper for $n \geq 3$) subset of $\mathbb{R}_n$ defined in terms of the euclidean scalar product $x \cdot y$ by the equations

$$x_K = 0, \ x \cdot (xe_K) = 0 \text{ for every } e_K \neq 1 \text{ such that } e_K^2 = 1.$$

The subspace of *paravectors* $\mathbb{R}^{n+1} := \{x \in \mathbb{R}_n \mid [x]_k = 0 \text{ for every } k > 1\}$ is contained in $\mathcal{Q}_n$. For $x \in \mathbb{R}^{n+1}$, $t(x) = 2x_0 \in \mathbb{R}$ and $n(x) = |x|^2 \geq 0$ (the euclidean norm). The $(n-1)$–dimensional sphere $\mathbb{S} = \{x = x_1 e_1 + \cdots x_n e_n \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_n^2 = 1\}$ of unit 1–vectors is (properly) contained in $\mathbb{S}_{\mathbb{R}_n}$. The normal cone $\mathcal{N}_n := \mathcal{N}_{\mathbb{R}_n}$ contains also the *Clifford group* Γ_n and its subgroups $\mathrm{Pin}(n)$ and $\mathrm{Spin}(n)$. If $n \geq 3$, $\mathrm{Im}(A)$ is not a subspace of A.

(3) The case of $\mathbb{R}_3$. A direct computation shows that

$$\mathcal{N}_3 = \{x \in \mathbb{R}_3 \mid x_0 x_{123} + x_2 x_{13} - x_1 x_{23} - x_3 x_{12} = 0\}$$

and the quadratic cone is the 6–dimensional real algebraic set

$$\mathcal{Q}_3 = \{x \in \mathbb{R}_3 \mid x_{123} = 0, \ x_2 x_{13} - x_1 x_{23} - x_3 x_{12} = 0\}.$$

Finally, $\mathbb{S}_{\mathbb{R}_3} = \{x \in \mathcal{Q}_3 \mid x_0 = 0, \sum_i x_i^2 + \sum_{j,k} x_{jk}^2 = 1\} \subset \mathrm{Im}(\mathbb{R}_3) \cap \mathcal{Q}_3$ is the intersection of a 5–sphere with the hypersurface $x_2 x_{13} - x_1 x_{23} - x_3 x_{12} = 0$.

(4) The case of $\mathbb{R}_4$. The normal cone $\mathcal{N}_4$ is the 11–dimensional real algebraic set with equations

$$x_1 x_{1234} + x_{124} x_{13} - x_{12} x_{134} - x_{123} x_{14} = 0, \ x_2 x_{1234} + x_{124} x_{23} - x_{12} x_{234} - x_{123} x_{24} = 0,$$

$$x_3 x_{1234} + x_{134} x_{23} - x_{13} x_{234} - x_{123} x_{34} = 0, \ x_4 x_{1234} - x_{14} x_{234} + x_{134} x_{24} - x_{124} x_{34} = 0,$$

$$x_{134} x_2 - x_1 x_{234} - x_{124} x_3 + x_{123} x_4 = 0, \ x_0 x_{1234} - x_{14} x_{23} + x_{13} x_{24} - x_{12} x_{34} = 0,$$

$$x_0 x_{234} + x_3 x_{24} - x_2 x_{34} - x_{23} x_4 = 0, \ x_0 x_{134} + x_3 x_{14} - x_1 x_{34} - x_{13} x_4 = 0,$$

$$x_0 x_{124} + x_2 x_{14} - x_1 x_{24} - x_{12} x_4 = 0, \ x_0 x_{123} + x_2 x_{13} - x_1 x_{23} - x_{12} x_3 = 0,$$

while the quadratic cone $\mathcal{Q}_4$ is the 8–dimensional real algebraic set defined by

$$x_{14} x_{23} - x_{13} x_{24} + x_{12} x_{34} = x_3 x_{24} - x_2 x_{34} - x_{23} x_4 = x_3 x_{14} - x_1 x_{34} - x_{13} x_4 = 0,$$

$$x_2 x_{14} - x_1 x_{24} - x_{12} x_4 = x_2 x_{13} - x_1 x_{23} - x_{12} x_3 = x_{123} = x_{124} = x_{134} = x_{234} = x_{1234} = 0.$$

(5) Let $A = \mathbb{R}_{1,2}$. Then $\mathcal{N}_{1,2}$ is the 7–dimensional real semi–algebraic set

$$\{x \in \mathbb{R}_{1,2} | x_0 x_{123} + x_2 x_{13} - x_1 x_{23} - x_3 x_{12} = 0, x_0^2 + x_2^2 + x_3^2 + x_{23}^2 \neq x_1^2 + x_{12}^2 + x_{13}^2 + x_{123}^2\} \cup \{0\}$$

and the quadratic cone $\mathcal{Q}_{1,2}$ is the 6–dimensional real semi–algebraic set

$$\mathcal{Q}_{1,2} = \mathbb{R} \cup \{x \in \mathbb{R}_{1,2} \mid x_{123} = 0, \ x_2 x_{13} - x_1 x_{23} - x_3 x_{12} = 0,$$
$$x_2^2 + x_3^2 + x_{23}^2 > x_1^2 + x_{12}^2 + x_{13}^2\}.$$

(6) The quadratic cone and the normal cone depend on the antiinvolution chosen in the algebra. For example, in the Clifford algebras $\mathbb{R}_n$, it is possible to take the *reversion* $x \mapsto x^*$ in place of the conjugation. The reversion is defined by

$$x^* = ([x]_0 + [x]_1 + [x]_2 + [x]_3 + [x]_4 + \cdots)^* = [x]_0 + [x]_1 - [x]_2 - [x]_3 + [x]_4 - \cdots.$$

In $\mathbb{R}_2 \simeq \mathbb{H}$, we get the normal and quadratic cones

$$\mathcal{N}_2^* = \{x \mid x_1 = x_2 = 0\} \cup \{x \mid x_0 = x_{12} = 0\}, \ \mathcal{Q}_2^* = \{x \in \mathbb{R}_2 \mid x_1 = x_2 = 0\} \simeq \mathbb{C}.$$

Here $\mathbb{S}_A = \{\pm e_{12}\}$.

In $\mathbb{R}_3$ the normal cone (w.r.t. reversion) is the 5–dimensional real semi–algebraic set

$$\mathcal{N}_3^* = \mathbb{R} \cup \{x \mid x_2 x_{12} + x_3 x_{13} = 0, \ x_1 x_{13} + x_2 x_{23} = 0, \ x_1 x_{12} - x_{23} x_3 = 0,$$
$$x_0 x_1 + x_{123} x_{23} = 0, \ x_0 x_2 - x_{123} x_{13} = 0, \ x_0 x_3 + x_{12} x_{123} = 0,$$
$$x_0^2 + x_{12}^2 + x_{13}^2 + x_{23}^2 \neq x_1^2 + x_2^2 + x_3^2 + x_{123}^2\}$$

and the quadratic cone is the 4–dimensional plane

$$\mathcal{Q}_3^* = \{x \mid x_1 = x_2 = x_3 = x_{123} = 0\} = \langle 1, e_{12}, e_{13}, e_{23} \rangle \simeq \mathbb{H}.$$

3. Slice functions

Let $A_{\mathbb{C}} = A \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of A. We will use the representation

$$A_{\mathbb{C}} = \{w = x + iy \mid x, y \in A\} \quad (i^2 = -1).$$

$A_{\mathbb{C}}$ is an alternative complex algebra with a unity w.r.t. the product:

$$(x + iy)(x' + iy') = xx' - yy' + i(xy' + yx').$$

The algebra A can be identified with the real subalgebra $A' := \{w = x + iy \mid y = 0\}$ of $A_{\mathbb{C}}$. In $A_{\mathbb{C}}$ two commuting operators are defined: the *complex–linear antiinvolution* $w \mapsto w^c = (x + iy)^c = x^c + iy^c$ and the *complex conjugation* defined by $\overline{w} = \overline{x + iy} = x - iy$.

Definition 4. Let $D \subseteq \mathbb{C}$ be an open subset. If a function $F : D \to A_{\mathbb{C}}$ is *complex intrinsic*, i.e. it satisfies the condition $F(\overline{z}) = \overline{F(z)}$ for every $z \in D$ such that $\overline{z} \in D$, then F is called an *A–stem function* on D.

In the preceding definition, there is no restriction to assume that D is symmetric w.r.t. the real axis, i.e. $D = \mathrm{conj}(D) := \{z \in \mathbb{C} \mid \overline{z} \in D\}$.

Remarks 1. (1) A function F is an A–stem function if and only if the A–valued components F_1, F_2 of $F = F_1 + iF_2$ form an *even–odd pair* w.r.t. the imaginary part of z.

(2) By means of a basis $\mathcal{B} = \{u_k\}_{k=1,\dots,d}$ of A, F can be identified with a complex intrinsic curve in $\mathbb{C}^d$.

Given an open subset D of $\mathbb{C}$, let Ω_D be the relatively open subset of $\mathcal{Q}_A$ obtained by the action on D of the square roots of -1:

$$\Omega_D := \{x = \alpha + \beta J \in \mathbb{C}_J \mid \alpha, \beta \in \mathbb{R}, \ \alpha + i\beta \in D, \ J \in \mathbb{S}_A\}.$$

Sets of this type will be called *circular sets* in A.

Definition 5. Any stem function $F : D \to A_{\mathbb{C}}$ induces a *left slice function* $f = \mathcal{I}(F) : \Omega_D \to A$. If $x = \alpha + \beta J \in D_J := \Omega_D \cap \mathbb{C}_J$, we set

$$f(x) := F_1(z) + JF_2(z) \quad (z = \alpha + i\beta).$$

There is an analogous definition for *right* slice functions when J is placed on the right of $F_2(z)$. In what follows, the term *slice functions* will always mean left slice functions. We will denote the real vector space of (left) slice functions on Ω_D by $\mathcal{S}(\Omega_D)$.

Examples 2. Assume $A = \mathbb{H}$ or $A = \mathbb{O}$, with the usual conjugation mapping.

(1) For any element $a \in A$, $F(z) := z^n a = \mathrm{Re}(z^n)a + i\,(\mathrm{Im}(z^n)a)$ induces the monomial $f(x) = x^n a \in \mathcal{S}(A)$.

(2) By linearity, we get all the *standard polynomials* $p(x) = \sum_{j=0}^{n} x^j a_j$ with right quaternionic or octonionic coefficients. More generally, every convergent *power series* $\sum_j x^j a_j$ is a slice function on an open ball of A centered in the origin with (possibly infinite) radius.

The above examples generalize to standard polynomials in $x = \mathcal{I}(z)$ and $x^c = \mathcal{I}(\bar{z})$ with coefficients in A. The domain of slice polynomial functions or series must be restricted to subsets of the quadratic cone.

For an element $J \in \mathbb{S}_A$, let $\mathbb{C}_J^+$ denote the upper half plane $\mathbb{C}_J^+ = \{x = \alpha + \beta J \in A \mid \beta \geq 0\}$.

Proposition 2. *Let $J, K \in \mathbb{S}_A$ with $J - K$ invertible. Every slice function $f \in \mathcal{S}(\Omega_D)$ is uniquely determined by its values on the two distinct half planes $\mathbb{C}_J^+$ and $\mathbb{C}_K^+$. Moreover, the following* representation formula *holds:*

$$f(x) = (I - K)\left((J - K)^{-1}f(\alpha + \beta J)\right) - (I - J)\left((J - K)^{-1}f(\alpha + \beta K)\right) \quad (3.1)$$

for every $I \in \mathbb{S}_A$ and for every $x = \alpha + \beta I \in D_I = \Omega_D \cap \mathbb{C}_I$.

Representation formulas for quaternionic Cullen regular functions appeared in [3, 4], and for slice monogenic functions of a Clifford variable in [6, 5].

Definition 6. Let $f \in \mathcal{S}(\Omega_D)$ be a slice function.
The *spherical value of f in $x \in \Omega_D$* is $v_s f(x) := \frac{1}{2}\left(f(x) + f(x^c)\right)$.
The *spherical derivative of f in $x \in \Omega_D \setminus \mathbb{R}$* is $\partial_s f(x) := \frac{1}{2}\mathrm{Im}(x)^{-1}(f(x) - f(x^c))$.

In this way, we get two A–valued slice functions associated with f, constant on every "sphere" $\mathbb{S}_x := \{y \in \mathcal{Q}_A \mid y = \alpha + \beta I,\ I \in \mathbb{S}_A\}$ $(x = \alpha + \beta J,\ J \in \mathbb{S}_A)$ such that $\partial_s f(x) = 0$ if and only if f is constant on $\mathbb{S}_x$. In this case, f has value $v_s f(x)$ on $\mathbb{S}_x$. If $\Omega_D \cap \mathbb{R} \neq \emptyset$, under mild regularity conditions on F, we get that $\partial_s f$ can be continuously extended as a slice function on Ω_D. By definition, the following identity holds for every $x \in \Omega_D$: $f(x) = v_s f(x) + \mathrm{Im}(x)\,\partial_s f(x)$.

Proposition 3. *Let $f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D)$ be a slice function. Then the following statements hold:*

(1) *If $F \in C^0(D)$, then $f \in C^0(\Omega_D)$.*
(2) *If $F \in C^{2s+1}(D)$ for a positive integer s, then f is of class $C^s(\Omega_D)$. As a consequence, if $F \in C^\infty(D)$, then f is of class $C^\infty(\Omega_D)$.*
(3) *If $F \in C^\omega(D)$, then f is of class $C^\omega(\Omega_D)$.*

We will denote by $\mathcal{S}^1(\Omega_D) := \{f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D) \mid F \in C^1(D)\}$ the real vector space of slice functions with stem function of class C^1.

Definition 7. Let $f = \mathcal{I}(F) \in \mathcal{S}^1(\Omega_D)$. We set

$$\frac{\partial f}{\partial x} := \mathcal{I}\left(\frac{\partial F}{\partial z}\right), \quad \frac{\partial f}{\partial x^c} := \mathcal{I}\left(\frac{\partial F}{\partial \bar{z}}\right).$$

These functions are continuous slice functions on Ω_D.

4. Slice regular functions

Left multiplication by i defines a complex structure on $A_\mathbb{C}$. With respect to this structure, a C^1 function $F = F_1 + iF_2 : D \to A_\mathbb{C}$ is holomorphic if and only if its components F_1, F_2 satisfy the Cauchy–Riemann equations.

Definition 8. A (left) slice function $f \in \mathcal{S}^1(\Omega_D)$ is *(left) slice regular* if its stem function F is holomorphic. We will denote the real vector space of slice regular functions on Ω_D by $\mathcal{SR}(\Omega_D) := \{f \in \mathcal{S}^1(\Omega_D) \mid f = \mathcal{I}(F), \ F : D \to A_\mathbb{C} \text{ holomorphic}\}$.

Polynomials with right coefficients in A can be considered as slice regular functions on the quadratic cone. Assume that on A is defined a positive scalar product $x \cdot y$ whose associated norm satisfies an inequality $|xy| \leq C|x||y| \ (C > 0)$ and such that $|x|^2 = n(x)$ for every $x \in \mathcal{Q}_A$. In this case we can consider also convergent power series $\sum_k x^k a_k$ as slice regular functions on the intersection of the quadratic cone with a ball centered in the origin. See for example [25, §4.2] for the quaternionic and Clifford algebra cases, where we can take as product $x \cdot y$ the euclidean product in $\mathbb{R}^4$ or $\mathbb{R}^{2^n}$, respectively.

Proposition 4. *Let $f = \mathcal{I}(F) \in \mathcal{S}^1(\Omega_D)$. Then f is slice regular on Ω_D if and only if the restriction $f_J := f_{|\mathbb{C}_J \cap \Omega_D} : D_J = \mathbb{C}_J \cap \Omega_D \to A$ is holomorphic for every $J \in \mathbb{S}_A$ with respect to the complex structures defined by left multiplication by J.*

Proposition 4 implies that if A is the algebra of quaternions or octonions, and D intersects the real axis, then f is slice regular on Ω_D if and only if it is *Cullen regular* in the sense introduced by Gentili and Struppa in [17, 18, 20, 16]. If A is the real Clifford algebra $\mathbb{R}_n$, slice regularity generalizes the concept of *slice monogenic functions* introduced by Colombo, Sabadini and Struppa in [8]. If $f = \mathcal{I}(F) \in \mathcal{SR}(\Omega_D)$, $F \in C^1(D)$ and D intersects the real axis, then the restriction of f to the subspace of paravectors is a slice monogenic function. Conversely, every slice monogenic function is the restriction of a unique slice regular function.

5. Product of slice functions

In general, the pointwise product of two slice functions is not a slice function. However, pointwise product in the algebra $A_\mathbb{C}$ of A–stem functions induces a natural product on slice functions.

Definition 9. Let $f = \mathcal{I}(F), g = \mathcal{I}(G) \in \mathcal{S}(\Omega_D)$. The *product* of f and g is the slice function $f \cdot g := \mathcal{I}(FG) \in \mathcal{S}(\Omega_D)$.

The preceding definition is well–posed, since the pointwise product of F and G is complex intrinsic. The product is distributive and also associative if A is an associative algebra. If the components F_1, F_2 of the *first* stem function F are real–valued, then $(f \cdot g)(x) = f(x)g(x)$ for every $x \in \Omega_D$. In this case, we will use also the notation fg in place of $f \cdot g$.

Definition 10. A slice function $f = \mathcal{I}(F)$ is called *real* if the A–valued components F_1, F_2 of its stem function are real–valued. Equivalently, f is real if the spherical value $v_s f$ and the spherical derivative $\partial_s f$ are real–valued.

Real slice functions are characterized by the following property: for every $J \in \mathbb{S}_A$, the image $f(\mathbb{C}_J \cap \Omega_D)$ is contained in $\mathbb{C}_J$.

Let $f(x) = \sum_j x^j a_j$ and $g(x) = \sum_k x^k b_k$ be polynomials or convergent power series with coefficients $a_j, b_k \in A$. The usual product of polynomials, where x is considered to be a commuting variable (cf. for example [27] and [14, 13]), can be extended to power series (cf. [15, 19] for the quaternionic case) by setting:

$$(f * g)(x) := \sum_n x^n \big(\sum_{j+k=n} a_j b_k \big).$$

Proposition 5. *Let f and g be polynomials or convergent power series. Then the product $f \cdot g$ coincides with the star product $f * g$, i.e. $\mathcal{I}(FG) = \mathcal{I}(F) * \mathcal{I}(G)$.*

We now associate to every slice function the *normal function*, which is useful when dealing with zero sets. Our definition is equivalent to the *symmetrization* of quaternionic power series given in [15].

Definition 11. Let $f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D)$. Then also $F^c(z) := F(z)^c = F_1(z)^c + i F_2(z)^c$ is an A–stem function. We set $f^c := \mathcal{I}(F^c)$, $CN(F) := FF^c = n(F_1) - n(F_2) + i\, t(F_1 F_2{}^c)$ and $N(f) := f \cdot f^c = \mathcal{I}(CN(F)) \in \mathcal{S}(\Omega_D)$. The slice function $N(f)$ will be called the *normal function* of f.

If f is slice regular, then also f^c and $N(f)$ are slice regular. If A is the algebra of quaternions or octonions, then $CN(F)$ is complex–valued and then the normal function $N(f)$ is real. For a general algebra A, this is not true for every slice function. This is the motivation for the following definition.

Definition 12. A slice function $f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D)$ is called *admissible* if $v_s f(x) \in \mathcal{N}_A$ for every $x \in \Omega_D$ and the real vector subspace $\langle v_s f(x), \partial_s f(x) \rangle \subseteq \mathcal{N}_A$ for every $x \in \Omega_D \setminus \mathbb{R}$. Equivalently, $\langle F_1(z), F_2(z) \rangle \subseteq \mathcal{N}_A$ for every $z \in D$.

If $A = \mathbb{H}$ or $\mathbb{O}$, then *every* slice function is admissible. Moreover, if f is real, then $f^c = f$, $N(f) = f^2$ and f is admissible. If f is admissible, then $CN(F)$ is complex–valued and then $N(f)$ is real.

Example 3. Consider the Clifford algebra $\mathbb{R}_3$ with the usual conjugation. Its normal cone contains the subspace $\mathbb{R}^4$ of paravectors. Every polynomial $p(x) = \sum_n x^n a_n$ with paravectors coefficients is admissible. The polynomial $p(x) = xe_{23} + e_1$ is an example of a non–admissible slice regular function.

Theorem 6. *Let A be associative or $A = \mathbb{O}$. Then $N(f \cdot g) = N(f)\,N(g)$ for every admissible $f, g \in \mathcal{S}(\Omega_D)$.*

Corollary 7. *Let A be associative or $A = \mathbb{O}$. Assume that $n(x) = n(x^c) \neq 0$ for every $x \in A \setminus \{0\}$ such that $n(x)$ is real. If f and g are admissible slice functions, then also the product $f \cdot g$ is admissible.*

Example 4. Let $A = \mathbb{R}_3$. Consider the admissible polynomials $f(x) = xe_2 + e_1$, $g(x) = xe_3 + e_2$. Then $(f \cdot g)(x) = x^2 e_{23} + x(e_{13} - 1) + e_{12}$ is admissible, $N(f) = N(g) = x^2 + 1$ and $N(f \cdot g) = (x^2 + 1)^2$.

6. Zeros of slice functions

The zero set $V(f) = \{x \in \Omega_D \mid f(x) = 0\}$ of an admissible slice function f has a particular structure. For every fixed $x \in \mathcal{Q}_A$, the "sphere" $\mathbb{S}_x$ is entirely contained in $V(f)$ or it contains at most one zero of f. Moreover, if f is not real, there can be isolated, non–real zeros. These different types of zeros of a slice function correspond to the existence of zero–divisors in the complexified algebra $A_{\mathbb{C}}$.

Theorem 8 (Structure of $V(f)$). *Let $f \in \mathcal{S}(\Omega_D)$ an admissible slice function. Let $x = \alpha + \beta J \in \Omega_D$ and $z = \alpha + i\beta \in D$. Then one of the following mutually exclusive statements holds:*

  (1) $\mathbb{S}_x \cap V(f) = \emptyset$.
  (2) $\mathbb{S}_x \subseteq V(f)$. *In this case x is called a* real *(if $x \in \mathbb{R}$) or* spherical *(if $x \notin \mathbb{R}$) zero of f.*
  (3) $\mathbb{S}_x \cap V(f)$ *consists of a single, non–real point. In this case x is called an $\mathbb{S}_A$–isolated non–real zero of f.*

Moreover, a real slice function has no $\mathbb{S}_A$–isolated non–real zeros, and for every admissible slice function f, we have $V(N(f)) = \bigcup_{x \in V(f)} \mathbb{S}_x$.

Theorem 9. *Let Ω_D be connected. If f is slice regular and admissible on Ω_D, and $N(f)$ does not vanish identically, then $\mathbb{C}_J \cap \bigcup_{x \in V(f)} \mathbb{S}_x$ is closed and discrete in $D_J = \mathbb{C}_J \cap \Omega_D$ for every $J \in \mathbb{S}_A$. If $\Omega_D \cap \mathbb{R} \neq \emptyset$, then $N(f) \equiv 0$ if and only if $f \equiv 0$.*

In the quaternionic case, the structure theorem for the zero set of slice regular functions was proved by Pogorui and Shapiro [29] for polynomials and by Gentili and Stoppato [15] for power series.

Remark 2. If Ω_D does not intersect the real axis, a not identically zero slice regular function f can have normal function $N(f) \equiv 0$. For example, let $J \in \mathbb{S}_{\mathbb{H}}$ be fixed. The admissible slice regular function defined on $\mathbb{H} \setminus \mathbb{R}$ by $f(x) = 1 - IJ$ for $x \in \mathbb{C}_I^+ = \{x = \alpha + \beta I \in A \mid \beta \geq 0\}$ has zero normal function.

Now we state a remainder theorem, which generalizes a result proved by Beck [1] for quaternionic polynomials and by Serôdio [33] for octonionic polynomials.

Definition 13. For any $y \in \mathcal{Q}_A$, the *characteristic polynomial* of y is the slice regular function on $\mathcal{Q}_A$

$$\Delta_y(x) := N(x - y) = (x - y) \cdot (x - y^c) = x^2 - x\, t(y) + n(y).$$

Proposition 10. *The characteristic polynomial Δ_y of $y \in \mathcal{Q}_A$ is real. Two characteristic polynomials Δ_y and $\Delta_{y'}$ coincides if and only if $\mathbb{S}_y = \mathbb{S}_{y'}$. Moreover, $V(\Delta_y) = \mathbb{S}_y$.*

Theorem 11. *Let $f \in \mathcal{SR}(\Omega_D)$ be an admissible slice regular function. Let $y \in V(f) = \{x \in \mathcal{Q}_A \mid f(x) = 0\}$. Then the following statements hold.*

(1) *If y is a real zero, then there exists an admissible $g \in \mathcal{SR}(\Omega_D)$ such that $f(x) = (x - y)\, g(x)$.*

(2) *If $y \in \Omega_D \setminus \mathbb{R}$, then there exists an admissible $h \in \mathcal{SR}(\Omega_D)$ and $a, b \in A$ such that $\langle a, b \rangle \subseteq \mathcal{N}_A$ and $f(x) = \Delta_y(x)\, h(x) + xa + b$. Moreover,*
   - *y is a spherical zero of f if and only if $a = b = 0$.*
   - *y is an $\mathbb{S}_A$–isolated non–real zero of f if and only if $a \neq 0$ (in this case $y = -ba^{-1}$).*

If f is real, then g, h are real and $a = b = 0$.

For every non–real $y \in V(f)$, the element $a \in \mathcal{N}_A$ which appears in the statement of the preceding theorem is the spherical derivative of f at $x \in \mathbb{S}_y$.

Corollary 12. *Let $f \in \mathcal{SR}(\Omega_D)$ be admissible. If $\mathbb{S}_y$ contains at least one zero of f, of whatever type, then Δ_y divides $N(f)$.*

Definition 14. Let $f \in \mathcal{SR}(\Omega_D)$ be admissible, with $N(f) \not\equiv 0$. Given a non–negative integer s and an element y of $V(f)$, we say that y is a *zero of f of multiplicity s* if $\Delta_y^s \mid N(f)$ and $\Delta_y^{s+1} \nmid N(f)$. We will denote the integer s, called *multiplicity of y*, by $m_f(y)$.

In the case of y real, the preceding condition is equivalent to $(x - y)^s \mid f$ and $(x - y)^{s+1} \nmid f$. If y is a spherical zero, then $m_f(y) \geq 2$. In the case of quaternionic polynomials, the definition is equivalent to the one given in [2] and in [19].

Proposition 13. *Let A be associative. Let $f, g \in \mathcal{S}(\Omega_D)$. Then $V(f) \subseteq V(f \cdot g)$.*

As shown in [33] for octonionic polynomials and in [23] for octonionic power series, if A is *not* associative the statement of the proposition is no more true. However, we can still say something about the location of the zeros of $f \cdot g$ (cf. [23] for the precise relation linking the zeros of f and g to those of $f \cdot g$). For the associative case, see also [27, §16].

Now we focus our attention on the zero set of slice regular admissible *polynomials*. If $p(x) = \sum_{j=0}^m x^j a_j$ is an admissible polynomial of degree m with coefficients $a_j \in A$, then the normal polynomial

$$N(p)(x) = (p * p^c)(x) = \sum_n x^n \left(\sum_{j+k=n} a_j a_k^c \right)$$

has degree $2m$ and real coefficients. A sufficient condition for the admissibility of p is that the real vector subspace $\langle a_0, \ldots, a_m \rangle$ is contained in $\mathcal{N}_A$.

Theorem 14 (Fundamental Theorem of Algebra with multiplicities). *Let* $p(x) = \sum_{j=0}^{m} x^j a_j$ *be an admissible polynomial of degree* $m > 0$ *with coefficients in* A. *Then* $V(p) = \{y \in \mathcal{Q}_A \mid p(y) = 0\}$ *is non–empty. More precisely, there are distinct "spheres"* $\mathbb{S}_{x_1}, \ldots, \mathbb{S}_{x_t}$ *such that*

$$V(p) \subseteq \bigcup_{k=1}^{t} \mathbb{S}_{x_k} = V(N(p)), \quad V(p) \cap \mathbb{S}_{x_j} \neq \emptyset \quad \text{for every } j,$$

and, for any choice of zeros $y_1 \in \mathbb{S}_{x_1}, \ldots, y_t \in \mathbb{S}_{x_t}$ *of* p, *it holds* $\sum_{k=1}^{t} m_p(y_k) = m$.

Remark 3. If r denotes the number of real zeros of the polynomial p, i the number of $\mathbb{S}_A$–isolated non–real zeros of p and s the number of "spheres" $\mathbb{S}_y$ ($y \notin \mathbb{R}$) containing spherical zeros of p, we have that $r + i + 2s \leq \deg(p)$.

7. Examples

The following examples show the relevance of the quadratic cone and of admissibility for the algebraic and topological properties of the zero set of a polynomial.

(1) Every polynomial $\sum_{j=0}^{m} x^j a_j$, with paravector coefficients a_j in the Clifford algebra $\mathbb{R}_n$ (with conjugation as antiinvolution), has m roots (counted with their multiplicities) in the quadratic cone $\mathcal{Q}_n$. If the coefficients are real, then it has at least one root in the paravector space $\mathbb{R}^{n+1}$, since every "sphere" $\mathbb{S}_y$ intersect $\mathbb{R}^{n+1}$ (cf. [37, Theorem 3.1]).

(2) In $\mathbb{R}_3$, the admissible polynomial $p(x) = x^2 + xe_3 + e_2$ (cf. [37, Ex. 3]) has two isolated zeros

$$y_1 = \tfrac{1}{2}(1 - e_2 - e_3 + e_{23}), \ y_2 = \tfrac{1}{2}(-1 + e_2 - e_3 + e_{23})$$

in $\mathcal{Q}_3 \setminus \mathbb{R}^4$. They can be computed by solving the complex equation $CN(P) = z^4 + z^2 + 1 = 0$ to find the two "spheres" $\mathbb{S}_{y_1}, \mathbb{S}_{y_2}$ and then using the Remainder Theorem (Theorem 11) with $\Delta_{y_1} = x^2 - x + 1$ and $\Delta_{y_2} = x^2 + x + 1$.

(3) In $\mathbb{R}_3$, the polynomial $p(x) = xe_{23} + e_1$ vanishes only at $y = e_{123} \notin \mathcal{Q}_3$. Note that p is *not* admissible: $e_1, e_{23} \in \mathcal{N}_3$, but $e_1 + e_{23} \notin \mathcal{N}_3$.

(4) An admissible polynomial of degree m, even in the case of non–spherical zeros, can have more than m roots in the whole algebra. For example, $p(x) = x^2 - 1$ has *four* roots in $\mathbb{R}_3$, two in the quadratic cone ($x = \pm 1$) and two outside it ($x = \pm e_{123}$).

(5) Outside the quadratic cone, an admissible polynomial can have infinite non–spherical zeros. For example, in $\mathbb{R}_{1,2}$ the polynomial $p(x) = x^2 - 1$ has zeros ± 1 in $\mathcal{Q}_{1,2}$, while in $\mathbb{R}_{1,2} \setminus \mathcal{Q}_{1,2}$ it vanishes on the 4–dimensional set

$$\{x \mid x_0 = x_{123} = 0, \ x_2 x_{13} - x_1 x_{23} - x_3 x_{12} = 0, \ x_1^2 + x_{12}^2 + x_{13}^2 - x_2^2 - x_3^2 - x_{23}^2 = 1\}.$$

(6) In $\mathbb{R}_3$, the admissible polynomial $p(x) = x^3 - 1$ has zero set $V(p) = \{1\} \cup \mathbb{S}_y$ ($y = -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} J$, $J \in \mathbb{S}_{\mathbb{R}_3}$) in $\mathcal{Q}_3$, while in $\mathbb{R}_3 \setminus \mathcal{Q}_3$ the polynomial p vanishes on

the two 2–spheres

$$S_\pm = \{x = \tfrac{1}{4} + x_1(e_1 \pm e_{23}) + x_2(e_2 \pm e_{13}) + x_3(e_3 \pm e_{12}) \pm \tfrac{3}{4}e_{123} \mid x_1^2 + x_2^2 + x_3^2 = \tfrac{3}{16}\}.$$

(7) In the algebras $\mathbb{H}, \mathbb{O}, \mathbb{R}_3$, the solutions of the equation $x^2 = -1$ are exactly the elements of $\mathbb{S}_{\mathbb{R}_3}$, i.e. they all belong to the quadratic cone. This is not the case for other algebras. In $\mathbb{R}_{1,2}$, $e_{123}^2 = -1$, but $e_{123} \notin \mathcal{Q}_{1,2}$. In $\mathbb{R}_4$, the equation $x^2 = -1$ has many solutions: the square roots of -1 in the "sphere" $\mathbb{S}_{\mathbb{R}_4}$, but also infinite other points outside the quadratic cone:

$$x = \sum_{i=1}^{4} x_i e_i + x_{1234} e_{1234}, \text{ with } \sum_{i=1}^{4} x_i^2 - x_{1234}^2 = 1.$$

(8) The admissibility of a slice function depends on the algebra. For example, the polynomial $p(x) = x^2 + xe_1 + 2$ is admissible on the algebra $\mathbb{R}_3$ (w.r.t. conjugation), where its zero set is the union of $V(p) = \{e_1, -2e_1\} \subseteq \mathcal{Q}_3$ and the subset $\{-\tfrac{1}{2}e_1 \pm \tfrac{3}{2}e_{23}\}$ of $\mathbb{R}_3 \setminus \mathcal{Q}_3$. The same polynomial is *not* admissible on the algebra $\mathbb{R}_{1,2}$, and it has no zeros in $\mathcal{Q}_{1,2}$.

(9) The admissibility depends also on the antiinvolution chosen in the algebra. For example, in the algebra $\mathbb{R}_n$ with the reversion as fixed antiinvolution, a polynomial with paravector coefficient can be *not* admissible. For $n = 2$, $p(x) = xe_1 + 1$ is not admissible, since $\langle 1, e_1 \rangle \not\subseteq \mathcal{N}_2^*$. This property reflects in the fact that the unique zero of p in $\mathbb{R}_2$ does not belong to the quadratic cone $\mathcal{Q}_2^* = \{x \in \mathbb{R}_2 \mid x_1 = x_2 = 0\}$. The same holds in every $\mathbb{R}_n$. Instead, the polynomials with coefficients in the real vector subspace of 1–vectors are admissible in $\mathbb{R}_n$ (w.r.t. reversion).

(10) The reality of $N(p)$ is not sufficient to get the admissibility of p. For example, in $\mathbb{R}_3$, the polynomial $p(x) = x^2 e_{123} + x(e_1 + e_{23}) + 1$ has real normal function $N(p) = (x^2 + 1)^2$, but the spherical derivative $\partial_s p = t(x)e_{123} + e_1 + e_{23}$ has $N(\partial_s p)$ not real. In particular, $\partial_s p(J) = e_1 + e_{23} \notin \mathcal{N}_3$ for every $J \in \mathbb{S}_{\mathbb{R}_3}$. The non–admissibility of p is reflected by the existence of a S^1 of distinct zeros on $\mathbb{S}_{\mathbb{R}_3}$, where p does not vanish identically.

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