Tangential form of the trace condition for
pluriharmonic functions in \( \mathbb{C}^n \)

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Abstract

In this paper we introduce a tangential form of the trace condition
for pluriharmonic functions on a smooth bounded domain in \( \mathbb{C}^n \) that has
some convexity. The condition obtained, in the bidimensional case, is
related to the complex components of quaternionic regular functions. In
the case of the unit ball of \( \mathbb{C}^n \), we rewrite the trace condition in terms of
spherical harmonics.

1 Introduction

Let \( D \) be a smooth bounded domain in \( \mathbb{C}^n \). Pluriharmonic functions
on \( D \) with some regularity on \( \partial D \) are characterized by a trace condition
introduced by Fichera in the papers [4],[5],[6] and investigated in [11], [12]
and [1]. This condition has a global character: a real function on \( \partial D \) is the
trace of a pluriharmonic function on \( D \) if it is orthogonal, in the \( L^2(\partial D) \)-
norm, with respect to a suitably chosen space of functions. This approach
is alternative to the local study of tangential differential conditions (see
for example [13]§18.3, [2] and [6] and the references given there). This
second approach requires some geometrical conditions on the boundary
\( \partial D \), essentially the non-vanishing of the Levi-form at every point of \( \partial D \).

In this paper we search for a characterization in tangential terms of
the space of functions orthogonal to pluriharmonic functions. We show
that this interpretation is possible on domains having some global convex-
ity. In the bidimensional case, the convexity conditions can be weakened.

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Moreover, in $\mathbb{C}^2$ we obtain an interesting relation between the two characterizations of the orthogonal space and the complex components of a quaternionic regular function on $D$.

The main results of this paper are the following: there exist tangential Cauchy-Riemann operators $L_{jk}$ and a real, tangential, first-order differential operator $T$ on $\partial D$, which allow to rewrite the trace condition in the form described by Theorem 1 and Theorem 2.

**Theorem 1.** (a) Let $H^1(D, \mathbb{R}) = 0$ and $\partial D$ of class $C^1$. If $u \in Ph^\alpha_{\partial D}(D)$, $\alpha > 0$, then $u$ satisfies the condition
\[
\int_{\partial D} u \left( \mathbf{T} \cdot \omega \right) d\sigma = 0
\]
for every form $\omega \in C^1_{n-2,n}(\overline{D})$ such that $T(\omega) = 0$ on $\partial D$.

Here $\mathbf{T} \cdot \omega$ denotes the function $\sum_{j<k} (-1)^{j+k-1} T_{jk}(K_{jk})$, where $K_{jk}$ are the coefficients of the form $\omega$.

(b) Let $H^1(D, \mathbb{R}) = 0$ and $\partial D$ of class $C^2$. If $u \in Ph^0_{\partial D}(D)$, then $u$ satisfies $\mathbf{T} \cdot \omega$.

The same result holds if $\omega \in C^1_{n-2,n}(D)$ and $\partial \omega \in C^0_{n-1,n}(\overline{D})$.

**Theorem 2.** Let $D$ be a smooth bounded pseudoconvex domain or a domain that satisfies the Hörmander condition $Z(n-1)$, i.e., at every point $\zeta \in \partial D$ the Levi form of $\partial D$ has at least 1 positive eigenvalue.

If $u \in C^0(\partial D)$ is a real function that satisfies condition $\mathbf{T} \cdot \omega$, then $u \in Ph^0_{\partial D}(D)$.

We refer to §§2 and 3 for notations and for the exact formulation of the results. In §4 we consider in more detail the situation in $C^n$, where the assumption on the domain can be relaxed. In §5 we write the trace condition for the unit sphere of $\mathbb{C}^n$ in terms of spherical harmonics.

## 2 Notations and preliminaries

### 2.1

Let $D = \{ z \in \mathbb{C}^n : \rho(z) < 0 \}$ be a bounded domain in $\mathbb{C}^n$ with boundary of class $C^m$, $m \geq 1$. We assume $\rho \in C^m$ on $\mathbb{C}^n$ and $d\rho \neq 0$ on $\partial D$.

For every $\alpha, 0 \leq \alpha \leq m$, we denote by $Ph^\alpha(D)$ the space of real pluriharmonic functions of class $C^\alpha(D)$ and similarly for holomorphic functions $A^\alpha(D)$ and complex harmonic functions $Harm^\alpha(D)$. We denote by $Ph^0_{\partial D}(D)$ the space of restrictions to $\partial D$ of pluriharmonic functions in $Ph^0(D)$ and by $Re A^\alpha(D)$ the space of real parts of element of $A^\alpha(D)$.

### 2.2

Let $\nu$ denotes the outer unit normal to $\partial D$ and $\tau = i \nu$. For every $F \in C^1(\overline{D})$, we set $\overline{\partial}_\nu F = \frac{1}{2} \left( \frac{\partial F}{\partial \nu} + i \frac{\partial F}{\partial \tau} \right)$ (see [7]§§3.3 and 14.2).

Then in a neighbourhood of $\partial D$ we have the decomposition of $\overline{\partial} F$ in the tangential and the normal parts
\[
\overline{\partial} F = \overline{\partial}_t F + \overline{\partial}_n F \frac{\overline{\partial} \rho}{|\overline{\partial} \rho|}.
\]
The normal part of $\overline{\partial} F$ on $\partial D$ can also be expressed in the form $\overline{\partial}_n F = \sum_k \frac{\partial F}{\partial \overline{\zeta}_k} \frac{1}{|\partial \rho|} \partial \rho / \partial \zeta_k$ or by means of the Hodge $*$-operator and the Lebesgue surface measure $d\sigma$: $\overline{\partial}_n F d\sigma = * \overline{\partial} F_{\partial D}$.

### 2.3

We shall denote by $\text{Harm}_1^0(D)$ the real subspace of $\text{Harm}^1(D)$

$$\text{Harm}_1^0(D) = \{ H \in \text{Harm}^1(D) : \overline{\partial}_n H \text{ is real on } \partial D \}.$$  

The space of $C^1(D)$-holomorphic functions on $D$ is the maximal complex subspace in $\text{Harm}_1^0(D)$ contained in $\text{Harm}_1^0(D)$. This follows from a theorem of Kytmanov and Aizenberg [8] (cf. also [7] §§14 and 15): a $C^1(D)$-harmonic function $F$ is holomorphic on $D$ if and only if $\overline{\partial}_n F = 0$ on $\partial D$.

We recall the integral orthogonality condition that characterizes the traces on $\partial D$ of pluriharmonic functions on $D$. It was introduced by Fichera in [4] (see also [5],[6]) in a form not involving explicitly the normal component of $\overline{\partial}$. It was investigated in [11] and in [1]:

$$\int_{\partial D} U \overline{\partial}_n H d\sigma = 0 \quad (*)$$

for every $H \in \text{Harm}_1^0(D)$.

It was shown in [11] that this condition is necessary for pluriharmonicity when $\partial D$ is of class $C^1$ and $U \in C^\alpha(D)$, $\alpha > 0$, or when $\partial D$ is of class $C^2$ and $U \in C^0(D)$. The trace condition is sufficient in the case when $U \in C^{1+\lambda} \cap C^\infty(D)$. If the boundary value $u = U_{\partial D}$ is only continuous, the same result holds on bounded strongly pseudoconvex domains and on bounded weakly pseudoconvex domains with real analytic boundaries.

Recently, A. Cialdea improved these results in [1], using some facts from potential theory. He gave a more general theorem allowing data to be in $L^p(\partial D)$. Note that the proof of this result given in [1] (cf. §23) shows that in condition $(*)$ it is sufficient to consider functions $H \in \text{Harm}_1^0(D) \cap C^\infty(D)$ to obtain the pluriharmonicity of the harmonic extension $U$ of $u \in L^p(\partial D)$.

We shall denote by $\text{Harm}_1^\infty(D)$ the space $\text{Harm}_1^0(D) \cap C^\infty(D)$.

**Remark 1.** When $n = 1$ and $H^1(D, \mathbb{R}) = 0$ condition $(*)$ is void, since $H$ belongs to $\text{Harm}_1^0(D)$ if and only if $H$ is holomorphic on $D$ (cf. [11]).

### 2.4

The space $\text{Harm}_1^0(D)$ can be characterized in terms of Bochner-Martinelli operator $M$. In [11]§4 it was shown that $F \in \text{Harm}_1^0(D)$ if and only if $\text{Im } M(F) = \text{Im } F$ in $D$.

Let $B$ be the unit ball of $\mathbb{C}^n$ and let $S = \partial B$. The space $L^2(S)$ is the direct sum of pairwise orthogonal spaces $H(s,t), s \geq 0, t \geq 0$, where $H(s,t)$ is the space of harmonic homogeneous polynomials of total degree $s$ in $z$ and total degree $t$ in $\overline{z}$ (see [13] §12). These spaces are the eigenspaces of the Bochner-Martinelli operator.
As it was shown in [11]§5, in this case the trace condition for pluriharmonic functions reduces to the orthogonality to the spaces \( H(s, t) \), \( s, t > 0 \), a theorem proved by Nagel and Rudin in [10]. Other results on this line have been given by Dzhuraev [3].

2.5

We recall the definition of tangential Cauchy-Riemann operators (see for example [13] §18). A linear first-order differential operator \( L \) is tangential to \( \partial D \) if \( (L\rho)(\zeta) = 0 \) for each point \( \zeta \in \partial D \). A tangential operator of the form

\[
L = \sum_{j=1}^{n} a_j \frac{\partial}{\partial \bar{z}_j}
\]

is called a tangential Cauchy-Riemann operator. The operators

\[
L_{jk} = \frac{1}{|\partial \rho|} \left( \frac{\partial \rho}{\partial \bar{z}_k} \frac{\partial}{\partial \bar{z}_j} - \frac{\partial \rho}{\partial \bar{z}_j} \frac{\partial}{\partial \bar{z}_k} \right), \quad 1 \leq j < k \leq n,
\]

are tangential and the corresponding vectors at \( \zeta \in \partial D \) span (not independently when \( n > 2 \)) the (conjugate) complex tangent space to \( \partial D \) at \( \zeta \). Then a function \( f \in C^1(\partial D) \) is a CR function if and only if \( L_{jk}(f) = 0 \) on \( \partial D \) for every \( j, k \), or, equivalently, if \( L(f) = 0 \) for each tangential Cauchy-Riemann operator \( L \).

3 The trace condition in tangential form

3.1

Let \( n > 1 \) and let \( H \) be a harmonic function on \( D \). The form \( \overline{\partial} H \) is a \( \partial \)-closed \((n-1, n)\)-form on \( D \), since \( \overline{\partial}(\overline{\partial} H) = -\overline{\partial} H = -\Box H = 2\Delta H = 0 \), where \( \overline{\partial} \) is the formal \( L^2(D) \)-adjoint operator to \( \partial \).

Let \( c_n \) be the constant such that \( \overline{\partial} H = c_n \sum_{k=1}^{n-1} (-1)^{k-1} \frac{\partial H}{\partial \bar{z}_k} dz[k] \wedge d\bar{z} \). Let \( \omega \) be a \((n-2, n)\)-form with \( C^1(\overline{\partial}D) \)-coefficients. We set \( \omega = c_n \sum_{j<k} K_{jk} dz[j][k] \wedge d\bar{z} \), where \( dz[j][k] \) denotes the product of the forms \( dz_1, \ldots, dz_n \) with \( dz[j] \) and \( dz[k] \) deleted, and \( d\bar{z} = d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n \).

An easy computation shows that

\[
\partial \omega = c_n \sum_{k=1}^{n} \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial K_{ih}}{\partial \bar{z}_i} dz[h] \wedge d\bar{z}.
\]

Here we extend by antisymmetry: \( K_{ih} = -K_{hi} \) when \( i > h \), \( K_{ii} = 0 \).

**Proposition 1.** Let \( H \in \text{Harm}^1(D) \) and the form \( \omega \in C^{1}_{n-2, n}(\overline{\partial}D) \) be as above. Let \( L \) be the tangential Cauchy-Riemann operator defined by \( L = \sum_{j<k} (-1)^{j+k} L_{jk} \). We shall denote by \( \overline{\partial} \circ \omega \) the operation of contraction

\[
\overline{\partial} \circ \omega = \sum_{j<k} (-1)^{j+k} \overline{\partial} L_{jk}(K_{jk}).
\]
Then \( \partial \omega|_{\partial D} = (\bar{T} \ast \omega) d\sigma \) on \( \partial D \). Moreover, if \( \partial \omega = \ast \bar{\partial} H \) on \( D \), then, for every \( \epsilon < 0 \) sufficiently small, the equality

\[
\bar{\partial}_\epsilon H = \bar{T} \ast \omega
\]

holds on \( \partial D_\epsilon = \{ z \in D : \rho(z) = \epsilon \} \).

**Proof.** On \( \partial D \) we have

\[
(\bar{T} \ast \omega) d\sigma = \sum_{j<k} (-1)^{j+k} \frac{1}{|\partial \rho|} \left( \frac{\partial \rho}{\partial z_k} \frac{\partial K_{jk}}{\partial z_j} - \frac{\partial \rho}{\partial z_j} \frac{\partial K_{jk}}{\partial z_k} \right) d\sigma
\]

\[
= \sum_{k=1}^n \frac{\partial \rho}{\partial z_k} \frac{1}{|\partial \rho|} (-1)^k \sum_{j=1}^n (-1)^j \frac{\partial K_{jk}}{\partial z_j} d\sigma
\]

\[
= c_n \sum_{k=1}^n \sum_{j=1}^n (-1)^{j-1} \frac{\partial K_{jk}}{\partial z_j} d\bar{z} \wedge dz_k,
\]

since \( \frac{\partial \rho}{\partial z_k} (-1)^k d\sigma \) equals the restriction of the form \(-c_n d\bar{z} \wedge dz_k\) on \( \partial D \) (see for example Lemma 3.5 in [7]). Then \( (\bar{T} \ast \omega) d\sigma = \partial \omega|_{\partial D} \).

If \( \partial \omega = \ast \bar{\partial} H \) on \( D \), the last assertion of the lemma follows from the first part applied on \( \partial D_\epsilon \), since \( \bar{\partial}_\epsilon H d\sigma = \ast \bar{\partial} H|_{\partial D_\epsilon} \).

**Remark 2.** (i) In the formula defining \( \bar{T} \ast \omega \) we used the generators \( L_{jk} \), even if they are not independent for \( n > 2 \) (since \( \frac{\partial \rho}{\partial z_k} L_{kj} + \frac{\partial \rho}{\partial z_j} L_{kj} + \frac{\partial \rho}{\partial z_i} L_{ij} = 0 \) on \( \partial D \)). This choice allows to obtain expressions that are more symmetric with respect to the coordinates.

(ii) If \( \ast \bar{\partial} H = \partial \omega \) on \( D \), the \((n-2,n)\)-form \( \omega \) satisfies the condition \( \partial^* \partial \omega = 0 \) on \( D \), since \( \bar{\partial} (\ast \partial \omega) = -\bar{\partial} \partial H = 0 \). When \( n = 2 \), the preceding condition is equivalent to \( \Delta \omega = 0 \). If \( n > 2 \), \( \omega \) is harmonic exactly when also the form \( \partial \partial^* \omega \) vanishes.

(iii) The equality \( \ast \bar{\partial} H = \partial \omega \) is equivalent to the following system of differential equations on \( D \):

\[
\frac{\partial H}{\partial z_k} = (-1)^k \sum_{j=1}^n (-1)^j \frac{\partial K_{jk}}{\partial z_j} \quad \text{for every } k = 1, \ldots, n.
\]

### 3.2

Let \( T \) be the real, tangential, first-order differential operator defined on forms \( \omega \in C^1_{n-2,n}(D) \) by

\[
T(\omega) = \text{Im}(\bar{T} \ast \omega) = (2i)^{-1} \sum_{j<k} (-1)^{j+k} (\bar{T}_{jk}(K_{jk}) - L_{jk}(K_{jk})) \quad \text{on } \partial D.
\]

Note that \( T(\omega) \) depends only on the restriction of the coefficients of \( \omega \) on \( \partial D \). We can now show that the trace condition for pluriharmonic functions can be reformulated in tangential terms.
Theorem 1. (a) Let $H^1(D, \mathbb{R}) = 0$ and $\partial D$ of class $C^1$. If $u \in Ph^\alpha_{\partial D}(D)$, $\alpha > 0$, then $u$ satisfies the condition

$$\int_{\partial D} u \left( \mathcal{T} \ast \omega \right) d\sigma = 0 \quad (\sharp)$$

for every form $\omega \in C^1_{n-2,n}(\overline{D})$ such that $T(\omega) = 0$ on $\partial D$.

(b) Let $H^1(D, \mathbb{R}) = 0$ and $\partial D$ of class $C^2$. If $u \in Ph^0_{\partial D}(D)$, then $u$ satisfies (\sharp).

The same result holds if $\omega \in C^1_{n-2,n}(D)$ and $\partial \omega \in C^0_{n-1,n}(\overline{D})$.

Proof. (a) Let $U \in Ph^\alpha(D)$ be the extension of $u$ and $F = U + iV \in A^n(D)$. Then from Proposition 1 we get

$$\int_{\partial D} F \left( \mathcal{T} \ast \omega \right) d\sigma = \int_{\partial D} F \partial \omega,$$

which vanishes since $F$ is a CR function on $\partial D$. The real part of the first integral is the left hand side of (\sharp).

(b) Let $F = U + iV$ be as above. We can proceed exactly as in the proof of the second part of Theorem 1 in [11], using Stout’s estimate (see [14]) to get that $F$ belongs to the Hardy space $H^2(D)$.

Remark 3. The hypothesis of regularity of the domain $D$ can be weakened using the techniques introduced in [1].

3.3 Here we show that condition (\sharp) characterizes the traces of pluriharmonic functions if the domain $D$ satisfies a global convexity condition.

Let the boundary $\partial D$ be at least of class $C^{1+\lambda}$, $\lambda > 0$. Assume that for every function $H \in Harm^\infty_0(D)$, the $\partial$-closed $(n-1,n)$-form $\ast \partial H$ is $\partial$-exact with regularity at the boundary: there exists a $(n-2,n)$-form $\omega$, with at least $C^1$ coefficients on $\overline{D}$, such that $\ast \partial H = \partial \omega$. This condition is satisfied for example in the following cases:

(i) $D$ is a smooth bounded pseudoconvex domain.

(ii) $D$ is a smooth bounded domain that satisfies the Hörmander condition $Z(n-1)$, i.e., at every point $\zeta \in \partial D$ the Levi form of $\partial D$ has at least 1 positive eigenvalue (see [9]).

Theorem 2. Let $D$ be a domain that satisfies the condition above. If $u \in C^0(\partial D)$ is a real function that satisfies condition (\sharp), then $u \in Ph^0_{\partial D}(D)$.

Proof. Let $H \in Harm^\infty_0(D)$ and let $\omega$ be a $C^1(\overline{D})$-form such that $\partial \omega = \ast \partial H$. From Proposition 1, we get that the equality $\square_n H = \mathcal{T} \ast \omega$ is valid on $\partial D$, for any small $\epsilon < 0$ and then, by continuity, also on $\partial D$. Moreover, $H$ belongs to the space $Harm^\infty_0(D)$ exactly when $T(\omega) = 0$. Then condition (\sharp) is equivalent to

$$\int_{\partial D} u \square_n H d\sigma = 0$$

for every $H \in Harm^\infty_0(D)$. 


If $u \in C^{1+\lambda}(\partial D)$, then the assertion of the theorem follows from Theorem 2 in [11]. If $u$ is only continuous on $\partial D$, then it follows from the results obtained in [1].

**Remark 4.** In view of what has been said in §3.1, the condition (♯) needs to be satisfied only for every $(n-2,n)$-form $\omega$ such that $\partial^* \partial \omega = 0$ on $D$, $T(\omega) = 0$ on $\partial D$.

## 4 The bidimensional case

When $n = 2$, the system (1) of differential equations of §3.1 arising from the equality $\partial \omega = \ast \partial H$ reduces to the equations

\[
\begin{cases}
\frac{\partial H}{\partial z_1} = \frac{\partial K}{\partial z_2} \\
\frac{\partial H}{\partial z_2} = -\frac{\partial K}{\partial z_1}
\end{cases}
\]

where $K = K_{12}$ is the unique coefficient of the $(0,2)$-form $\omega = c_2 K d\bar{z}$.

The preceding equations imply that when $H \in\text{Harm}^1(D)$, also $K$ is harmonic on $D$ and the derivatives $\frac{\partial K}{\partial z_j}$ are continuous on $\overline{D}$ for $j = 1, 2$.

From this property it follows that the function

\[
L \ast \omega = -L_{12}(K) = 1 \left| \frac{\partial \rho}{\partial \rho} \right| \left( \frac{\partial \rho}{\partial z_1} \frac{\partial K}{\partial z_2} - \frac{\partial \rho}{\partial z_2} \frac{\partial K}{\partial z_1} \right)
\]

is continuous on the intersection of $\overline{D}$ with a neighborhood of $\partial D$. The tangent vector obtained from $L_{12}$ at $\zeta \in \partial D$ is a basis of the complex tangent space to $\partial D$ at $\zeta$.

We now show that, in the bidimensional case, we can obtain the result stated in Theorem 2 under a weaker convexity assumption on $D$.

**Theorem 3.** Let $n = 2$. Let $\partial D$ be of class $C^{1+\lambda}$. Assume that the Dolbeault cohomology group $H^1_{\lambda,0}(D)$ vanishes. If $u \in C^0(\partial D)$ is a real function that satisfies the condition

\[
\int_{\partial D} u \, L_{12}(K) d\sigma = 0
\]

for every complex harmonic function $K$ on $D$, such that $\frac{\partial K}{\partial z_j} \in C^0(\overline{D})$ ($j = 1, 2$) and with $L_{12}(K)$ real on $\partial D$, then $u \in Ph^0_{\partial D}(D)$. If $H^1(D, \mathbb{R}) = 0$ and $\partial D$ is of class $C^2$, the condition stated above is also necessary in order to have $u \in Ph^0_{\partial D}(D)$.

**Proof.** Let $H \in\text{Harm}^0(D)$. Since $H^1_{\lambda,0}(D) = 0$, we can find $\omega = c_2 K d\bar{z} \in C^\infty_{0,2}(D)$ such that $\partial^* \partial \omega = \ast \partial H$ (the form $d\bar{z}$ can always be factored out). The discussion above says that $L_{12}(K)$ is continuous on a domain $\{z \in D : \epsilon_0 < \rho(z)\}$, and then the equality $\overline{\partial}_u H = -L_{12}(K)$ is valid on $\partial D$, also for $\epsilon = 0$. Then

\[
\int_{\partial D} u \, \overline{\partial}_u H d\sigma = -\int_{\partial D} u \, L_{12}(K) d\sigma = 0
\]
since \( T_{12}(K) \) is real on \( \partial D \). Then condition (\*) is satisfied by \( u \), and \( u \in \text{Ph}^0_0(D) \).

Conversely, let \( U \in \text{Ph}^0(D) \) be the extension of \( u \). Then \( U = \text{Re} F \), with \( F = U + iV \in \mathcal{H}^2(D) \). The integral \( \int_{\partial D_\epsilon} \overline{T_{12}(K)} d\sigma = - \int_{\partial D_\epsilon} T \partial \omega \) vanishes for every \( \epsilon < 0 \) sufficiently small. Then \( \int_{\partial D_\epsilon} \text{Re}(T_{12}(K)) d\sigma + \int_{\partial D_\epsilon} \text{Im}(T_{12}(K)) d\sigma = 0 \). The first integral tends to \( \int_{\partial D} u \overline{T_{12}(K)} d\sigma \) as \( \epsilon \) tends to 0, while the second has limit \( \int_{\partial D} \text{Im}(T_{12}(K)) d\sigma = 0 \).

\[ \text{Remark 5.} \] The functions \( H \) and \( K \) related by the system (2), can be considered as the two complex components of a (left-)regular function of one quaternionic variable. We refer to [15] for the basic facts of quaternionic analysis. We identify the algebra \( \mathbb{H} \) of quaternions with the space \( \mathbb{C}^2 \) by means of the map that associates to the pair \((z_1, z_2) = (x_1 + iy_1, x_2 + iy_2)\) the quaternion \( q = z_1 + jz_2 = x_1 + iy_1 + jx_2 - ky_2 \), where \( i, j, k \) denote the basic quaternions. The equations (2) then become the Cauchy-Riemann-Fueter system for the quaternionic-valued function \( H + jK \) of one quaternionic variable \( q \).

\section{5 The case of the unit ball of \( \mathbb{C}^n \)}

\subsection{5.1}

If the domain is the unit ball \( B \) of \( \mathbb{C}^n \), the trace condition (2) has a more explicit form in terms of spherical harmonics. Let \( N_0 \) be the real linear projection in \( L^2(S) \) introduced in [11]. It is defined for \( P_{s,t} \in H(s, t) \) by

\[
N_0(P_{s,t}) = \begin{cases} \frac{s}{s+t} P_{s,t} + \frac{t}{s+t} \overline{P_{s,t}}, & \text{for } t > 0 \\ P_{s,t}, & \text{for } t = 0. \end{cases}
\]

The space \( \text{Harm}^0_0(B) \) coincides with \( \text{Fix}(N_0) = \{ H \in \text{Harm}^1(B) : N_0(H) = H \} \).

\subsection{5.2}

We first consider the bidimensional case. Let \( H, K \in \text{Harm}^1(B) \) be a pair of functions that solve the system (2). Then the equality \( \partial_n H = -T_{12}(K) \) holds on the unit sphere \( S \). The system is satisfied also by the pair \( -R, \overline{R} \). Then also the equality \( \partial_n K = L_{12}(H) \) is valid on \( S \). This fact allows to obtain the projection \( N_1 \) on the space of the functions \( K \in \text{Harm}^1(B) \) such that \( T_{12}(K) \) is real on \( S \). From the equation \( \partial_n K = L_{12}(N_0(P_{s,t})) \) for \( P_{s,t} \in H(s, t) \), we get that \( N_1 \) can be defined as follows:

\[
N_1(P_{s,t}) = \frac{s}{(s+1)(s+t)} L_{12}(P_{s,t}) + \frac{t}{(t+1)(s+t)} L_{12}(\overline{P_{s,t}}),
\]

where \( L_{12}(F) = z_1^2 \frac{\partial F}{\partial z_1} - z_1 \frac{\partial F}{\partial z_2} \) on \( S \). Note that \( N_1(P_{s,t}) = 0 \) when \( s = 0 \) or \( t = 0 \).
Remark 6. Let $U_{s+t}$ be the space of polynomials in the quaternionic variable $q$ that are regular and homogeneous of order $s + t$ over $\mathbb{R}$. In view of what has been said in §4, for every $P_{s,t} \in H(s, t)$, the function $R(P_{s,t}) = N_0(P_{s,t}) + jN_1(P_{s,t})$ belongs to $U_{s+t}$.

As a consequence of Theorems 1,2 and 3, we get the following result.

**Corollary 1.** Let $B$ be the unit ball of $\mathbb{C}^2$ and let $S$ be the unit sphere. A function $u \in C^0(S)$ has a pluriharmonic extension to $B$ if and only if

$$
\int_S u \, T_{12}(N_1(P_{s,t})) \, d\sigma = 0
$$

for every $s,t > 0$ and for every $P_{s,t} \in H(s,t)$.

5.3

Now we come to the general case $n \geq 2$, using the bidimensional case as a guide. An easy computation shows that the $(n-2, n)$-form

$$
\omega = -c_n \frac{1}{n+s-1} \sum_{j<k} (-1)^{j+k} L_{jk}(P_{s,t}) \, dz[j,k] \wedge d\bar{z}
$$

satisfies the equation $\partial \omega = \ast \partial P_{s,t}$. Then we can define, for any $s,t \geq 0$,

$$
\omega_{s,t} = -c_n \sum_{j<k} (-1)^{j+k} \left( \frac{sL_{jk}(P_{s,t})}{(n+s-1)(s+t)} + \frac{tL_{jk}(P_{s,t})}{(n+t-1)(s+t)} \right) \, dz[j,k] \wedge d\bar{z}.
$$

The form $\omega_{s,t}$ vanishes when $s = 0$ or $t = 0$, and the equality $\partial \omega_{s,t} = \ast \partial N_0(P_{s,t})$ holds on $\mathbb{C}^n$. From Proposition 1, we get that $\nabla^{\sharp} \omega_{s,t} = \nabla N_0(P_{s,t})$ and $T(\omega_{s,t}) = 0$ on $S$.

As a consequence of Theorems 1 and 2, we get the following result, that reduces to Corollary 1 in the case $n = 2$.

**Corollary 2.** Let $B$ be the unit ball of $\mathbb{C}^n$ and let $S$ be the unit sphere. A function $u \in C^0(S)$ has a pluriharmonic extension to $B$ if and only if

$$
\int_S u \, (\nabla^{\sharp} \omega_{s,t}) \, d\sigma = 0
$$

for every $\omega_{s,t}$ ($s,t > 0$).
References


