DIRICHLET PROBLEM FOR PLURIHARMONIC
FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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Abstract. In this paper we consider the Dirichlet problem for real pluriharmonic functions on a smooth, bounded domain in $\mathbb{C}^n$. We start from a result announced by Fichera some years ago, and relate it to the properties of the normal component on $\partial D$ of the $\overline{\partial}$-operator. We prove, for some classes of domains, that an orthogonality condition, in the $L^2(\partial D)$-norm, with respect to a suitably chosen space of functions is necessary and sufficient for pluriharmonicity of the harmonic extension of the boundary value.

1. Introduction

Let $D$ be a smoothly bounded domain in $\mathbb{C}^n$. In this paper we investigate the Dirichlet problem for real valued pluriharmonic functions on $D$. Given a real, continuous function $u$ on $\partial D$, we examine necessary and sufficient conditions for the pluriharmonicity of the harmonic extension $U$ in $D$ of the boundary value $u$. There are two different approaches to this problem. The local approach, in which tangential differential conditions on $u$ are searched, has been pursued by many authors on domains whose Levi form has at least one positive eigenvalue at every boundary point (see for example [R]§18.3 and [F3] and the references given there). The global approach consists in giving integral conditions, which impose orthogonality of $u$, in the $L^2(\partial D)$-norm, with respect to a suitably chosen space of functions. When $D$ is the unit ball of $\mathbb{C}^n$, a solution to the Dirichlet problem for pluriharmonic functions is contained in the results of Nagel and Rudin [NR] (see also §13 in the book by Rudin [R]). Other results on this line have been given by Dzhuraev [D].

Our starting point is the following result announced by Fichera in the papers [F1],[F2],[F3]. Let $D$ be a simply connected domain, with boundary of class $C^{1+\lambda}$, $\lambda > 0$. Let $\nu$ be the outer unit normal to $\partial D$ and $\tau = i\nu$.

Let $A, B$ be a pair of real $C^1(\overline{D})$ functions, harmonic on $D$, such that

$$\frac{\partial A}{\partial \tau} + \frac{\partial B}{\partial \nu} = 0$$

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on $\partial D$. Given $u \in C^0(\partial D)$, there exists one (and only one) pluriharmonic function $U \in C^0(D)$ such that $U|_{\partial D} = u$ if and only if

$$\int_{\partial D} u \left( \frac{\partial A}{\partial \nu} - \frac{\partial B}{\partial \tau} \right) \, d\sigma = 0$$

for any pair $A, B$.

In this paper, after rewriting the above integral condition in terms of the complex normal component of the $\bar{\partial}$-operator, we show (Theorem 1) that this condition is necessary for pluriharmonicity when $\partial D$ is of class $C^1$ and $u \in C^\alpha(\partial D)$, $\alpha > 0$, or when $\partial D$ is of class $C^2$ and $u \in C^0(\partial D)$. We then prove (Theorem 2) sufficiency of the condition in the case when $u \in C^{1+\lambda}, \lambda > 0$. If the boundary value $u$ is only continuous, we are able to get the result on strongly pseudoconvex domains and on weakly pseudoconvex domains with real analytic boundary (Theorem 3). We give then in §4 a characterization of the pairs $A, B$ appearing in the orthogonality condition in terms of the Bochner-Martinelli integral operator. In the case of the ball, this result can be further precised and allows to obtain another proof of Nagel-Rudin theorem.

2. Notations and preliminaries

2.1. Let $D = \{ z \in \mathbb{C}^n : \rho(z) < 0 \}$ be a bounded domain in $\mathbb{C}^n$ with boundary of class $C^m, m \geq 1$. We assume $\rho \in C^m$ on $\mathbb{C}^n$ and $d\rho \neq 0$ on $\partial D$.

We denote by $Ph(D), A(D)$ and $Harm(D)$ respectively the spaces of real pluriharmonic, holomorphic and complex harmonic functions on $D$. For every $\alpha, 0 \leq \alpha \leq m$, $Ph^\alpha(D)$ is the space $Ph(D) \cap C^\alpha(\overline{D})$ and similarly for $A^\alpha(D)$ and $Harm^\alpha(D)$. We denote by $Ph^\alpha_{\partial D}(D)$ the space of restrictions to $\partial D$ of pluriharmonic functions in $Ph^\alpha(D)$ and by $ReA^\alpha_{\partial D}(D)$ the restrictions of real parts of functions in $A^\alpha(D)$. Finally, $ReA^\alpha(D)$ is the space of real parts of element of $A^\alpha(D)$.

Remark. If $D$ is a simply connected domain or, more precisely, $H^1(D, \mathbb{R}) = 0$, then $Ph^\alpha(D) = ReA^\alpha(D)$ for every $\alpha \geq 1$, since Cauchy-Riemann equations imply that the harmonic conjugate of a function $U \in Ph^\alpha(D)$ belongs to $C^\alpha(\overline{D})$. In general, this fact is no longer true when $\alpha = 0$, but it remains valid for $0 < \alpha < 1$. In fact, the first derivatives of $U \in Ph^\alpha(D)$ satisfy a Hardy-Littlewood estimate (see [L] §2 for example) and then the same holds for the harmonic conjugate.

2.2. For every $F \in C^1(\overline{D})$, in a neighbourhood of $\partial D$ we have the decomposition of $\partial F$ in the tangential and the normal parts

$$\partial F = \partial_b F + \partial_n F \frac{\partial \rho}{|\partial \rho|}$$

where $|\partial \rho| = \left( \sum_k \left| \frac{\partial \rho}{\partial \zeta_k} \right| \right)^{1/2}$ and $\partial_n F = \sum_k \frac{\partial F}{\partial \zeta_k} \frac{\partial \rho}{\partial \zeta_k} \frac{1}{|\partial \rho|}$. 

If \( \nu \) denotes the outer unit normal to \( \partial D \) and \( \tau = i\nu \), then we can also write \( \overline{\partial}_n F = \frac{1}{2} \left( \frac{\partial F}{\partial \nu} + i \frac{\partial F}{\partial \tau} \right) \). The normal part of \( \overline{\partial} F \) on \( \partial D \) can also be expressed by means of the Hodge *-operator and the Lebesgue surface measure \( d\sigma \): \( \overline{\partial}_n F d\sigma = \star \overline{\partial} F|_{\partial D} \) (see [K]§§3.3 and 14.2). Note that \( \overline{\partial} \)-Neumann problem for functions \( \square F = \psi \) in \( D \), \( \overline{\partial}_n F = 0 \) on \( \partial D \)
is equivalent, at least in the smooth case, to the problem \( \overline{\partial}_n F = \phi \) on \( \partial D \) (see [K]§14). The compatibility condition for this problem becomes

\[ \int_{\partial D} \phi d\sigma = 0 \]

for every \( f \) holomorphic in a neighbourhood of \( D \).

2.3. Now assume that \( A \) and \( B \) are real harmonic functions on \( D \), of class \( C^1 \) on \( \overline{D} \). Let \( H = A + iB \). Then \( \overline{\partial}_n H \) is real on \( \partial D \) if and only if \( \frac{\partial A}{\partial \tau} + \frac{\partial B}{\partial \nu} = 0 \) on \( \partial D \). If this is the case, \( \overline{\partial}_n H = \frac{1}{2} \left( \frac{\partial A}{\partial \nu} - \frac{\partial B}{\partial \tau} \right) \) on \( \partial D \) and Fichera’s condition on \( u \in C^0(\partial D) \) becomes the following condition

\[
(*) \quad \int_{\partial D} u \overline{\partial}_n H d\sigma = 0
\]

for every complex harmonic function \( H \in C^1(\overline{D}) \) such that \( \overline{\partial}_n H \) is real on \( \partial D \).

Note that the integral above can be rewritten as \( \int_{\partial D} u \star \overline{\partial} H \), with \( \star \overline{\partial} H \) a closed \((n-1,n)\)-form on \( D \), since \( \star \overline{\partial} \star \overline{\partial} H = -\overline{\partial} \overline{\partial} H = -\Box H = 2\Delta H = 0 \).

We shall denote by \( \text{Harm}^1(D) \) the real subspace of \( \text{Harm}^1(D) \)

\[ \text{Harm}^1_0(D) = \{ H \in \text{Harm}^1(D) : \overline{\partial}_n H \text{ is real on } \partial D \} \]

**Remark.** When \( n = 1 \) and \( H^1(D,\mathbb{R}) = 0 \) condition (*) is void: if \( H = A + iB \) as above, let \( B' \) be a harmonic conjugate of \( A \). Then \( B' \in C^1(\overline{D}) \) from Cauchy-Riemann equations and \( F = A + iB' \) is holomorphic on \( D \). This implies that \( \frac{\partial A}{\partial \tau} + \frac{\partial B'}{\partial \nu} = 0 \) on \( \partial D \) and so \( \frac{\partial B}{\partial \nu} = \frac{\partial B'}{\partial \nu} \) on \( \partial D \). Then \( B - B' \) is a constant and so \( \overline{\partial}_n H = \overline{\partial}_n F = 0 \) on \( \partial D \).

3. Condition (*) and pluriharmonic functions

3.1. In this section we examine the validity of condition (*) for boundary values of pluriharmonic functions.

**Theorem 1.** (a) Let \( H^1(D,\mathbb{R}) = 0 \) and \( \partial D \) of class \( C^1 \). If \( u \in \text{Ph}^\alpha_{\overline{\partial}D}(D) \), \( \alpha > 0 \), then \( u \) satisfies condition (*);

(b) Let \( H^1(D,\mathbb{R}) = 0 \) and \( \partial D \) of class \( C^2 \). If \( u \in \text{Ph}^0_{\overline{\partial}D}(D) \), then \( u \) satisfies condition (*).
Theorem 1 when the boundary value $u \star (b)$ shows that $(\ast)$ implies pluriharmonicity without any topological assumption on $D$.

Proof. (a) Let $U \in Ph^0(D)$ be the extension of $u$ and $F = U + iV \in A^0(D)$. Then

$$\int_{\partial D} F \bar{\partial}_n H d\sigma = \int_{\partial D} F \ast \bar{\partial} H = \int_D \partial (F \ast \bar{\partial} H) = \int_D F \partial (\ast \bar{\partial} H) = 0$$

The real part of the first integral is $\int_{\partial D} u \bar{\partial}_n H d\sigma$.

(b) Let $U \in Ph^0(D)$ be the extension of $u$. Then $U = \text{Re} F$, with $F = U + iV$ holomorphic on $D$. From Stout’s estimate (see [S]) it follows that $F$ belongs to the Hardy space $\mathcal{H}^2(D)$. Then $F$ has boundary value $f \in L^2(\partial D)$. Let $D_\epsilon = \{ z : \rho(z) < \epsilon \}$ for small $\epsilon < 0$. We get as before that the integral $\int_{\partial D_\epsilon} F \bar{\partial}_n H d\sigma$ vanishes. Then $\int_{\partial D_\epsilon} U \text{Re}(\bar{\partial}_n H) d\sigma + \int_{\partial D_\epsilon} V \text{Im}(\bar{\partial}_n H) d\sigma = 0$. The first integral tends to $\int_{\partial D} u \bar{\partial}_n H d\sigma$ as $\epsilon$ tends to 0, while the second has limit $\int_{\partial D} V \text{Im}(\bar{\partial}_n H) d\sigma = 0$.

Remarks. (i) The proof of (a) shows that condition $(\ast)$ is satisfied by any $u \in \text{Re} A^0_{\partial D}(D)$ even without the topological restriction on $D$;
(ii) If $\text{Re} A^0_{\partial D}(D)$ is dense in $Ph^0_{\partial D}(D)$ then condition $(\ast)$ is satisfied by every $u \in Ph^0_{\partial D}(D)$. This happens for example when $D$ is a $n$-circular domain;
(iii) If $u \in L^2(\partial D)$ extends to a pluriharmonic function on $D$, then the proof of (b) shows that $(\ast)$ still holds.

3.2. Now we show that if the boundary value is sufficiently regular, condition $(\ast)$ implies pluriharmonicity without any topological assumption on $D$.

Theorem 2. Let $\partial D$ be of class $C^{1+\lambda}$, $\lambda > 0$. If $u \in C^{1+\lambda}(\partial D)$ is a real function which satisfies condition $(\ast)$, then $u \in \text{Re} A^{1+\lambda}_{\partial D}(D)$. In particular, $u \in Ph^{1+\lambda}_{\partial D}(D)$.

Proof. Let $U \in C^{1+\lambda}(\overline{D})$ be the harmonic extension of $u$. Let $V \in C^1(\overline{D})$ be a solution of the (classical) Neumann problem with boundary condition $\frac{\partial V}{\partial \nu} = -\frac{\partial u}{\partial \tau}$ on $\partial D$. We show that the function $F = U + iV$ is holomorphic on $D$.

Since $\bar{\partial}_n F$ is real on $\partial D$, condition $(\ast)$ implies that $\int_{\partial D} u \bar{\partial}_n F d\sigma = 0$. The integral

$$\int_{\partial D} F \ast \bar{\partial} F = \int_D \frac{\partial F}{\partial \bar{\zeta}} \frac{\partial F}{\partial \zeta} = 2^{1-n} i^n \int_D \sum_k \left| \frac{\partial F}{\partial \zeta_k} \right|^2 dv \int_D \sum_k \left| \frac{\partial F}{\partial \zeta_k} \right|^2 dv$$

is real. It vanishes because $\text{Re} \int_{\partial D} (U - iV) \bar{\partial}_n F d\sigma = \int_{\partial D} u \bar{\partial}_n F d\sigma$. Then $\bar{\partial} F = 0$ on $D$. The regularity of $F$ follows from that of $U$ and Cauchy-Riemann equations.

Corollary 1. Let $\partial D$ be connected, of class $C^{1+\lambda}$, $\lambda > 0$. If $u \in C^{1+\lambda}(\partial D)$, then condition $(\ast)$ holds if and only if $u$ is the real part of a CR function on $\partial D$.

Remark. Theorem 2 shows that the condition $H^1(D, \mathbb{R}) = 0$ can not be omitted in Theorem 1 when the boundary value $u$ is of class $C^{1+\lambda}$.

3.3. In the case when $D$ is strongly pseudoconvex, condition $(\ast)$ implies that the harmonic extension is pluriharmonic even for continuous functions.
Theorem 3. Let $D$ be a smoothly bounded strongly pseudoconvex domain. If $u \in C^0(\partial D)$ is a real function which satisfies condition $(\ast)$, then $u \in Ph^0_{\partial D}(D)$.

Proof. Let $D'$ be any simply connected, smoothly bounded domain contained in the interior of $D$. Let $U$ be the harmonic extension of $u$ on $D$ and let $H \in Harm^1(D')$ have real normal part $\overline{\partial}_n'H$ of $\overline{\partial}H$ on $\partial D'$. Then

$$\int_{\partial D'} U \overline{\partial}_n'H \, d\sigma' = \int_{\partial D} u(\eta) P_D(\zeta, \eta) \overline{\partial}_n'H(\zeta) d\sigma(\eta) d\sigma'(\zeta) = \int_{\partial D} u(\phi) d\sigma$$

where $P_D(\zeta, \eta)$ denotes the Poisson kernel for $D$ and $\phi(\eta)$ is the $C^\infty$ real valued function on $\partial D$ given by the integral $\int_{\partial D'} P_D(\zeta, \eta) \overline{\partial}_n'H(\zeta) d\sigma'(\zeta)$.

On $D$ the $\overline{\partial}$-Neumann problem $\overline{\partial}_n F = \phi$ can be solved if $\phi$ is orthogonal to antiholomorphic functions with respect to integration on $\partial D$. Let $f$ be holomorphic in a neighbourhood of $D'$. Then

$$\int_{\partial D} f(\phi) d\sigma = \int_{\partial D} f(\eta) \int_{\partial D'} P_D(\zeta, \eta) \overline{\partial}_n'H(\zeta) d\sigma(\eta) d\sigma'(\zeta)$$

$$= \int_{\partial D'} \overline{\partial}_n'H(\zeta) \int_{\partial D} f(\eta) P_D(\zeta, \eta) d\sigma(\eta) d\sigma'(\zeta)$$

$$= \int_{\partial D'} f(\zeta) \overline{\partial}_n'H(\zeta) d\sigma(\zeta) = 0$$

since $f$ is pluriharmonic on a neighbourhood of $D'$.

Then there exists $F \in Harm(D) \cap C^\infty(\overline{D})$ such that $\overline{\partial}_n F = \phi$ on $\partial D$. It follows that

$$\int_{\partial D'} U \overline{\partial}_n'H \, d\sigma' = \int_{\partial D} u \overline{\partial}_n F \, d\sigma = 0$$

From Theorem 2 we get that $U$ is pluriharmonic on $D'$, and so on the whole domain $D$.

Remark. The proof shows that the same result holds on any weakly pseudoconvex domain $D$ for which $\overline{\partial}$-Neumann problem can be solved for $C^\infty$ forms (see [K] §18). For example, on weakly pseudoconvex domains with real analytic boundary.

4. Characterization of $Harm^1_0(D)$

Let $M$ denote the Bochner-Martinelli operator on $D \subset \mathbb{C}^n$, $n > 1$:

$$M(F)(z) = \int_{\partial D} F(\zeta) U(\zeta, z), \quad \text{for } z \notin \partial D$$

where $U(\zeta, z)$ is the Bochner-Martinelli kernel.
Proposition 1. Let $\partial D$ be of class $C^1$. Then a function $F \in \text{Harm}^1(D)$ is in $\text{Harm}^1_0(D)$ if and only if $\text{Im}(M(F)) = \text{Im}(F)$ in $D$.

Proof. We adapt the proof of a theorem of Aronov ([A]) given in [K] (Theorem 14.1). From the Bochner-Martinelli formula for harmonic functions

$$
\int_{\partial D} F(\zeta)U(\zeta,z) + \int_{\partial D} g_0(\zeta,z)\overline{\partial}_n F(\zeta) d\sigma = \begin{cases} F(z), & \text{for } z \in D \\ 0, & \text{for } z \in \mathbb{C}^n \setminus \overline{D} \end{cases}
$$

where $g_0(\zeta,z) = \frac{(n-2)!}{2\pi^n} |\zeta - z|^{2-2n}$.

If $\overline{\partial}_n F$ is real on $\partial D$, then $\text{Im}(M(F)) = \text{Im}(F)$ in $D$. Conversely, assume that $\text{Im}(M(F)) = \text{Im}(F)$ in $D$. From the formula we get

$$
\int_{\partial D} \text{Im}(\overline{\partial}_n F(\zeta)) \frac{|z - \zeta|^{2n-2}}{|\zeta - z|^{2n-2}} d\sigma = 0
$$

when $z \in D$. These integrals vanish also for $z \notin \overline{D}$, because from the jump formula for $M(F)$ on $\partial D$ we get that $\text{Im}(M(F)) = 0$ on $\mathbb{C}^n \setminus \overline{D}$. A density argument gives $\text{Im}(\overline{\partial}_n F) = 0$ on $\partial D$.

Let $N$ be the real linear operator defined for $F \in \text{Harm}^1(D)$ by

$$
N(F) = \text{Re}(F) + i \text{Im}(M(F))
$$

Proposition 1 says that $\text{Harm}^1_0(D)$ is the set $\text{Fix}(N) = \{ F \in \text{Harm}^1(D) : N(F) = F \}$. Note that $N(F)$ is of class $C^\alpha$ for every $\alpha$, $0 < \alpha < 1$, and that $N$ can be extended as a real operator from the space $L^2(\partial D)$ to the Hardy space of harmonic functions $h^2(D)$.

5. The case of the ball

5.1. Let $B$ be the unit ball of $\mathbb{C}^n$ and let $S = \partial B$ be the unit sphere. We consider the space $L^2(S)$ identified with the space of harmonic extensions of $L^2$ functions from $S$ into $B$. $L^2(S)$ is the direct sum of pairwise orthogonal spaces $H(s,t)$, $s \geq 0$, $t \geq 0$, where $H(s,t)$ is the space of harmonic homogeneous polynomials of total degree $s$ in $z$ and total degree $t$ in $\overline{z}$ (see [R] §12). Romanov [R1] proved that the Bochner-Martinelli operator $M$ is a bounded self-adjoint operator in the space $L^2(S)$, since

$$
M(P_{s,t}) = \frac{n+s-1}{n+s+t-1} P_{s,t}
$$

for any $P_{s,t} \in H(s,t)$.

This spectral decomposition allows to compute the limit of the iterates of the operator $N$ given in §4. Let $N_0$ be the real linear projection in $L^2(S)$ defined for $P_{s,t} \in H(s,t)$ by

$$
N_0(P_{s,t}) = \begin{cases} \frac{s}{s+t} P_{s,t} + \frac{t}{s+t} \overline{P_{s,t}}, & \text{for } t > 0 \\ P_{s,t}, & \text{for } t = 0 \end{cases}
$$
Proposition 2. \( \lim_{k \to \infty} N^k = N_0 \) in the strong operator topology of \( L^2(S) \).

**Proof.** Assume that \( M(F) = \lambda F \), \( M(\overline{F}) = \mu F \). Let \( \alpha = (\lambda + \mu)/2 \), \( \beta = (\lambda - \mu)/2 \). An easy computation shows that \( N^k(F) = \text{Re}(F) + i\lambda^{(k)} \text{Im}(F) \), where \( \lambda^{(0)} = 1 \), and \( \lambda^{(k)} = \alpha \lambda^{(k-1)} + \beta \). If \( |\alpha| < 1 \), \( \lambda^{(k)} \) tends to \( \frac{\beta}{1-\alpha} \) as \( k \to \infty \). Setting \( F = P_{s,t} \), we get \( \frac{\beta}{1-\alpha} = \frac{s-t}{s+t} \), from which it follows that \( N^k(P_{s,t}) \to N_0(P_{s,t}) \) for every \( s, t \).

5.2. Taking \( \rho(\zeta) = |\zeta|^2 - 1 \) as a defining function for \( B \), we see that \( \overline{\partial}_n P_{s,t} = \sum_k \frac{\partial P_{s,t}}{\partial \zeta_k} \frac{s-t}{|\zeta|} = tP_{s,t}/|\zeta| \). Then \( \overline{\partial}_n N_0(P_{s,t})|_S = \frac{st}{s+t} (P_{s,t} + \overline{P_{s,t}}) \) for every \( s, t \).

Proposition 3. Let \( \text{Fix}(N_0) = \{ F \in \text{Harm}^1(B) : N_0(F) = F \} \). Then \( \text{Fix}(N_0) = \text{Harm}^0(B) \).

**Proof.** If \( N(F) = F \), then \( N_0(F) = F \). Conversely, if \( F \in \text{Fix}(N_0) \), then \( \overline{\partial}_n F|_S = \overline{\partial}_n N_0(F)|_S \) is real, since \( \overline{\partial}_n N_0(P_{s,t})|_S \) is real for every \( s, t \). Then \( F \in \text{Fix}(N) = \text{Harm}^0(B) \).

This result allows to obtain a proof of a theorem of Nagel and Rudin (see [NR] or [R]§13):

**Corollary 2.** A real function \( u \in C^0(S) \) has a pluriharmonic extension on \( B \) if and only if \( u \) is orthogonal to the spaces \( H(s, t) \) in \( L^2(S) \) for any \( s > 0 \), \( t > 0 \).

**Proof.** We show that condition (*) is equivalent to Nagel-Rudin condition. If \( u \) satisfies condition (*), we can take \( H = N_0(P_{s,t}) \) and get that \( u \) is orthogonal to \( \text{Re}(P_{s,t}) \) for every \( s > 0 \), \( t > 0 \). Taking \( iP_{s,t} \) in place of \( P_{s,t} \), we get orthogonality with respect to \( \text{Im}(P_{s,t}) \) for every \( s > 0 \), \( t > 0 \), and then with respect to the spaces \( H(s, t) \). Conversely, orthogonality with respect to \( \overline{\partial}_n N_0(P_{s,t}) \) for any \( s, t \) implies that \( u \) is orthogonal to \( \overline{\partial}_n H|_S \) for every \( H = N_0(H) \in \text{Fix}(N_0) \). From Proposition 3, this is condition (*). Theorems 1 and 3 give the result.

**Remarks.** (i) If \( F \in \text{Harm}^1(B) \) and \( N_0(F) \in \text{Harm}^0(B) \), then the imaginary part of \( N_0(F) \) is the unique classical solution of the interior Neumann problem with boundary datum \( \frac{\partial \text{Re} F}{\partial \sigma} \) and such that \( N_0(F)(0) = F(0) \).

(ii) The real projection \( N_0 \) is the identity on holomorphic functions, it is the conjugation operator on non-constant antiholomorphic functions and has the property \( \text{Re} N_0(F) = \text{Re} F \) for every \( F \). Assume that \( N_0 \) is a real linear operator with the same properties on a smooth domain \( D \). Let \( N_0(u) \in \text{Harm}^0(D) \), \( u \) a real function continuous on \( \partial D \) as usual, \( u \) is identified with its harmonic extension \( U \). Then \( u \in \text{Re} A^0_{\partial D}(D) \) if and only if \( \overline{\partial}_n N_0(u) = 0 \) on \( \partial D \). In fact, from the above cited theorem of Aronov and from a theorem of Kytmanov and Aizenberg [KA], \( \overline{\partial}_n N_0(u) = 0 \) on \( \partial D \) if and only if \( N_0(u) \) is holomorphic on \( D \).

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References


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