Geography on 3-folds of General Type

Meng Chen
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September 9, 2010
• Let \( V \) be a nonsingular projective variety. Consider the canonical line bundle \( \omega_V = \mathcal{O}_V(K_V) \). One task of birational geometry is to study the geometry induced from linear system \( |mK_V| \) or \( |-mK_V| \), \( \forall m \in \mathbb{Z}^+ \).
• Let $V$ be a nonsingular projective variety. Consider the canonical line bundle $\omega_V = \mathcal{O}_V(K_V)$. One task of birational geometry is to study the geometry induced from linear system $|mK_V|$ or $|-mK_V|$, $\forall m \in \mathbb{Z}^+$. 

• Assume that $V$ is of general type, i.e. $\kappa(V) = \dim(V)$. Set

$$\mathcal{V}_n := \{n\text{-dimensional variety of general type}\}.$$
A Classical Problem

- Let $V$ be a nonsingular projective variety. Consider the canonical line bundle $\omega_V = \mathcal{O}_V(K_V)$. One task of birational geometry is to study the geometry induced from linear system $|mK_V|$ or $|-mK_V|$, $\forall m \in \mathbb{Z}^+$. 

- Assume that $V$ is of general type, i.e. $\kappa(V) = \dim(V)$. Set

$$\mathcal{V}_n := \{\text{n-dimensional variety of general type}\}.$$ 

- Post-MMP Problem: how to classify $\mathcal{V}_n$?
• In 2006, Hacon-McKernan, Takayama $\Rightarrow \exists \ r_n$ such that $\varphi_m$ is birational $\forall \ m \geq r_n$ and $\forall \ V \in \mathcal{V}_n$. 
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• Chen-Chen $\Rightarrow$

  (1) $r_3 \leq 73$;
  (2) $\text{Vol}(V) \geq 1/2660$ $\forall V \in \mathcal{V}_3$. 
• In 2006, Hacon-McKernan, Takayama ⇒ ∃ \( r_n \) such that \( \varphi_m \) is birational \( \forall \ m \geq r_n \) and \( \forall \ V \in \mathcal{V}_n \).

• Chen-Chen ⇒
  
  (1) \( r_3 \leq 73 \);
  
  (2) \( \text{Vol}(V) \geq 1/2660 \ \forall \ V \in \mathcal{V}_3 \).

• The aim of this talk—geography ⇒ to improve the above results.
Let $X$ be a ($\mathbb{Q}$FT) minimal projective 3-fold of general type. Reid $\Rightarrow \exists!$ weighted basket $\mathbb{B}_X := \{B_X, P_2, O_X\}$ such that all the birational invariants of $X$ are uniquely determined by $\mathbb{B}_X$, where $B_X = \{\frac{1}{r_i}(1, -1, b_i)|i = 1, \ldots, t\}$. 

Open problem: to find exact relations between $V_3 \leftrightarrow \{\text{weighted baskets}\}$. 

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• Open problem: to find exact relations between the sets

\[ \mathcal{V}_3 \leftrightarrow \{\text{weighted baskets}\} \]
Two geographical inequalities

- Miyaoka-Reid inequality:

\[ K_X^3 \leq 72\chi(\omega_X) + 3 \sum_i (r_i - \frac{1}{r_i}). \]

- Inequalities of Noether type (Chen-Chen):

\[ K_X^3 \geq a_m P_m(X) - b_m \]

where \( a_m, b_m \in \mathbb{Q}^+ \), and \( m \geq 1 \).
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• The fact: general type 3-folds with $p_g \leq 1$
form an infinite family.
Numerical genus

- The fact: general type 3-folds with $p_g \leq 1$ form an infinite family.
- When $p_g(X) \leq 1$, $n_0(X) := \min\{m|P_m(X) \geq 2\}$. Chen-Chen \( \Rightarrow \) $2 \leq n_0(X) \leq 18$.

**Definition**

The numerical genus of $X$ is defined as:

$$g(X) := \begin{cases} p_g(X); & p_g(X) \geq 2 \\ \frac{1}{n_0(X)}; & \text{otherwise.} \end{cases}$$
The Noether function $\mathcal{N}(g)$

- Chen-Chen $\Rightarrow g(X) \geq \frac{1}{18}$. 

For all minimal 3-fold $X$ of general type, Noether inequality $K^3 X \geq \mathcal{N}(g(X))$. 

What is the Noether function $\mathcal{N}(g)$?
The Noether function $N(g)$

- Chen-Chen $\Rightarrow g(X) \geq \frac{1}{18}$.
- The Noether function
  
  $N(g) := \inf\{K_X^3|g(X) = g\}.$
The Noether function $\mathcal{N}(g)$

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- What is the Noether function $\mathcal{N}(g)$?
• In 1992, Kobayashi constructed a family of canonically polarized 3-folds satisfying:
\[ K_X^3 = \frac{4}{3} p_g(X) - \frac{10}{3}. \]
Noether inequalities in narrow sense

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• In 2004, Chen \[ K_X^3 \geq \frac{4}{3} p_g(X) - \frac{10}{3} \] for canonically polarized 3-folds.

• In 2006, Catanese-Chen-Zhang \[ K_X^3 \geq \frac{4}{3} p_g(X) - \frac{10}{3} \] for nonsingular minimal 3-folds of general type.

• Conjecture: \[ K_X^3 \geq \frac{4}{3} p_g(X) - \frac{10}{3} \] holds for Gorenstein minimal 3-folds of general type.
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Known value of $\mathcal{N}(g)$

- In 2007, Chen $\Rightarrow K_X^3 \geq \frac{1}{3}$ when $g = p_g(X) \geq 2$. 
Known value of $\mathcal{N}(g)$

- In 2007, Chen $\Rightarrow K_X^3 \geq \frac{1}{3}$ when $g = p_g(X) \geq 2$.
- Chen $\Rightarrow$
  
  \[
  \mathcal{N}(2) = \frac{1}{3};
  \]
  
  \[
  \mathcal{N}(3) = 1;
  \]
  
  \[
  \mathcal{N}(4) = 2;
  \]
  
  \[
  \mathcal{N}(g) \geq g - 2 \text{ for } g \geq 5.
  \]

  due to supporting examples of Fletcher-Reid.
The strategy to get the lower bound of $K_X^3$

- Fletcher-Reid’s example:
  $X_{46} \subset \mathbb{P}(4, 5, 6, 7, 23)$, $K^3 = \frac{1}{420}$. 
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\( X_{46} \subset \mathbb{P}(4, 5, 6, 7, 23), \ K^3 = \frac{1}{420}. \)

• When \( p_g(X) \leq 1, \ \frac{1}{18} \leq g \leq \frac{1}{2}. \)
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- When $K_X^3 < \frac{1}{420}$, Reid’s weighted baskets can be completely listed, but the list is too big!
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- When $K_X^3 < \frac{1}{420}$, Reid’s weighted baskets can be completely listed, but the list is too big!

- To find a function $c(g)$ such that $K_X^3 \geq c(g)$ with $g(X) = g$. 

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Geography on 3-folds of General Type
• Chen-Chen $\Rightarrow \exists$ a very effective function $v(g)$ ($g < 2$) satisfying $K_X^3 \geq v(g(X))$. 
The main statements

- Chen-Chen ⇒ ∃ a very effective function \( v(g) \) (\( g < 2 \)) satisfying \( K_X^3 \geq v(g(X)) \).

- Set \( g = 1/n_0 \), here is part of the description:

<table>
<thead>
<tr>
<th>( n_0 )</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v(n_0) )</td>
<td>1/420</td>
<td>1/450</td>
<td>1/630</td>
<td>1/825</td>
<td>1/1089</td>
<td>1/1404</td>
</tr>
<tr>
<td>( n_0 )</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>18</td>
</tr>
<tr>
<td>( v(n_0) )</td>
<td>1/1728</td>
<td>1/2152.5</td>
<td>1/2640</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>
• Chen-Chen \implies \exists \text{ a very effective function } \nu(g) (g < 2) \text{ satisfying } K_X^3 \geq \nu(g(X)).

• Set \( g = 1/n_0 \), here is part of the description:

<table>
<thead>
<tr>
<th>( n_0 )</th>
<th>7</th>
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</thead>
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<td>( \nu(n_0) )</td>
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<td>( \frac{1}{450} )</td>
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<td>( \frac{1}{825} )</td>
<td>( \frac{1}{1089} )</td>
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<th>14</th>
<th>15</th>
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<td>–</td>
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<td>–</td>
</tr>
</tbody>
</table>

• \( \mathcal{N}(\frac{1}{2}) = \nu(\frac{1}{2}) = \frac{1}{12} \). (optimal)
Conclusions

- Fletcher-Reid examples with $g = 1/2$ and $K^3 = 1/12$:
  \[ X_{22} \subset \mathbb{P}(1, 2, 3, 4, 11) \]
  \[ X_{6,18} \subset \mathbb{P}(2, 2, 3, 3, 4, 9) \]
  \[ X_{10,14} \subset \mathbb{P}(2, 2, 3, 4, 5, 7) \]
Conclusions

• Fletcher-Reid examples with $g = 1/2$ and $K^3 = 1/12$:

\[ X_{22} \subset \mathbb{P}(1, 2, 3, 4, 11) \]
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\[ X_{10,14} \subset \mathbb{P}(2, 2, 3, 4, 5, 7) \]

Theorem

Let $X$ be a minimal projective 3-fold of general type. Then

1. $K_X^3 \geq \frac{17}{30030} > \frac{1}{1767}$. Furthermore, $K_X^3 = \frac{17}{30030}$ if and only if $\mathbb{B}(X) = \{B_{3a}, 0, 3\}$.
2. (announcement) $\varphi_m$ is birational for $m \geq 65$. 
The method

- We study the $m_0$-canonical map of $X$:

$$\varphi_{m_0} : X \dashrightarrow \mathbb{P}^{P_{m_0}-1}.$$ 

By Hironaka’s big theorem, we can take successive blow-ups $\pi : X' \rightarrow X$ such that:

1. $X'$ is smooth;
2. the movable part of $|m_0K_{X'}|$ is base point free;
3. the support of the union of $\pi^*(K_{m_0})$ and the exceptional divisors is of simple normal crossings.

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Geography on 3-folds of General Type
• Set $g_m^0 := \varphi_m^0 \circ \pi$. Then $g_m^0$ is a morphism by assumption. Let $X' \xrightarrow{f} \Gamma \xrightarrow{s} W'$ be the Stein factorization of $g_m^0$ with $W'$ the image of $X'$ through $g_m^0$. 

\[
\begin{array}{ccc}
X' & \xrightarrow{f} & \Gamma \\
\downarrow \pi & & \downarrow s \\
X & \xrightarrow{\varphi_m^0} & W'
\end{array}
\]
The method

- Denote by $M_{m_0}$ the movable part of $|m_0K_{X'}|$. One has

$$m_0\pi^*(K_X) = M_{m_0} + E_{m_0}'$$

for an effective $\mathbb{Q}$-divisor $E_{m_0}'$. In total, since

$$h^0(X', \lceil m_0\pi^*(K_X) \rceil) = h^0(X', \lceil m_0\pi^*(K_X) \rceil) = P_{m_0}(X') = P_{m_0}(X),$$

one has:

$$m_0K_{X'} = M_{m_0} + Z_{m_0}$$

where $Z_{m_0}$ is the fixed part of $|m_0K_{X'}|$. 
The method

- If $\dim(\Gamma) \geq 2$, a general member $S$ of $|M_{m_0}|$ is a nonsingular projective surface of general type. Set $p = 1$.
• If \( \dim(\Gamma) \geq 2 \), a general member \( S \) of \( |M_{m_0}| \) is a nonsingular projective surface of general type. Set \( p = 1 \).

• If \( \dim(\Gamma) = 1 \), a general fiber \( S \) of \( f \) is an irreducible smooth projective surface of general type. We may write

\[
M_{m_0} = \sum_{i=1}^{a_{m_0}} S_i \equiv a_{m_0} S
\]

where \( S_i \) are smooth fibers of \( f \) for all \( i \) and \( a_{m_0} \geq \min\{2P_{m_0} - 2, P_{m_0} + g(\Gamma) - 1\} \). Set \( p = a_{m_0} \).
The method

- Let $S$ be a generic irreducible element of $|m_0 K_{X'}|$. Let $|G|$ be a base point free linear system on $S$. Let $C$ be a generic irreducible element of $|G|$. Kodaira Lemma $\Rightarrow \exists \beta > 0$ such that $\pi^*(K_X)|_S \geq \beta C$. 

Inequality (1): $K^3_X \geq p \beta m_0 \xi$ (1) where $\xi = \pi^*(K_X) \cdot C$. 

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• Inequality (1):

\[ K_X^3 \geq \frac{p\beta}{m_0} \xi \]  \hspace{1cm} (1)

where $\xi = \pi^*(K_X) \cdot C$. 
• Inequality (2):

\[ \xi \geq \frac{\text{deg}(K_C)}{1 + \frac{m_0}{p} + \frac{1}{\beta}}. \]  

(2)
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\[ \xi \geq \frac{\deg(K_C)}{1 + \frac{m_0}{p} + \frac{1}{\beta}}. \]  

\[ (2) \]

• Inequality (3): For any positive integer \( m \) such that \( \alpha_m := (m - 1 - \frac{m_0}{p} - \frac{1}{\beta})\xi > 1 \), one has

\[ \xi \geq \frac{\deg(K_C) + \lceil \alpha_m \rceil}{m}. \]  

\[ (3) \]
• When $\dim \Gamma > 1$, take $|G| := |S_s|$. Thus $\beta = \frac{1}{m_0}$.
• When $\dim \Gamma > 1$, take $|G| := |S|_S$. Thus $\beta = \frac{1}{m_0}$.

• When $\dim \Gamma = 1$, take $G = q\sigma^*(K_{S_0})$ for $q \geq 1$ where $\sigma : S \to S_0$ is the contraction onto the minima model. Here is a key inequality:

$$\pi^*(K_X)|_S \geq \frac{p}{m_0 + p} \sigma^*(K_{S_0}).$$
Technical applications

• When \( \text{dim } \Gamma > 1 \), take \( |G| := |S|_S \). Thus \( \beta = \frac{1}{m_0} \).

• When \( \text{dim } \Gamma = 1 \), take \( G = q \sigma^*(K_{S_0}) \) for \( q \geq 1 \) where \( \sigma : S \to S_0 \) is the contraction onto the minima model. Here is a key inequality:

\[
\pi^*(K_X)|_S \geq \frac{p}{m_0 + p} \sigma^*(K_{S_0}).
\]

• Here is the complete list for 3-folds with small invariants:
<table>
<thead>
<tr>
<th>No.</th>
<th>((P_3, \ldots, P_{11}))</th>
<th>(P_{18})</th>
<th>(P_{24})</th>
<th>(\mu_1)</th>
<th>(x)</th>
<th>(B^{(12)} = (n_{1,2}, n_{5,11}, \ldots, n_{1,5})) or (B_{min})</th>
<th>(K^3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0, 0, 0, 0, 0, 0, 0, 1, 0)</td>
<td>4</td>
<td>8</td>
<td>14</td>
<td>2</td>
<td>(5, 0, 0, 1, 0, 3, 0, 0, 0, 1, 0, 0, 0)</td>
<td>(17)</td>
</tr>
<tr>
<td>2</td>
<td>(0, 0, 0, 0, 0, 1, 0, 0, 0)</td>
<td>3</td>
<td>7</td>
<td>15</td>
<td>2</td>
<td>(4, 0, 1, 0, 0, 2, 1, 0, 3, 0, 0, 2, 0, 0)</td>
<td>(360)</td>
</tr>
<tr>
<td>2a</td>
<td></td>
<td>2</td>
<td>3</td>
<td>18</td>
<td>({(2, 5), (3, 8), \ast}) (\succ) ({(5, 13), \ast})</td>
<td>(147)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(0, 0, 0, 0, 0, 1, 0, 1, 0)</td>
<td>3</td>
<td>7</td>
<td>15</td>
<td>3</td>
<td>(6, 1, 0, 0, 4, 1, 0, 4, 0, 1, 0, 2, 0, 0)</td>
<td>(9240)</td>
</tr>
<tr>
<td>3a</td>
<td></td>
<td>2</td>
<td>3</td>
<td>18</td>
<td>({(2, 5), (3, 8), \ast}) (\succ) ({(5, 13), \ast})</td>
<td>(30030)</td>
<td></td>
</tr>
<tr>
<td>4</td>
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<td>4</td>
<td>9</td>
<td>14</td>
<td>3</td>
<td>(7, 0, 1, 0, 0, 4, 0, 1, 3, 0, 1, 0, 2, 0, 0)</td>
<td>(3605)</td>
</tr>
<tr>
<td>4.5</td>
<td></td>
<td>1</td>
<td>2</td>
<td>14</td>
<td>({(4, 11), (1, 3), \ast}) (\succ) ({(5, 14), \ast})</td>
<td>(2630)</td>
<td></td>
</tr>
<tr>
<td>5</td>
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<td>5</td>
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<td>14</td>
<td>3</td>
<td>(7, 0, 1, 0, 0, 4, 1, 0, 4, 0, 1, 0, 1, 0)</td>
<td>(3960)</td>
</tr>
<tr>
<td>5a</td>
<td></td>
<td>4</td>
<td>3</td>
<td>15</td>
<td>({(8, 20), (3, 8), \ast}) (\succ) ({(11, 28), \ast})</td>
<td>(1386)</td>
<td></td>
</tr>
<tr>
<td>5b</td>
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<td>3</td>
<td>3</td>
<td>15</td>
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<td>(1170)</td>
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<td>6</td>
<td>14</td>
<td>3</td>
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<tr>
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<td>3</td>
<td>5</td>
<td>14</td>
<td>2</td>
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<td>(630)</td>
</tr>
<tr>
<td>7a</td>
<td></td>
<td>2</td>
<td>3</td>
<td>14</td>
<td>({(4, 9), (3, 7), \ast}) (\succ) ({(7, 16), \ast})</td>
<td>(1680)</td>
<td></td>
</tr>
<tr>
<td>8</td>
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<td>3</td>
<td>5</td>
<td>14</td>
<td>3</td>
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<td>(70)</td>
</tr>
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<td>6</td>
<td>14</td>
<td>3</td>
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<td></td>
</tr>
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<td>No.</td>
<td>((P_3, \ldots, P_{11}))</td>
<td>(P_{18})</td>
<td>(P_{24})</td>
<td>(\mu_1)</td>
<td>(\chi)</td>
<td>((n_{1,2}, n_{4,9}, \ldots, n_{1,5})) or (B_{\min})</td>
<td>(K^3)</td>
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<td>14</td>
<td>{((3, 10), (2, 7), *} \succ {(5, 17), *}}</td>
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<td>37</td>
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<td>{((4, 9), (3, 7), *} \succ {(7, 16), *}}</td>
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<td>{((3, 10), (2, 7), *} \succ {(5, 17), *}}</td>
<td>(\frac{5355}{4})</td>
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<td>(\frac{1260}{3})</td>
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<td>{(7, 16), (5, 18), *}</td>
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<td>(4, 11), (1, 3), *] (\leadsto) [(5, 14), *]</td>
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Let $X$ be a nonsingular projective 3-fold of general type. When the geometric genus $p_g \geq 2$, the canonical map $\varphi_1 := \Phi|_{K_X}$ is usually a key tool for birational classification.
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Chen-Hacon

When $X$ is Gorenstein minimal and $\varphi_1$ is of fiber type, then $X$ is canonically fibred by surfaces or curves with bounded invariants.

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Geography on 3-folds of General Type
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• Chen-Hacon $\Rightarrow$ When $X$ is Gorenstein minimal and $\varphi_1$ is of fiber type, then $X$ is canonically fibred by surfaces or curves with bounded invariants.
Theorem

Let $X$ be a Gorenstein minimal projective 3-fold of general type. Assume that $X$ is canonically of fiber type. Let $F$ be a smooth model of the generic irreducible component in the general fiber of $\varphi_1$. Then

(i) $g(F) \leq 91$ when $F$ is a curve and $p_g(X) \geq 183$;
(ii) $p_g(F) \leq 37$ when $F$ is a surface and $p_g(X) \gg 0$, say $p_g(X) \geq 3890$. 
**New examples**

**Standard construction.** Let \( S \) be a minimal surface of general type with \( p_g(S) = 0 \). Assume there exists a divisor \( H \) on \( S \) such that \(|K_S + H|\) is composed with a pencil of curves and that \( 2H \) is linearly equivalent to a smooth divisor \( R \). Let \( \hat{C} \) be a generic irreducible element of the movable part of \(|K_S + H|\). Assume \( \hat{C} \) is smooth. Set \( d := \hat{C} \cdot H \) and \( D := \hat{C} \cap H \). Let \( C_0 \) be a fixed smooth projective curve of genus 2. Let \( \theta \) be a 2-torsion divisor on \( C_0 \). Set \( Y := S \times C_0 \). Take \( \delta := p_1^*(H) + p_2^*(\theta) \) and pick a smooth divisor \( \Delta \sim p_1^*(2H) \). Then the pair \((\delta, \Delta)\) determines a smooth double covering \( \pi : X \to Y \) and \( K_X = \pi^*(K_Y + \delta) \).
Since $K_Y + \delta = p_1^*(K_S + H) + p_2^*(K_{C_0} + \theta)$, $p_g(Y) = 0$ and $h^0(K_{C_0} + \theta) = 1$, one sees that $|K_X| = \pi^*|K_Y + \delta|$ and that $\Phi|_{K_X}$ factors through $\pi$, $p_1$ and $\Phi|_{K_S + H}$. Since $|K_S + H|$ is composed with a pencil of curves $\hat{C}$, $X$ is canonically fibred by surfaces $F$ and $F$ is a double covering over $T := \hat{C} \times C_0$ corresponding to the data $(q_1^*(D) + q_2^*(\theta), q_1^*(2D))$ where $q_1$ and $q_2$ are projections. Denote by $\sigma : F \to T$ the double covering. Then $K_F = \sigma^*(K_T + q_1^*(D) + q_2^*(\theta))$. By calculation, one has $p_g(F) = 3g(\hat{C})$ when $d = 0$ and $p_g(F) = 3g(\hat{C}) + d - 1$ whenever $d > 0$. 
Let $S$ be any smooth minimal projective surface of general type with $p_g(S) = 0$. Assume $\mu : S \to \mathbb{P}^1$ is a genus 2 fibration. Let $H$ be a general fiber of $\mu$. Then $|K_S + H|$ is composed with a pencil of curves $\hat{C}$ of genus $g(\hat{C})$ and $\hat{C}.H = 2$. 

We take a pair $(S, H)$ which was found by Xiao, where $S$ is a numerical Compadelli surface with $K^2_S = 2$, $p_g(S) = q(S) = 0$ and $\text{Tor}(S) = (\mathbb{Z}/2)^3$. 

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Geography on 3-folds of General Type
New examples

 Lemma

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- We take a pair $(S, H)$ which was found by Xiao, where $S$ is a numerical Compedelli surface with $K_S^2 = 2$, $p_g(S) = q(S) = 0$ and $\text{Tor}(S) = (\mathbb{Z}_2)^3$. 
Let $P = \mathbb{P}^1 \times \mathbb{P}^1$. Take four curves $C_1$, $C_2$, $C_3$ and $C_4$ defined by the following equations, respectively:

\begin{align*}
C_1 : \quad & x = y; \\
C_2 : \quad & x = -y; \\
C_3 : \quad & xy = 1; \\
C_4 : \quad & xy = -1.
\end{align*}

These four curves intersect mutually at 12 ordinary double points:

\begin{align*}
(0, 0), \quad & (\infty, \infty), \quad (0, \infty), \quad (\infty, 0) \\
(\pm 1, \pm 1), \quad & (\pm \sqrt{-1}, \pm \sqrt{-1}).
\end{align*}
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Geography on 3-folds of General Type
Xiao \implies \text{There exists a divisor } R_1 \text{ of bidegree } (14, 6) \text{ which has exactly 12 simple singularities of multiplicity 4. Then the data } (\delta_1, R_1) \text{ determines a singular double covering onto } P.

\[ S \xleftarrow{\sigma} \tilde{S} \xrightarrow{\theta} \tilde{P} \]

\[ f \downarrow \quad \tilde{f} \downarrow \quad \tau \]

\[ \mathbb{P}^1 \quad \mathbb{P}^1 \quad \mathbb{P}^1 \xleftarrow{\varphi} P \]

\[ K^2_S = 2 \text{ and } p_g(S) = q(S) = 0. \]
Let $H$ be a general fiber of $f$. Calculations imply that $|K_S + H|$ has exactly 6 base points, but no fixed parts. Clearly a general member $\hat{C} \in |K_S + H|$ is a smooth curve of genus 6.
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Now we take the triple $(S, H, \hat{C})$ and run standard construction. What we get is the 3-fold $X_{S,19}$ which is canonically fibred by surfaces $F$ with $p_g(F) = 19$. 
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Thanks very much!