Classification & Construction of general singular fibers of proper holomorphic Lagrangian fibrations

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Joint work with

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References

1) ArXiv: 0710.2376

2) ArXiv: 0907.4869

3) ArXiv: 1007.2043

Definition

(M, σ): (holomorphic) symplectic manifold

$\Leftrightarrow \begin{cases} M: \text{complex manifold} \\ \quad (\text{connected, not necessarily compact}) \\ \sigma: d\text{-closed holomorphic 2-form on } M \\ \quad \text{s.t. everywhere non-degenerate} \end{cases}$

$$\begin{array}{ccc} TM \times TM & \longrightarrow & \mathcal{O}_M \\ (v_1, v_2) & \longmapsto & \sigma(v_1, v_2) \end{array}$$

at $\forall x \in M$
non-deg.
alternating

$\Rightarrow$ 1) $\dim_{\mathbb{C}} M = 2d$ even

2) $TM \underset{\iota_\sigma := \jmath_\sigma^{-1}}{\overset{\jmath_\sigma \atop \cong}{\rightleftarrows}} T^*M \quad v \mapsto \sigma(v, -)$

- $V \subset (M^{2d}, \sigma)$ closed subvariety

V: Lagrangian $\Leftrightarrow \begin{cases} \sigma|V_{reg} \equiv 0 & \text{— (i)} \\ \dim V = d & \text{— (ii)} \end{cases}$

<u>Rem.</u> (i) $\Rightarrow \dim V \leqq d$.

$f : (M^{2d}, \sigma) \twoheadrightarrow B$ (B : smooth)

proper, surjective, holomorphic map with connected fibers

- f : Lagrangian fibration

$\Leftrightarrow$ $\forall$ irred. comp. of f is Lagrangian & Class $\mathcal{C}$ (bimeromorphic to Kähler mfd)

($\Rightarrow$ $\dim {}^\forall$fiber $= d$ & $\dim B = d$)

Rem. f : smooth at $x \in M$, $b = f(x) \in B$

$\Rightarrow$

$$\begin{tikzcd}
 & T_b^*B \arrow[d, phantom, "\wr\wr"] & & & \\
0 \arrow[r] & N^*_{M/M_b} \text{ at } x \arrow[r] & T_x^*M \arrow[r] \arrow[d, shift left, "\iota_{\sigma,x}"] & T_x^*M_b \arrow[r] & 0 \\
0 \arrow[r] & T_xM_b \arrow[r] & T_xM \arrow[r] \arrow[u, shift left, "c_{\sigma,x}"] & N_{M/M_b,x} \arrow[r] \arrow[d, phantom, "\wr\wr"] & 0 \\
 & & & T_bB &
\end{tikzcd}$$

$\underset{\text{Lagrangian}}{\Rightarrow}$ $\begin{cases} T_b^*B \xrightarrow[\iota_{\sigma,x}]{\cong} T_xM_b \\ T_bB \xrightarrow[c_{\sigma,x}]{\cong} T_x^*M_b \end{cases}$

In particular: For smooth $M_b = f^{-1}(b)$

$(z_1, \cdots, z_d)$ local coord. at $b \in B$

$\Rightarrow \iota_\sigma f^* dz_1, \cdots, \iota_\sigma f^* dz_d \in H^0(M_b, TM_b)$

linearly independent at $\forall x \in M_b$

$\Rightarrow TM_b \cong \mathcal{O}_{M_b}^{\oplus d} \Rightarrow T^*M_b \cong \mathcal{O}_{M_b}^{\oplus d}$

$\Rightarrow M_b \cong$ d-dim. complex torus

classe (Liouville's Thm)

Albanese map

Fact (H.-O.) $D := \{ b \in B \mid f^{-1}(b) : \text{singular} \} \subset B$ is pure of codimension 1 if $D \neq \phi$.

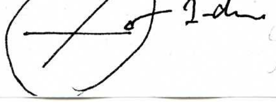
(Rem. <u>In general</u>, this is false even if $f : X \to Y$ is flat & X, Y smooth. (Mumford-Ramanujan))

<u>Question</u> How $f^{-1}(b)$ ($b \in D$: general) looks like?

Local Question on the base B May/Will assume:

$B = \underset{(z_1, \cdots, z_d)}{\Delta^d} \supset D = (z_d = 0) \cong \Delta^{d-1}$

$f^* D = \sum a_i H_i$, $n = GCD(a_i)$ n: multiplicity of gen. fiber over D



Ex1 ($d=1$, Kodaira)

$f: S=S^2 \to \Delta^1 \ni 0$

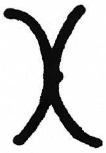
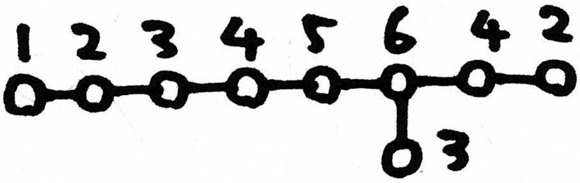
S : symplectic $\Leftrightarrow K_S \cong \mathcal{O}_S$

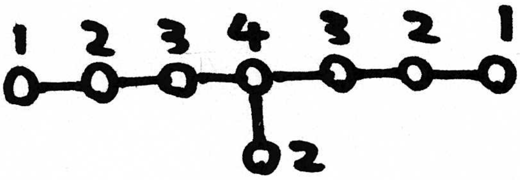
(& f : automatically Lagrangian)

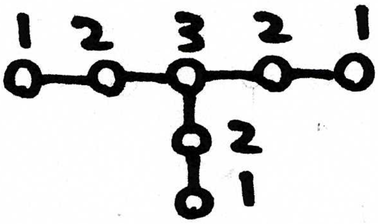
$\Rightarrow$ f : relatively minimal elliptic fibration without multiple fiber

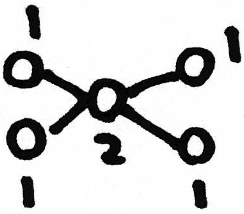
$f^{-1}(0)$ is either:

$\left(\mathrm{I}_0 : \text{smooth elliptic curve} \right)$, I_1  II  (irreducible)

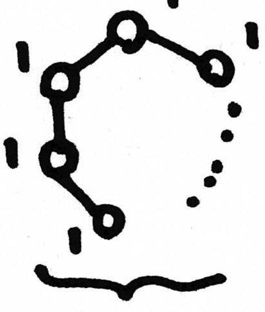
III  IV  II^* 

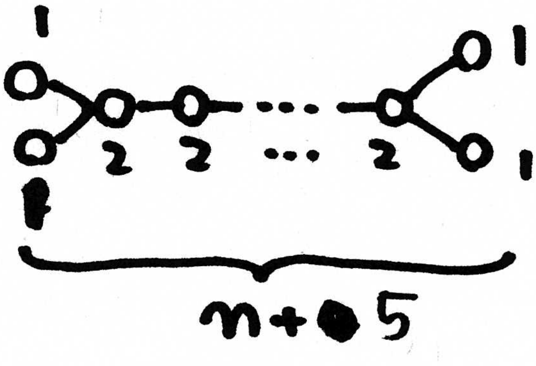
III^* 

IV^* 

I_0^* 

(finite monodromy)

I_n ($n \geq 2$) 

I_n^* ($n \geq 1$) 

(infinite monodromy)