Abel-Jacobi map, integral Hodge classes, and decomposition of the diagonal

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Decomposition of the diagonal

\( X = \) smooth projective complex variety.

- Assume \( CH_0(X) = \mathbb{Z} \) (equivalently \( CH_0(X)_\mathbb{Q} = \mathbb{Q} \)).

**Theorem**

*(Bloch-Srinivas)* For some integer \( N > 0 \), one has an equality *(Chow decomposition of the diagonal)*

\[
N \Delta_X = Z + Z' \text{ in } CH^n(X \times X), \ n = \text{dim} \ X
\]

with \( Z' \) supported on \( X \times \text{pt} \), \( Z \) supported on \( D \times X \), for some \( D \not\subseteq X \).

In particular, one has then such a decomposition at the level of cohomology classes, i.e. in \( H^{2n}(X \times X, \mathbb{Z}) \).

**Definition**

\( X \) has an *integral cohomological* decomposition of the diagonal if a decomposition as above holds in \( H^{2n}(X \times X, \mathbb{Z}) \), with \( N = 1 \).
Question (Q0)

For which $X$ with trivial $CH_0$ group does there exist an integral cohomological decomposition of the diagonal?

More generally, study the following invariant $N(X)$ of $X$: $N(X)$ is the GCD of the integers $N$ appearing in a cohomological decomposition of the diagonal as above. (Can also study the similar invariant defined using the Chow decomposition of the diagonal.)

Remark

This is a birational invariant of $X$. Indeed, under blow-up $\tau : Y \to X$, there is a decomposition

$$[\Delta_Y] = (\tau, \tau)^* [\Delta_X] + [\Delta_\tau]$$

where the cycle $\Delta_\tau$ is supported over $E \times E$, $E=\text{exceptional divisor}$. 
1-cycles on threefolds with trivial $CH_0$

From now on, $dim X = 3$.

The Chow decomposition of the diagonal $N\Delta_X \equiv_{rat} Z + N(X \times pt)$ with $Z$ supported over $D \times X$, $D \subsetneq X$ implies:

a) $CH^2(X)_{hom}/CH^2(X)_{alg}$ is of torsion (annihilated by $N$).

b) $\phi_X$ is surjective and $\text{Ker} (\phi_X : CH^2(X)_{hom} \rightarrow J(X))$ is of torsion (annihilated by $N$).

Here $\phi_X$ is the Abel-Jacobi map of $X$. $J(X) =$Griffiths’ intermediate Jacobian (an abelian variety in this case).

Much more is true:

**Theorem (Bloch, Bloch-Srinivas)**

Under the same assumptions:

a) $CH^2(X)_{hom}/CH^2(X)_{alg} = \{0\}$.

b) $\text{Ker} (\phi_X : CH^2(X)_{hom} \rightarrow J(X)) = \{0\}$.

(Uses Bloch-Ogus theory and Merkureev-Suslin theorem).
For such $X$, 1-cycles look very much like 0-cycles on curves. Namely $\phi_X$ induces an isomorphism: $CH^2(X)_{\text{hom}} \cong J(X)$.

Note $CH^2(X)_{\text{hom}}$ is an abstract group, a priori not a variety. $J(X)$ is a variety. The group morphism $\phi_X$ is algebraic in the following sense (this can be taken as an universal definition of $J(X)$):

For any smooth variety $B$, for any cod. 2 cycle $Z_B \subset B \times X$, s.t. $Z_b$ is cohomologous to 0, $\forall b \in B$, the map

$$\phi_{Z_B} : B \to J(X), \ b \mapsto \phi_X(Z_b)$$

\text{is a morphism of alg. varieties.}

**Question (Q1)**

Same assumptions on $X$. Does there exist a cod. 2 cycle $Z_{J(X)} \subset J(X) \times X$, s.t. $Z_t$ is cohomologous to 0, $t \in J(X)$ and

$$\phi_{Z_{J(X)}} : J(X) \to J(X), \ t \mapsto \phi_X(Z_t)$$

is the identity of $J(X)$?

NB. For 0-cycles on curves, the analogous question has a positive answer 
(the universal divisor on $Pic^0(C) \times C$).
Remarks

**Remark**

There is an integral Hodge class of degree 4 on $J(X) \times X$, which corresponds to the isomorphism of Hodge structures $H_1(J(X), \mathbb{Z}) \cong H^3(X, \mathbb{Z})/\text{torsion}$. Thus (Q1) has an affirmative answer if the Hodge conjecture holds for degree 4 integral Hodge classes on $J(X) \times X$.

**Note**: The Hodge conjecture does not hold in general for integral degree 4 Hodge classes (Atiyah-Hirzebruch, Kollár...), even on unirational varieties (Colliot-Thélène-Voisin 2010).

**Remark**

Answer to (Q1) is birationally invariant. More generally: the GCD of $\deg f: B \to J(X)$, $f$ onto gen. finite, induced by a cycle $Z \subset B \times X$, i.e. $f(b) = \phi_X(Z_b)$, is a birational invariant of $X$. 
There are useful variants of the previous question:

**Question (Q2)**

Same assumptions on $X$. Does there exist a smooth projective variety $B$, and a cod. 2 cycle $Z_B \subset B \times X$, s.t. $Z_b$ is cohomologous to 0, $b \in B$ and

$$
\phi_{Z_B} : B \to J(X), \ b \mapsto \phi_X(Z_b)
$$

is surjective with rationally connected general fibers?

(Compare with the case of zero cycles on curves: the Abel map

$$
z \mapsto alb_C(z - z_0), \ C^{(n)} \to J(C)
$$

is surjective with RC fibers for $n > g$).

**Note**: Positive answer to (Q1) $\Rightarrow$ Positive answer to (Q2). Take $B = J(X)$. 

Proposition (Voisin 2010)

Assume (Q2) has an affirmative answer, and that there exists a 1-cycle \( \Gamma \in CH_1(J) \) such that \( \Gamma^g = g!J(X) \), \( g = \dim J(X) \). Then (Q1) also has an affirmative answer.

Remark

As \( \dim X = 3 \), \( J(X) \) is a ppav. (\( \Theta \) divisor given by the unimodular intersection pairing on \( H^3(X, \mathbb{Z})/torsion \)).

There is thus an integral Hodge class \( \gamma = \frac{[\Theta]^{(g-1)}}{(g-1)!} \) on \( J(X) \), where \( g = \dim J(X) \). It satisfies \( \gamma^g = g![J(X)] \), but is not known in general to be algebraic. This is known if \( J(X) \) is a product of Jacobians of curves, for example if \( g \leq 3 \).
Proof of Proposition (sketch)

Assume for simplicity $\Gamma$ is effective. By assumption, there exist a smooth projective variety $B$, and a cod. 2 cycle $Z \subset B \times X$, s.t. $Z_b$ is cohomologous to 0, $b \in B$ and $\phi_{ZB} : B \to J(X)$ is surjective with rationally connected general fibers.

- May assume by translating $\Gamma \subset J(X)$ that general fibers over $\Gamma$ are rationally connected.
- The Graber-Harris-Starr theorem then says : there exists a lift $\gamma : \Gamma \to B$ of $\phi_B$ over $\Gamma$.
- Let $Z_\Gamma := (\gamma, Id_X)^* Z_B$. Then $\phi_\Gamma : \Gamma \to J(X)$, $\gamma \mapsto \phi_X(Z_{\Gamma,\gamma})$ is the inclusion of $\Gamma$ in $J(X)$.
- $Z_\Gamma$ induces an obvious cod. 2 cycle $Z^{(g)}_\Gamma \subset \Gamma^{(g)} \times X$,
  $$Z^{(g)}_{\Gamma^{(g)},\gamma_1 + \ldots + \gamma_g} := Z_{\Gamma,\gamma_1} + \ldots + Z_{\Gamma,\gamma_g}.$$
- Now use the sum map, which is by assumption birational:
  $$\mu : \Gamma^{(g)} \to J(X).$$

Let $Z_{J(X)} := (\mu, Id_X)_*(Z^{(g)}) \subset J(X) \times X$. Check that $\phi_{Z_{J(X)}} = Id_{J(X)}$. 
Second variant

Under our assumptions on $X$, any degree 4 class $\alpha \in H^4(X, \mathbb{Z})$ is Hodge. Then get a torsor $J(X)_\alpha$ in which the Deligne cycle class map $\phi_{X,D}$ on cod. 2 cycles of class $\alpha$ takes value. (Analogue of $Pic^d(C)$). Concretely, for any smooth variety $B$, and cod 2 cycle $Z_B \subset B \times X$ s.t. $Z_b$ is of class $\alpha$, get a morphism

$$\phi_{Z_B} : B \to J(X)_\alpha, \ b \mapsto \phi_{X,D}(Z_B,b).$$

**Question (Q3)**

*Does there exist a smooth projective variety $B_\alpha$ canonically defined up to birational transformations, and a cod. 2 cycle $Z_\alpha \subset B_\alpha \times X$, s.t. $Z_b$ is of class $\alpha$, $b \in B_\alpha$ and

$$\phi_{Z_\alpha} : B_\alpha \to J(X)_\alpha$$

is surjective with rationally connected general fibers?*

Eg. It could be that, if $X$ is rationally connected, for $\alpha$ sufficiently positive: ($B_\alpha=$Hilbert scheme of rational curves of class $\alpha$, $Z_\alpha=$universal curve) works (question by Jason Starr).
A motivation for variant (Q3)

If $B_\alpha$, $Z_\alpha$ are canonically defined, can put them in family.

- Let $\pi : \mathcal{X} \to \Gamma$, $\Gamma=$ smooth proj. curve. $\mathcal{X}$ smooth proj. fourfold, $\pi$ smooth over $\Gamma_0$.
- Let the generic fiber satisfy $H^2(\mathcal{O}_{\mathcal{X}_\eta}) = H^3(\mathcal{O}_{\mathcal{X}_\eta}) = 0$ (eg, $\mathcal{X}_t$ has $CH_0$ supported on a curve, $t \in \Gamma$ general).

This gives an algebraic family of abelian varieties $\mathcal{J} \to \Gamma_0$. For $\alpha$ section of $R^4\pi_*\mathbb{Z}$ over $\Gamma_0$, get twisted family $\mathcal{J}_\alpha \to \Gamma_0$.

- Assume $H^3(X_t, \mathbb{Z})$ has no torsion for any $t \in \Gamma_0$ and singular fibers of $\pi$ have at most ordinary quadratic singularities.

**Theorem (Colliot-Thélène-Voisin 2010)**

Assume for any section $\alpha$, there exists a family of codimension 2-cycles of class $\alpha$ in fibers of $\pi$:

$$B_\alpha \to \Gamma_0, \ Z_\alpha \subset B_\alpha \times_{\Gamma} \mathcal{X}$$

s.t. $\phi_{Z_\alpha} : B_\alpha \to \mathcal{J}_\alpha$ is surjective with rationally connected general fibers. Then the Hodge conjecture is true for integral Hodge classes of degree 4 on $\mathcal{X}$. 
• Use the theory of normal functions. A Hodge class $\beta$ on $\mathcal{X}$ induces a section $\alpha$ of $R^4\pi_*\mathbb{Z}$ which has a lift $\nu_\beta$ to an algebraic section of $J_\alpha$.

• By assumption, have $B_\alpha, Z_\alpha \subset B_\alpha \times \Gamma \mathcal{X}$, such that $\phi_{Z_\alpha} : B_\alpha \to J_\alpha$ is surjective with RC fibers. By Graber-Harris-Starr, $\nu_\beta$ has a lift to a section $\sigma : \Gamma \to B_\alpha$.

• Let $\mathcal{Z} := (\sigma, Id_{\mathcal{X}})^*Z_\alpha \subset \Gamma \times \Gamma \mathcal{X} = \mathcal{X}$. The normal function associated to $\mathcal{Z}$ is equal to $\nu_\beta$.

• As $H^3(X_t, \mathbb{Z})$ has no torsion, equality of normal functions implies that the degree 4 classes $\beta$ and $[\mathcal{Z}]$ agree on $\mathcal{X}_0 := \pi^{-1}(\Gamma_0)$.

• The difference $[\mathcal{Z}] - \beta$ comes then from homology of singular fibers $H_4(X_{t_i}, \mathbb{Z})$.

• Assumptions $H^2(X_t, \mathcal{O}_{X_t}) = 0 +$ singularities of $X_{t_i}$ are at worst nodes $\Rightarrow$ this homology is generated by homology classes of 2-cycles on $X_{t_i}$. Thus $[\mathcal{Z}] - \beta$ is algebraic and so is $\beta$. 

Proof of Theorem (sketch)
Application: cubic fibrations over curves

**Theorem (Voisin 2010)**

Let $X \to \Gamma$ be a smooth projective model of a cubic threefold in $\mathbb{P}^4(\mathcal{C}(\Gamma))$. Assume sing. fibers have at most ordinary quad. singularities. Then HC is true for integral Hodge classes of degree 4 on $X$.

Check hypotheses: cubic threefolds $X$ have trivial $CH_0$ (they are RC). No torsion in $H^3(X, \mathbb{Z})$ by Lefschetz. Note: $H^4(X, \mathbb{Z}) \cong \mathbb{Z}$ (degree).

**Theorem**

a) (Iliev-Markushevich 2002) The morphism induced by Abel-Jacobi map of $X$ is surjective with RC fibers for the families $B_4$ of degree 4 rational curves and the family $B_5$ of degree 5 elliptic curves on $X$.

b) (Voisin 2010) The morphism induced by Abel-Jacobi map is surjective with RC fibers for the family $B_6$ of degree 6 elliptic curves on $X$.

$\Rightarrow$ existence of $B_\alpha, Z_\alpha$ for all degrees. Indeed, use the cycle $h^2$ of degree 3 on fibers and its multiples to get then the result for all degrees.
Integral cohomological decomposition of the diagonal

Assume the existence of an integral cohomological decomposition

$$[\Delta_X] = [Z] + [Z']$$

with $Z$ supported on $D \times X$, $D \subseteq X$, $Z'$ supported on $X \times pt$. This implies that $H^i(X, \mathcal{O}_X) = 0$, $i > 0$ by applying $[\Delta_X]^*$ to $H^i(X, \mathcal{O}_X)$, noticing that $[Z]^* = 0$ on $H^i(X, \mathcal{O}_X)$. (=Bloch-Srinivas’ proof of Mumford’s theorem).

**Proposition (Voisin 2010)**

**Under this assumption, $X$ satisfies:**

a) $H^*(X, \mathbb{Z})$ has no torsion.

b) Positive answer to (Q1): there exists a codim 2 cycle $Z_J \subset J(X) \times X$ such that $\phi_{Z_J} : J(X) \to J(X)$ is $\text{Id}_{J(X)}$.

c) $H^4(X, \mathbb{Z})$ is generated over $\mathbb{Z}$ by classes of algebraic cycles.
Cycle classes act on integral cohomology and on Jacobians. 
$[\Delta]^*$ acts as identity on integral cohomology and on Jacobians. 
One has $[\Delta]^* = [Z]^*$ on $H^{* > 0}(X, \mathbb{Z})$ and on $J(X)$.

- For a), in degree 3, get that $Id_{H^3(X, \mathbb{Z})}$ factors through $H^1(\tilde{D}, \mathbb{Z})$. The later group has no torsion. Other degrees work similarly.

- For b), get that $Id_{J(X)}$ factors through $Z^*: J(X) \to Pic^0(\tilde{D})$. Here $j: \tilde{D} \to X$ is a desing. of $D$. $Z$ is lifted to a codim 2 cycle in $\tilde{D} \times X$. Let $\mathcal{D}$: universal divisor on $Pic^0(\tilde{D}) \times \tilde{D}$.
- Let $Z_J = (Id_{J(X)}, j)_*((Z^*, Id_{\tilde{D}})^*\mathcal{D}) \subset J(X) \times X$.
- Check that $\phi_{Z_J} = Id_{J(X)}$.

- For c), get for any $\alpha \in H^4(X, \mathbb{Z})$, by applying $[\Delta]^*$, that $\alpha = j_*(Z^*\alpha)$, where $Z$ is seen as a correspondence between $\tilde{D}$ and $X$. But $[Z]^*\alpha$ is a degree 2 integral Hodge class on $\tilde{D}$, hence algebraic by Lefschetz.
Partial converse

Assume $X$ = smooth proj. threefold with $H^i(X, \mathcal{O}_X) = 0$, $i > 0$. Hence the Hodge structures on $H^2(X, \mathbb{Z})$ and $H^4(X, \mathbb{Z})$ are trivial. $J(X)$ is a ppav.

**Theorem (Voisin 2010)**

Assume

i) $H^*(X, \mathbb{Z})$ has no torsion.

ii) The intermediate Jacobian $J(X)$ has a 1-cycle of class $\left[\Theta\right]^{g-1}/(g-1)!$.

iii) question (Q1) has affirmative answer for $X$, i.e. there is a codim 2 cycle $Z_J \subset J(X) \times X$ st. $Z_t$ cohomologous to 0 on $X$ for all $t$, with $\phi_{Z_J} = Id : J(X) \to J(X)$.

iv) $H^4(X, \mathbb{Z})$ is algebraic.

Then $X$ admits an integral cohomological decomposition of the diagonal.

**Remark**

When $X$ is a uniruled threefold with $H^2(X, \mathcal{O}_X) = 0$, it is known (Voisin 2006) that $H^4(X, \mathbb{Z})$ is algebraic, i.e. iv) holds.
Sketch proof

- For a topological manifold with no torsion in $H^*(X, \mathbb{Z})$, there is a Künneth decomposition of cohomology of $X \times X$. Thus $[\Delta_X] = \delta_{6,0} + \delta_{5,1} + \delta_{4,2} + \delta_{3,3} + \delta_{2,4} + \delta_{1,5} + \delta_{0,6}$.
- As $H^1(X, \mathcal{O}_X) = 0$, $\delta_{5,1} = \delta_{1,5} = 0$.
- As $H^2(X, \mathbb{Z})$ and $H^4(X, \mathbb{Z})$ are generated by cycle classes, both $\delta_{4,2}$ and $\delta_{2,4}$ are classes of algebraic cycles supported over $D \subsetneq X$. It only remains to construct a cycle $Z_3 \subset X \times X$ s.t. $Z_3$ is supported over some $D \subsetneq X$ and $[Z_3]$ acts as identity on $H^3(X, \mathbb{Z})$.
- There is a 1-cycle $\Gamma$ in $J(X)$ with class $[\Gamma] = \frac{\Theta^{g-1}}{(g-1)!}$. Assume for simplicity $\Gamma$ effective (so $J(X)$ is a Jacobian).
- There is $Z_J \subset J(X) \times X$ codim 2 cycle st. $\phi_{Z_J} = Id : J(X) \to J(X)$. Let $Z_\Gamma := Z_J|_{\Gamma \times X}$.
- Let $Z_3 := Z_\Gamma \circ t Z_\Gamma$. $Z_3$ is supported over a surface in $X$, as $t Z_\Gamma$. Check that $[Z_3]$ acts as identity on $H^3(X, \mathbb{Z})$. 