Rigid but not infinitesimally rigid compact complex manifolds

joint work with I. Bauer (Bayreuth)

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2. Our main results
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Rigid compact complex manifolds

Definition

A compact complex manifold M is *rigid* if for each deformation of M, $f\colon(\mathfrak{X}, M) \to (B, b_0)$ there is an open neighbourhood $U \subset B$ of b_0 such that $M_t := f^{-1}(t) \cong M$ for all $t \in U$.

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The situation in higher dimension is much more complicated: for example the Hirzebruch surface $\mathbb{F}_2$ is not rigid and homeomorphic to the rigid surface $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$.

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How do we check the rigidity of a manifold?

Let M be a compact complex manifold.

Kuranishi constructed a deformation $\pi\colon (\mathcal{X}, M) \to (\mathrm{Def}(M), 0)$ of M where $(\mathrm{Def}(M), 0)$ is a germ of analytic subspace of the vector space[1] $H^1(M, \Theta)$, inverse image of the origin under a local holomorphic map $k\colon H^1(M, \Theta) \to H^2(M, \Theta)$ whose differential vanishes[2] at the origin.

[1]Here Θ is the sheaf of holomorphic vector fields on M.

[2]Then $H^1(M, \Theta)$ is the Zariski tangent space of $(\mathrm{Def}(M), 0)$. In particular $(\mathrm{Def}(M), 0)$ is smooth if and only if $k = 0$, in which case we say that M has *unobstructed deformations*.

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Theorem (Kuranishi)

The Kuranishi family is semiuniversal, and universal if $H^0(M, \Theta) = 0$.

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Theorem (Kuranishi)

The Kuranishi family is semiuniversal, and universal if $H^0(M, \Theta) = 0$. The quadratic term in the Taylor development of k is given by the bilinear map $H^1(M, \Theta) \times H^1(M, \Theta) \to H^2(M, \Theta)$ called Schouten bracket, which is the composition of cup product followed by Lie bracket of vector fields.

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In particular $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ is (infinitesimally) rigid.

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Morrow-Kodaira's Problem

Morrow and Kodaira asked if the converse implication also hold[3]:

THEOREM 3.2. If $H^1(M, \Theta) = 0$, then M is rigid. We will give a proof of this using elementary methods. We have the following:

PROBLEM. Find an example of an M which is rigid, but $H^1(M, \Theta) \neq 0$. (Not easy?)

REMARK. $\mathbb{P}^n$ is rigid. For $n \geq 2$ the only known proof is to show $H^1(\mathbb{P}^n, \Theta) = 0$ [Bott (1957)]. Let us proceed to the proof.

A solution of the M-K Problem is a manifold M such that $\mathrm{Def}(M)$ is a *fat* point, a *singular* point.

[3] This is a screenshot of the book *Complex Manifolds* by James Morrow and Kunihiko Kodaira (1971), *Holt, Rinehart and Winston, Inc.*

The main result

Theorem

For every even $n \geq 8$ such that $3 \nmid n$ there is a minimal regular surface S_n of general type with

$$K^2_{S_n} = 2(n-3)^2, \quad p_g(S_n) = \left(\frac{n}{2} - 2\right)\left(\frac{n}{2} - 1\right),$$

such that S_n is rigid, but not infinitesimally rigid: $h^1(S_n, \Theta) = 6$.

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The canonical models of these surfaces have exactly 6 singular points, all nodes[4]. The hard part is proving their rigidity, since Kuranishi's rigidity criterium fails.

[4] A node is a singular point locally isomorphic to the quotient of a 2−dimensional disc by $p \mapsto -p$.

Generalization to higher dimension

Lemma

Let M, N be compact complex manifolds, such that

$$h^0(M,\Theta)h^1(N,\mathcal{O}) = h^0(N,\Theta)h^1(M,\mathcal{O}) = 0.$$

Then $\mathrm{Def}(M \times N) = \mathrm{Def}(M) \times \mathrm{Def}(N)$*.*

Then, if M is a regular surface of general type solving the M-K Problem and N is a rigid manifold, by Künneth formula $M \times N$ is a solution too.

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Then, if M is a regular surface of general type solving the M-K Problem and N is a rigid manifold, by Künneth formula $M \times N$ is a solution too. Using some known rigid manifolds[5] we obtain

Theorem

There are rigid manifolds of dimension d and Kodaira dimension κ that are not infinitesimally rigid for all possible pairs (d,κ) with $d \geq 5$ and $\kappa \neq 0, 1, 3$ and for $(d,\kappa) = (3,-\infty), (4,-\infty), (4,4)$.

[5] listed in Ingrid Bauer and Fabrizio Catanese, *On rigid compact complex surfaces and manifolds*, *Adv. Math.* **333**, 620–669 (2018).

We need M rigid: these are rare manifolds.

Theorem (Ingrid Bauer and Fabrizio Catanese, *On rigid compact complex surfaces and manifolds*, *Adv. Math.* **333**, 620–669 (2018).)

Let M be a smooth compact complex surface, which is rigid. Then either

1. *M is a minimal surface of general type, or*
2. *M is a Del Pezzo surface of degree $d \geq 5$*
3. *M is an Inoue surface of type S_M or $S^-_{N,p,q,r}$*

Rigid Del Pezzo and Inoue surfaces are infinitesimally rigid, so every surface solving M-K Problem has Kodaira dimension 2.

The minimal model of any rigid surface of general type whose canonical model is singular does the job.

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We need $\dim M \geq 2$. Several examples of manifolds M of dimension 2 with obstructed deformations are now known.

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Burns and Wahl[6] show how to associate to each smooth rational curve E with $E^2 = -2$ in a complex surface M a $1-$dimensional subspace $H^1_E(M)$ of $H^1(M, \Theta)$.

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Note that in particular if M is the minimal resolution of the singularities of a *nodal* surface[7], M can't be infinitesimally rigid.

[6] D. M. Burns Jr. and Jonathan M. Wahl, *Local contributions to global deformations of surfaces*, *Invent. Math* **26**, 67 – 88 (1974).

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A necessary condition for M to be rigid is that it is obstructed along this line: $H^1_E(M) \not\subset \mathrm{Def}(M)$. A way to check it has been provided by Kas[8].

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[8] Arnold Kas, *Ordinary double points and obstructed surfaces*, *Topology* **16** (1), 51 – 64 (1977).

The Kas maps

Let now X be a compact complex surface with a node ν, $M \to X$ be the minimal resolution of singularities of M, let E be exceptional curve mapping to ν and let θ be a generator of $H^1_E(M) \subset H^1(M, \Theta)$.

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Then we can write

$$k(t\theta) = \alpha_\nu t^2 + O(3) \in H^2(M, \Theta),$$

where, by Serre duality we can see α_ν as a map $\alpha_\nu \colon H^0(M, \Omega^1 \otimes \Omega^2) \to \mathbb{C}$.
A neighbourhood of ν in X is the quotient of a disc $\Delta \subset \mathbb{C}^2$ by the involution $(z_1, z_2) \mapsto (-z_1, -z_2)$.
Pulling-back we get an inclusion $H^0(M, \Omega^1 \otimes \Omega^2) \subset H^0(\Delta, \Omega^1 \otimes \Omega^2)^+$ allowing to write locally every $\eta \in H^0\left(M, \Omega^1 \otimes \Omega^2\right)$ as

$$\eta = (f_1 dz_1 + f_2 dz_2) \otimes (dz_1 \wedge dz_2)$$

Then Kas shows

$$\alpha_\nu(\eta) = \left(\frac{\partial f_2}{\partial z_1} - \frac{\partial f_1}{\partial z_2}\right)(0,0). \quad (1)$$

Interesting examples with obstructed deformations

1. Burns and Wahl construct[9] many examples of smooth surfaces with obstructed deformations by resolving the singularities of certain nodal hypersurfaces in $\mathbb{P}^3$.
2. Catanese[10] constructs surfaces M whose Kuranishi family $\mathrm{Def}(M)$ is *everywhere nonreduced* by resolving the singularities of certain quotients $(C_1 \times C_2)/G$ (C_i curves, G finite group) with rational double points.

Still, all these examples are not rigid.

[9] D. M. Burns Jr. and Jonathan M. Wahl, *Local contributions to global deformations of surfaces*, *Invent. Math* **26**, 67 – 88 (1974).

[10] Fabrizio Catanese, *Everywhere nonreduced moduli spaces*, *Invent. Math.* **98** (2), 293 – 310 (1989).

Strategy to the proof of the main theorem

Rigid manifolds are rare. I know a short list of examples of rigid surfaces of general type, all infinitesimally rigid: ball quotients, irreducible bi-disk quotients, Beauville surfaces, Mostow-Siu surfaces, some Kodaira fibrations constructed by Catanese and Rollenske.

Example (Beauville surfaces)

Consider two projective curves C_1, C_2, a finite group G and two injective homomorphisms $G \subset Aut(C_i)$.
Assume that the induced action $g(x, y) = (gx, gy)$ of G on $C_1 \times C_2$ is free. Then $M := (C_1 \times C_2)/G$ is smooth.
If (C_i, G) are *triangle curves*[a], then M is a *Beauville surface*.

[a] *i.e.* $C_i/G \cong \mathbb{P}^1$ and $p_i \colon C_i \to C_i/G$ has exactly three branching points.

Catanese's lemma

Lemma (Fabrizio Catanese, *Everywhere nonreduced moduli spaces*, *Invent. Math.* **98** (2), 293 – 310 (1989))

Let Z be a smooth algebraic surface and G a finite group acting on it freely in codimension 1. Set $p\colon Z \to X := Z/G$.
Then $H^1(X,\Theta) \cong H^1(Z,\Theta)^G$.

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Corollary

Consider two projective curves C_1, C_2, a finite group G and two injective homomorphisms $G \subset Aut(C_i)$. Set $X := (C_1 \times C_2)/G$.
If (C_i, G) are triangle curves, *then $H^1(X,\Theta) = 0$.*

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If (C_i, G) are triangle curves, *then $H^1(X,\Theta) = 0$.*

This implies that Beauville surfaces are infinitesimally rigid.
Note however that here G may act not freely, and then X has isolated singularities. Then the minimal resolution M of the singularities of X may still be neither rigid nor infinitesimally rigid.

A criterion to prove rigidity

Theorem

Let M be the minimal res. of the sing. of a nodal surface X. Assume that

1. $H^1(X, \Theta) = 0$;
2. *the maps α_{ν_i} associated to the nodes ν_i of X locally described in (1) are linearly independent in $H^0(M, \Omega^1 \otimes \Omega^2)^\vee$.*

Then M is rigid and $h^1(M, \Theta)$ equals the number of nodes of X.

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Sketch of the proof.

By condition 1 and a remark of Pinkham[a] $H^1(M,\Theta)\cong\bigoplus H^1_{E_i}(M)$.
Choose $0\neq\theta_i\in H^1_{E_i}(M)$: they form a basis of $H^1(M,\Theta)$. Then
$k(\sum t_i\theta_i)=\sum_1^r t_i^2\alpha_{\nu_i}+O(3)$. The rigidity follows now by condition 2. □

[a] Henry Pinkham, *Some local obstructions to deforming global surfaces*, *Nova Acta Leopoldina (N.F.)* **52** (1981), 173-178.

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We pick two triangle curves (C_1, G), (C_2, G) for the same finite group, we set $X := (C_1 \times C_2)/G$ the quotient by the diagonal action.

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We need X to be nodal and have at least one node. In other words, there is at least one point of $C_1 \times C_2$ whose stabilizer has order 2, and no points whose stabilizer has a higher order.

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Then by Catanese's Lemma the first condition in the rigidity criterion $H^1(X, \Theta) = 0$ is fulfilled, and we only need to check the second one.

The Fermat curves

Which triangle curves do the job?

The Fermat curve of degree n, $C := \{\sum_{j=0}^{2} x_j^n = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$ admits a natural action of the group $G \cong (\mathbb{Z}/n\mathbb{Z})^2$:

$$(a_1, a_2)(x_0 : x_1 : x_2) = (x_0 : e^{a_1 \frac{2\pi i}{n}} x_1 : e^{a_2 \frac{2\pi i}{n}} x_2).$$

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This $G-$action has only three orbits of cardinality different by n^2, all of cardinality n:

- $C \cap \{x_0 = 0\}$ with stabilizer $\langle(1,1)\rangle \cong \mathbb{Z}/n\mathbb{Z}$
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By Hurwitz formula $C/G \cong \mathbb{P}^1$ so (C, G) is a triangle curve.

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For the first copy (C_1, G) we take the action as described in the previous slide, whereas for the second copy (C_2, G) we twist the action by the matrix

$$A := \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

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that for all n not divisible by 3 defines an automorphism of G.

The surface S_n is the minimal resolution of the singularities of $(C_1 \times C_2)/G$.

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For n odd, the intersection of these two sets is $\{(0,0)\}$. Then the induced action on $C_1 \times C_2$ is free: S_n is a Beauville surface, infinitesimally rigid. Then we need to assume n even.

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For n even the non-trivial elements of G fixing some points of $C_1 \times C_2$ are $(n/2, 0)$, $(0, n/2)$ and $(n/2, n/2)$, all of order 2 fixing n^2 points: then X is a nodal surface with $3 \cdot 2 \cdot n^2/n^2 = 6$ nodes.

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The proof

We skip the computation of the invariants of S_n, that is standard.

We need to check if the six maps

$$\alpha_{\nu_i} \colon H^0(S_n, \Omega^1 \otimes \Omega^2) \to \mathbb{C}$$

associated to the nodes of X are linearly independent.

For this we need $h^0(S_n, \Omega^1 \otimes \Omega^2) \geq 6$: indeed this excludes the case $n \leq 4$ giving $n \geq 8$.

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When $h^0(S_n, \Omega^1 \otimes \Omega^2) \geq 6$ may to check the idependence of the α_{ν_i} by restricting to a suitable $6-$dimensional subspace.

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Decomposition of $H^0(S_n, \Omega^1 \otimes \Omega^2)$

We need a basis of $H^0(S_n, \Omega^1 \otimes \Omega^2)$ as explicit as possible, in order to be able to compute their image via the Kas map. By

$$H^0(S_n, \Omega^1 \otimes \Omega^2) \cong H^0(C_1 \times C_2, \Omega^1 \otimes \Omega^2)^G \cong$$
$$\cong \left(H^0(C_1, \omega_{C_1}^{\otimes 2}) \otimes H^0(C_2, \omega_{C_2})\right)^G \oplus \left(H^0(C_1, \omega_{C_1}) \otimes H^0(C_2, \omega_{C_2}^{\otimes 2})\right)^G \cong$$
$$\cong \bigoplus_{\chi \in G^*} \left(\left(H^0(\omega_{C_1}^{\otimes 2})^\chi \otimes H^0(\omega_{C_2})^{-\chi}\right) \oplus \left(H^0(\omega_{C_1})^\chi \otimes H^0(\omega_{C_2}^{\otimes 2})^{-\chi}\right)\right) \cong$$
$$\cong \bigoplus_{\chi \in G^*} \left(\left(H^0(\omega_{C_1}^{\otimes 2})^\chi \otimes H^0(\omega_{C_1})^{\chi'}\right) \oplus \left(H^0(\omega_{C_1})^\chi \otimes H^0(\omega_{C_1}^{\otimes 2})^{\chi'}\right)\right)$$

where. writing χ, χ' as a column, $\chi' := -{}^tA^{-1}\chi$

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Six good characters are enough

Lemma

Set $k_0 = k_1 = (1,0)$, $k_\infty = (0,1) \in G$. *Assume that there is a set of six characters* $\mathcal{C} := \{\chi_0, \chi_0', \chi_1, \chi_1', \chi_\infty, \chi_\infty'\} \subset G^*$, *such that*

1. $\chi_0 \equiv \chi_0' \equiv (0,1)$, $\chi_1 \equiv \chi_1' \equiv (1,1)$, $\chi_\infty \equiv \chi_\infty' \equiv (1,0)$ *mod* 2;
2. $\forall p \in \{0,1,\infty\}$, $\chi_p(k_p) \neq \chi_p'(k_p)$;
3. *if* $\chi \in \mathcal{C}$, *then* $H^0(\omega_C)^{(\chi)} \neq \{0\}$, $H^0(\omega_C^{\otimes 2})^{(\chi')} \neq \{0\}$.

Then condition 2 in the rigidity criterion holds.

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Sketch of the proof - part 1

We need to check the linear independence of the six maps α_{ν_j}.
We decomposed $H^0(S_n, \Omega^1 \otimes \Omega^2)$ obtaing addenda of the form $H^0(\omega_{C_1})^\chi \otimes H^0(\omega_{C_1}^{\otimes 2})^{\chi'}$. When $\chi \in \mathcal{C}$, by condition 3, the addendum is not trivial. Picking one general element in each of them, we get six different elements in $H^0(S_n, \Omega^1 \otimes \Omega^2)$.

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3. *if* $\chi \in \mathcal{C}$, *then* $H^0(\omega_C)^{(\chi)} \neq \{0\}$, $H^0(\omega_C^{\otimes 2})^{(-\chi')} \neq \{0\}$.

Then condition 2 in the rigidity criterion holds.

Sketch of the proof - part 2.

Computing explicitely the six Kas maps (1) in them we get the following

$$\begin{array}{ccc} (\chi_0(k_0),1,0,0,0,0) & (0,0,\chi_1(k_1),1,0,0) & (0,0,0,0,\chi_\infty(k_\infty),1) \\ (\chi'_0(k_0),1,0,0,0,0) & (0,0,\chi'_1(k_1),1,0,0) & (0,0,0,0,\chi'_\infty(k_\infty),1) \end{array}$$

□

Decomposition of $p_*\omega_C$

The Fermat triangle curve $p\colon C = C_1 \to \mathbb{P}^1$ is an abelian cover, with group G. We compute the decomposition of $p_*\omega_C$ by Pardini's[11] formula

n-1	-1	-1	-1	-1	-1	*	-1	-1	-1	-1	-1
n-2	-1	0	-1	-1	-1	*	-1	-1	-1	-1	-1
	-1	0	0	-1	-1	*	-1	-1	-1	-1	-1
	-1	0	0	0	-1	*	-1	-1	-1	-1	-1
	-1	0	0	0	0	*	-1	-1	-1	-1	-1
	*	*	*	*	*	*	*	*	*	*	*
	-1	0	0	0	0	*	0	-1	-1	-1	-1
3	-1	0	0	0	0	*	0	0	-1	-1	-1
2	-1	0	0	0	0	*	0	0	0	-1	-1
1	-1	0	0	0	0	*	0	0	0	0	-1
0	-2	-1	-1	-1	-1	*	-1	-1	-1	-1	-1
	0	1	2	3							n-1

Figure: The degrees of $(p_*\omega_C)^{(\alpha,\beta)}$

[11] Rita Pardini, *Abelian covers of algebraic varieties*, *J. Reine Angew. Math.* 417, 191-213.

Decomposition of $p_*\omega_C^{\otimes 2}$

Similarly we compute the decomposition $p_*\omega_C^{\otimes 2}$.

n-1	-1	-1	0	0	0	*	0	0	0	0	0
n-2	0	-1	0	0	0	*	0	0	0	0	0
n-3	0	0	0	0	0	*	0	0	0	0	0
	0	0	1	0	0	*	0	0	0	0	0
	0	0	1	1	0	*	0	0	0	0	0
	*	*	*	*	*	*	*	*	*	*	*
	0	0	1	1	1	*	0	0	0	0	0
3	0	0	1	1	1	*	1	0	0	0	0
2	0	0	1	1	1	*	1	1	0	0	0
1	-1	-1	0	0	0	*	0	0	0	-1	-1
0	-1	-1	0	0	0	*	0	0	0	0	-1
	0	1	2	3							n-1

Figure: The degrees of $(p_*\omega_C^{\otimes 2})^{(\alpha,\beta)}$

Note that $\forall n \geq 4$ the degree is negative only for 10 characters.

End of the proof of the main theorem

We need then to find six characters such that

1. $\chi_0 \equiv \chi'_0 \equiv (0,1)$, $\chi_1 \equiv \chi'_1 \equiv (1,1)$, $\chi_\infty \equiv \chi'_\infty \equiv (1,0) \bmod 2$;
2. $\forall p \in \{0,1,\infty\}$, $\chi_p(k_p) \neq \chi'_p(k_p)$;
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3. if $\chi \in \mathcal{C}$, then $H^0(\omega_C)^{(\chi)} \neq \{0\}$, $H^0(\omega_C^{\otimes 2})^{(\chi')} \neq \{0\}$.

End of the proof.

We pick the following six characters

$$\chi_0 = (2,1) \qquad \chi_1 = (1,3) \qquad \chi_\infty = (1,2)$$
$$\chi'_0 = (4,1) \qquad \chi'_1 = (3,1) \qquad \chi'_\infty = (1,4)$$

The only check that is not trivial is that $H^0(\omega_C^{\otimes 2})^{(\chi')} \neq \{0\}$: this indeed fails for $n = 4$ but a tedious computation shows that it holds for $n \geq 8$. □

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Problem (4)

Construct rigid surfaces to which our criterion does not apply.

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