Quasi-étale quotients of products of two curves

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8 June 2012
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How do we check that a surface is "new"?
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We look at surfaces modulo birational equivalence, the equivalence relation generated by the blow-up in a point.
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- the geometric genus
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- The (topological or algebraic) fundamental group.
Surfaces of general type

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In every birational class of surfaces of general type there is exactly one *minimal* surface. If $S$ is a surface of general type, $S$ is obtained by the only minimal surface in its birational class $\tilde{S}$ by a sequence of $K_S^2$ blow ups.
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Inequalities for surfaces of general type
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If $S$ is of general type, then the Riemann-Roch formula computes all $P_n(S)$ from $\chi$ and $K_S^2$. The possible values of the pair $(\chi, K_S^2)$ are almost all the integral points of the unbounded green region below.
In general, the surfaces with the most interesting geometry are the ones when "the inequalities are equalities", as for the boundary of the picture. This includes the surfaces with $\chi = 1$ and, among those, the surfaces with $p_g = 0$. 

\[
\begin{align*}
&K^2 \\
&\chi-1 \\
&K^2-9\chi \\
&K^2-2\chi-6 \\&K^2=1
\end{align*}
\]
Beauville’s idea

Beauville suggestion: take $S = (C_1 \times C_2)/G$ where $C_i$ are Riemann Surfaces of genus $g_i \geq 2$ and $G$ is a free group of automorphisms of order $(g_1 - 1)(g_2 - 1)$; $S$ is automatically minimal of general type with $\chi = 1$ and $K^2 = 8$. 
A surface is isogenous to a (higher) product if \( S = (C_1 \times C_2)/G \) where \( C_i \) are Riemann Surfaces of genus \( g_i \geq 2 \) and \( G \) is a free group of automorphisms; \( S \) is automatically minimal of general type, with \( K^2 = 8 \chi \).
A *quasi-étale surface* is

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where \( C_i \) are Riemann Surfaces of genus \( g_i \geq 2 \) and \( G \) is a group of automorphisms acting freely out of a finite set of points.
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\textbf{Disadvantages:}

- $X$ is singular, we need to consider a resolution of its singularities $S$.
- We lose every rigidity property.
Quasi-étale quotients

Advantage: it may be $K^2 < 8\chi$. We may in principle fill most of the picture. This gives a powerful tool to answer (positively) existence conjectures.
Mixed and unmixed structures

We know that

- either $\text{Aut}(C_1 \times C_2) = \text{Aut}(C_1) \times \text{Aut}(C_2)$,
- or $C_1 \cong C_2 \cong C$ and $\text{Aut}(C^2) \cong (\text{Aut}(C))^2 \rtimes \mathbb{Z}/2\mathbb{Z}$.
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Following Catanese, for \( G < \text{Aut}(C_1 \times C_2) \) we define \( G^{(0)} = G \cap (\text{Aut}(C_1) \times \text{Aut}(C_2)) \). There are two possibilities

- either \( G = G^{(0)} \) (the \textit{unmixed case}, the case of the \textit{product-quotient surfaces}, the \textit{standard isotrivial fibrations});
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- either $G = G^{(0)}$ (the *unmixed case*, the case of the *product-quotient surfaces*, the *standard isotrivial fibrations*);
- or (*mixed case*) there is an exact sequence

\[(\#) \quad 1 \to G^{(0)} \to G \to \mathbb{Z}/2\mathbb{Z} \to 1.\]
Theorem (Frapporti)

\( \pi \) is not quasi-étale if and only if \( G^{(0)} \cong G \) and

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Mixedness and quasi-étaleness

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Assuming quasi-étale we consider a class larger than the class of the standard isotrivial fibrations (=the unmixed quasi-étale surfaces). The quotient surfaces we are excluding are dominated by the symmetric product of a curve.
By Riemann Existence Theorem, to give an action of a group $G^{(0)}$ on a curve $C$ is equivalent to give
- the curve $C/G^{(0)}$;
Constructing curves with group actions

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For later use: to each suitable system of generators we associate its signature, which is the unordered list of the orders of some of these generators. The genus of $C$ is a function (Hurwitz' formula) of $|G^{(0)}|$, the signature, and the genus of $C/G^{(0)}$. 

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Constructing quotients

To construct an unmixed surface I need two curves with an action of the same group, so two systems of generators of the same group $G = G^{(0)}$. 
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$$\begin{align*}
g(x, y) &= (gx, \tau' g \tau'^{-1} y) \quad \forall g \in G^{(0)} \\
\tau' g(x, y) &= (\tau' g \tau'^{-1} y, \tau'^2 gx) \quad \forall g \in G^{(0)}
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and all mixed actions come in this way.
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and all mixed actions come in this way. Moreover, different choices of $\tau'$ give isomorphic constructions.
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The easier formula is for the irregularity:

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\begin{cases}
q(S) = g(C_1/G) + g(C_2/G) & \text{in the unmixed case} \\
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To compute the other invariants we need a better understanding of the singularities of \( X = (C_1 \times C_2)/G \).
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- in the unmixed case $X$ has only cyclic quotient singularities, locally biregular to the quotient of $C^2$ by the automorphism \( \begin{pmatrix} \omega & 0 \\ 0 & \omega^q \end{pmatrix} \) where $\omega$ is a $n$-th primitive root of 1, $0 < q < n$ and $(q, n) = 1$. We say that these singularities are of type $C_{n,q}$.
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- In the mixed case we have an intermediate unmixed quotient $Y = C^2 / G^{(0)}$ and an involution $i$ on $Y$ with $Y/i = X$. $\text{Sing}X$ is determined by $\text{Sing}Y$ and the action of $i$ on it:
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The exceptional divisor of the minimal resolution of a singularity \( C_{n,q} \) is a chain of rational curves \( A_1, \ldots, A_k \) with self intersections \(-b_1, \ldots, -b_k\) given by the continued fraction:

\[
\frac{n}{q} = \left[ b_1, \ldots, b_k \right] = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \ldots}}.
\]

The dual graph is

\[
\begin{array}{ccc}
  -b_1 & -b_2 & -b_{k-1} & -b_k \\
  \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

If \( qq' \equiv 1 \mod n \), then \( \frac{n}{q'} = \left[ b_k, \ldots, b_1 \right] \) and therefore \( C_{n,q} \cong C_{n,q'} \).
Proposition (Frapporti)

If $P \in Y$ is a fixed point of $Y$, then $P$ is a singular point of type $C_{n,q}$ with $q^2 \equiv 1 \pmod{n}$, and the lift of $i$ to a resolution of the singularity exchanges the ends of the string

\[-b_1 \quad -b_2 \quad -b_2 \quad -b_1\]

The resolution graph of a singularity of type $D_{n,q}$ is

\[
-k = 2h + 1
\]

\[-b_1 \quad -b_2 \quad -(1 + \frac{b_{h+1}}{2}) \quad -2 \quad -2\]
There are explicit formulas

\[ K_S^2 = \frac{8(g(C_1) - 1)(g(C_2) - 1)}{|G|} - \sum_{x \in \text{Sing}X} k_x \]

\[ e(S) = \frac{4(g(C_1) - 1)(g(C_2) - 1)}{|G|} + \sum_{x \in \text{Sing}X} e_x = 12\chi - K_S^2 \]

where \( k_x \) and \( e_x \) are positive rational numbers depending only on the type of the singularity. It follows

\[ K_S^2 = 8\chi - \sum_x \frac{2e_x + k_x}{3} \leq 8\chi. \]
The algorithms: idea

Now we are able to construct every quasi-étale surface and compute its invariants $p_g$, $q$ and $K^2_S$ (which is often but not always equal to $K^2_S$).
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We are interested in the inverse procedure: if we are interested in constructing surfaces with certain $p_g$, $q$ and $K^2$, what can we do? Reversing the above formula we can compute by them

- the possible $g(C_i/G^{(0)})$ (by $q$);
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Reversing the above formula we can compute by them

- the possible $g(C_i/G^{(0)})$ (by $q$);
- $\sum_x 2e_x + k_x = 24\chi - 3K_S^2$: there are finitely many possible configurations ("baskets") of singularities for each value of this;
The algorithms: idea

Now we are able to construct every quasi-étale surface and compute its invariants $p_g$, $q$ and $K_S^2$ (which is often but not always equal to $K_S^2$). We can compute $\pi_1(S)$ by a lemma of Armstrong.

We are interested in the inverse procedure: if we are interested in constructing surfaces with certain $p_g$, $q$ and $K^2$, what can we do?

Reversing the above formula we can compute by them

- the possible $g(C_i/G^{(0)})$ (by $q$);
- $\sum_x 2e_x + k_x = 24\chi - 3K_S^2$: there are finitely many possible configurations ("baskets") of singularities for each value of this;
- Hurwitz formula yields an equation involving $|G|$, $K_S^2$, $\sum k_x$, $g(C_i/G^{(0)})$ and the "signatures" of the actions of $G^{(0)}$ on the $C_i$. 
The algorithms: procedure

We proved some inequalities to bound the possible signatures.
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1. find all (finitely many) possible pairs of genera $(g(C_1/G(0)), g(C_2/G(0)))$ (equal in the mixed case) and configurations ("baskets") of singularities with $\sum x(2e_x + k_x) = 24 - 3K_S^2$; 
2. for every "basket" and pair of genera, list all "signatures" satisfying those inequalities; 
3. to each (pair of) signature(s), search all groups $G(0)$ of the order prescribed by the Hurwitz formula for set of generators of the prescribed signatures; 
4. in the mixed case, consider all the unsplit degree 2 extensions of $G(0)$; 
5. check the singularities of the surfaces in the output.
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**Theorem (Bauer, Catanese, Grunewald, -)**

*Unmixed quasi-étale surfaces of g. t. with $p_g = 0$ form*

1. exactly 13 irreducible families of surfaces for the case in which $G$ acts freely: they form 13 irreducible connected components of the moduli space;
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Similar results have been obtained by the same authors mentioned before for $p_g = q \geq 1$ only in the étale case.
Some corollaries

The Campedelli surfaces are the min. surf. of g. t. with $p_g = 0, K^2 = 2$.

**Conjecture**

The possible $\pi_1$ of the Campedelli surfaces are all abelian groups of order $\leq 9$ and the quaternion group.

This is now proved for $\pi_1^{alg}$ (Reid+...).
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**Corollary (1)**

*There are Campedelli surfaces with $\pi_1$ equal $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$.*

Park, Park and Shin found similar results for $\pi_1^{\text{alg}}$. 
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Corollary (2)

Minimal surfaces of general type with $p_g = 0$, $3 \leq K^2_S \leq 6$ realize at least 47 topological types.
Problem

*We have to run a search on all groups of a given order: sometimes there are too many even for a computer, sometimes we do not have a complete list of them. We used some group theory to exclude the cases that the computer could not do.*
Algorithmic problems

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*The algorithm is very time and memory consuming. We need some help in computational algebra to get results for different values of the invariants.*
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Theoretical problems

Problem

*How do we determine the minimal model of $S$?*
Theoretical problems

Problem

How do we determine the minimal model of $S$?

And, related to it is

Problem

Can we find all quasi-étale surfaces of general type with $p_g = 0$, or, more generally, with given $p_g$ and $q$?

If we could find an explicit bound $K_S^2 \geq k(p_g, q)$...
Searching rational curves

To answer the last questions we need to study the rational curves on a quasi-étale surface.

We would like to be able to locate all exceptional curves of the first kind (if any).
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Rational curves on $S$ are

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Searching rational curves

To answer the last questions we need to study the rational curves on a quasi-étale surface.

We would like to be able to locate all exceptional curves of the first kind (if any).

**Remark**

Rational curves on $S$ are

- either exceptional for the resolution $S \to X$
- or pass through the singular points of $X$ at least three times.
In this example $p_g(S) = q(S) = 1$, $K_S^2 = 1$ and the basket of singularities is $\{ \frac{1}{7}, 2 \times \frac{2}{7} \}$. 
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In this case, all exceptional divisors for $S \to X$ are mapped to the same point $p \in \alpha(S)$, so all rational curves are in $\alpha^{-1}(p)$. 
Mistretta-Polizzi’s example

In this example $p_g(S) = q(S) = 1$, $K_S^2 = 1$ and the basket of singularities is $\{\frac{1}{7}, 2 \times \frac{2}{7}\}$. Since $S$ is irregular, the Albanese map $\alpha$ contracts all rational curves. In this case, all exceptional divisors for $S \to X$ are mapped to the same point $p \in \alpha(S)$, so all rational curves are in $\alpha^{-1}(p)$. $\alpha^{-1}(p)$ is made of rational curves, with dual graph

```
     -7
    /   \
   /     \
-1      -2  -4
   \     / \
     \ /   \
-4      -2
   \   /  \
    \ /   \
     \ 
```

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In this example $p_g(S) = q(S) = 1$, $K_S^2 = 1$ and the basket of singularities is $\left\{ \frac{1}{7}, 2 \times \frac{2}{7} \right\}$.

Then the minimal model has $K_S^2 = 3$. This strategy works in every irregular case.

Question

Can we use this argument to get an inequality $K^2 \geq k(p_g, q)$ for the irregular case?
Assume that the singularities are mild, for example just $k$ nodes. Then $S$ has $k (-2)$ curves, every further rational curve should meet them at least three times.
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If we had a $(-1)$ curve, contracting them I would get either two $(-1)$ curves intersecting, or a singular rational curve intersecting negatively the canonical system. This implies that the surface is rational.
How to prove the minimality in the regular case

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If the surface is not simply connected, I have a contradiction, and the surface is minimal.
The fake Godeaux surface

- $G = \text{PSL}(2, 7)$;
- $\Psi_1 : \mathbb{T}(7, 3, 3) \to G$, $\Psi_2 : \mathbb{T}(7, 4, 2) \to G$;
- $p_g(S) = 0$, $K_S^2 = 1$, $\pi_1(S) = \mathbb{Z}/6\mathbb{Z}$;
- $\mathcal{B}(X) = \{\frac{1}{7}, 2 \times \frac{2}{7}\}$.

How do we find the $(-1)$-curves?
The first exceptional curve

\[ \text{branch pts of } \hat{f}_1 \]
\[ \hat{C}_1 \xrightarrow{\xi} C_1 \]
\[ (7, 7, 7) \]

\[ \text{branch pts of } f_1 \]
\[ \bar{C}_2 \xrightarrow{\eta} \bar{C}_2 \]
\[ (7, 3, 3) \]
The first exceptional curve

\[ \begin{array}{ccc}
\text{branch pts of } \hat{f}_1 & \xrightarrow{\hat{C}_1} & \text{branch pts of } f_1 \\
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\end{array} \]

\[ \begin{array}{ccc}
\text{branch pts of } \hat{f}_2 & \xrightarrow{\hat{C}_2} & \text{branch pts of } f_2 \\
(7, 7, 7) & \xrightarrow{\hat{f}_2} & (7, 4, 2)
\end{array} \]
Proposition

1. \((\hat{C}_1, \hat{f}_1)\) and \((\hat{C}_2, \hat{f}_2)\) are isomorphic as \(G\)-covers of \(\mathbb{P}^1\) (hence we write \(\hat{C} := \hat{C}_1 = \hat{C}_2\)).

2. The curve
\[
C' := (\hat{\xi}, \hat{\eta})(\hat{C}) \subset C_1 \times C_2,
\]
is \(G\)-invariant and the quotient is a rational curve \(D' \subset X\).
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   \[ C' := (\hat{\xi}, \hat{\eta})(\hat{C}) \subset C_1 \times C_2, \]
   is \(G\)-invariant and the quotient is a rational curve \(D' \subset X\).

3. The strict transform \(E'\) of \(D'\) is a \((-1)\)-curve on \(S\).
The second $(-1)$-curve on $S$

$$\begin{align*}
(7, 7, 7, 7) & \longrightarrow (7, 7, 7) \longrightarrow (7, 3, 3) \\
\mathbb{P}^1 & \longrightarrow \mathbb{P}^1 \\
\mathbb{P}^1 & \longrightarrow \mathbb{P}^1
\end{align*}$$
The second $(-1)$-curve on $S$

\[
(7, 7, 7, 7) \to (7, 7, 7) \to (7, 3, 3)
\]

\[
P^1 \overset{(2:1)}\to P^1 \overset{(3:1)}\to P^1
\]

\[
(7, 7, 7, 7) \to (7, 4, 2)
\]

\[
P^1 \overset{(4:1)}\to P^1
\]
The second \((-1)\)-curve on \(S\)

\[(7, 7, 7, 7) \to (7, 7, 7) \to (7, 3, 3)\]

\[\mathbb{P}^1 \to \mathbb{P}^1 \to \mathbb{P}^1\]

\[(7, 7, 7, 7) \to (7, 4, 2)\]

\[\mathbb{P}^1 \to \mathbb{P}^1\]

Proposition

The two \(G\)-coverings (with branching indices \((7, 7, 7, 7)\)) of \(\mathbb{P}^1\) are isomorphic, and give a further \((-1)\)-curve on \(S\).
The rational curves we have found on $S$ (5 from the resolution, 2 from the above construction) have dual graph

```
  -7  -1
   |   |
   v   v
-1   -4 -2
   |   |
   v   v
-4   -2
   |   |
   v   v
   -4
```
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\begin{align*}
-7 & \quad -1 \\
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\]

Exercise: the surface obtained by contracting the two $(-1)$-curves is minimal.
We have 73 families of unmixed quasi-étale surfaces with $p_g = 0$ and $K^2 > 0$; 72 families of minimal surfaces, and the fake Godeaux.
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By inspecting the list, we noticed that all the minimal surfaces have $H^2(X) \cong \mathbb{C}^2$, generated by the classes of the fibres of the two fibrations.
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Question

Is there a reason for that?
For sake of simplicity we assume from now on $q = 0$ and unmixedness.

**Proposition**

Let $X := (C_1 \times C_2)/G$ be the quotient model of an unmixed quasi-étale surface. Then

- $\dim H^2(X) \geq 2$,
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Hodge theoretic information

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- $\dim H^2(X) \geq 2$,  
- $\dim H^2(X) \equiv 0 \mod 2$,

Let $\sigma : S \to X$ be the minimal resolution of the singularities of $X$. Then $H^2(S, \mathbb{C}) \cong H^2(X, \mathbb{C}) \oplus \mathbb{C}^l$, where $l = \text{numb. of irr. comp.s of } \text{Exc}(\sigma)$.  


Global definition of $\gamma$

Let $X := (C_1 \times C_2)/G$ be the quotient model of an unmixed quasi-étale surface with $q = 0$. We set

$$\gamma(X) := \frac{h^2(S, \mathbb{C}) - l}{2} - 1 - 2p_g(S) \in \mathbb{Z}$$
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$\gamma \geq -p_g$. Indeed $\gamma + p_g$ is half of the codimension in $H^{1,1}(S)$ of the subspace generated by the classes we know (fibres + exceptional).
Lemma

\(\gamma\) depends only on the basket of \(X\). More precisely

\[
\gamma(X) = \sum_{x \in \mathcal{B}(X)} \gamma_x
\]

where, for a singular point of type \(C_{n,q}\) with \(\frac{n}{q} = [b_1, \ldots, b_l]\),

\[
\gamma_x = \frac{1}{6} \left( \frac{q + q'}{n} + \sum_{i=1}^{l} (b_i - 3) \right),
\]

where \(1 \leq q' \leq n - 1\) and \(qq' \equiv 1 \mod n\).
Local definition of \( \gamma \)

**Lemma**

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\]

where \( 1 \leq q' \leq n - 1 \) and \( qq' \equiv 1 \mod n \).

**Remark**

\[
K_S^2 = 8 \chi - 2\gamma - l.
\]

We have implemented a similar algorithm constructing all product-quotient surfaces with \( q = 0 \), given \( pg \), and \( \gamma \) (and looks much quicker than the other one!)
Let $S$ be a product-quotient surface with quotient model

$$X = (C_1 \times C_2)/G.$$ 

We assume furthermore that $S$ is regular, i.e., $q(S) = 0$. 

The dual surface

**The dual surface**
The dual surface

Let $S$ be a product-quotient surface with quotient model

$$X = (C_1 \times C_2)/G.$$  

We assume furthermore that $S$ is regular, i.e., $q(S) = 0$. Suppose that $S$ is given by a pair of spherical systems of generators: $(a_1, \ldots, a_s), (b_1, \ldots, b_t)$ of $G$.

**Definition**

*The dual surface $S'$ is the product-quotient surface given by the pair of spherical systems of generators: $(a_1, \ldots, a_s), (b_1^{-1}, \ldots, b_t^{-1})$.***
The invariants of $S$ and $S'$

Remark

$$C_{n,q} \in \mathcal{B}(X) \iff C_{n,n-q} \in \mathcal{B}(X').$$
The invariants of $S$ and $S'$

Remark

\[ C_{n,q} \in \mathcal{B}(X) \iff C_{n,n-q} \in \mathcal{B}(X'). \]

Proposition

1. \( \gamma' := \gamma(S') = -\gamma(S) = -\gamma; \)
2. \( q(S') = q(S) \)
3. \( p_g(S') = p_g(S) + \gamma; \)
Back to the original problem: bounding $K^2$ or equivalently, $\gamma$. Can we find an explicit function $C(p_g, q)$ such that for all unmixed quasi-étale surfaces of general type, $\gamma \leq C(p_g, q)$?

We have

$$H^2(S) = H^2(X) \oplus L,$$

where $L = \langle A_1, \ldots, A_l \rangle \cong \mathbb{C}^l$ is the subspace generated by the classes of the $l$ irreducible rational curves of the exceptional locus of $\sigma$. 
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We have

\[
H^2(S) = H^2(X) \oplus L,
\]

where \( L = \langle A_1, \ldots, A_l \rangle \cong \mathbb{C}^l \) is the subspace generated by the classes of the \( l \) irreducible rational curves of the exceptional locus of \( \sigma \).

It is easy to show that the exceptional divisors of the first kind do not belong to \( H^2(X) \).
Consider the subspace $W \subset H^2(S, \mathbb{C})$ generated by the exceptional divisors of the first kind.

**Conjecture**

$$W \cap H^2(X, \mathbb{C}) = \{0\}.$$
Consider the subspace \( W \subset H^2(S, \mathbb{C}) \) generated by the exceptional divisors of the first kind.

**Conjecture**

\[ W \cap H^2(X, \mathbb{C}) = \{0\}. \]

Assume the conjecture to be true. Then:

\[ l = \dim L \geq \dim W \geq 2\chi(S) - 6 - K_S^2 = l + 2\gamma - 6(\chi(S) + 1), \]

whence

\[ \gamma(S) \leq 3(\chi(S) + 1). \]

(and, with a similar argument \( \gamma < 4\chi \)).
[BCG]
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I. Bauer, R. Pignatelli (unpublished notes)