Rigid but not infinitesimally rigid compact complex manifolds

joint work with I. Bauer (Bayreuth)

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XXI Congresso dell'Unione Matematica Italiana

Sezione 18: Geometria Algebrica

Pavia, September 5^{th}, 2019

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Overview

1. The Question
    - Rigidities
    - Morrow-Kodaira's question
2. Our main results
    - Answers in every dimension ≥ 2
3. Why is that difficult?
    - Rigid manifolds
    - Manifolds with obstructed deformations
    - The construction
4. Open problems

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Rigid compact complex manifolds

Definition

A compact complex manifold M is *rigid* if for each deformation of M, $f: (\mathfrak{X}, M) \to (B, b_0)$ there is an open neighbourhood $U \subset B$ of b_0 such that $M_t := f^{-1}(t) \cong M$ for all $t \in U$.

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How do we check the rigidity of a manifold?

The Kuranishi family

Let M be a compact complex manifold.

Kuranishi constructed a deformation $\pi\colon (\mathcal{X}, M) \to (\mathrm{Def}(M), 0)$ of M where $(\mathrm{Def}(M), 0)$ is a germ of analytic subspace of the vector space[1] $H^1(M, \Theta)$, inverse image of the origin under a local holomorphic map, the *Kuranishi map*, $k\colon H^1(M, \Theta) \to H^2(M, \Theta)$ whose differential vanishes[2] at the origin.

[1] Here Θ is the sheaf of holomorphic vector fields on M.

[2] Then $H^1(M, \Theta)$ is the Zariski tangent space of $(\mathrm{Def}(M), 0)$.

In particular $(\mathrm{Def}(M), 0)$ is smooth if and only if $k = 0$, in which case we say that M has *unobstructed deformations*.

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Theorem (Kuranishi)

The Kuranishi family is semiuniversal, and universal if $H^0(M, \Theta) = 0$.

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In particular $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ is (infinitesimally) rigid.

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Morrow-Kodaira's Problem

Morrow and Kodaira asked if the converse implication also holds[3]:

THEOREM 3.2. If $H^1(M, \Theta) = 0$, then M is rigid. We will give a proof of this using elementary methods. We have the following:

PROBLEM. Find an example of an M which is rigid, but $H^1(M, \Theta) \neq 0$. (Not easy?)

REMARK. $\mathbb{P}^n$ is rigid. For $n \geq 2$ the only known proof is to show $H^1(\mathbb{P}^n, \Theta) = 0$ [Bott (1957)]. Let us proceed to the proof.

A solution of the M-K Problem is a manifold M such that Def(M) is a *fat* point, a *singular* point.

[3] This is a screenshot of the book *Complex Manifolds* by James Morrow and Kunihiko Kodaira (1971), *Holt, Rinehart and Winston, Inc.*

The main result

Theorem

For every even $n \geq 8$ such that $3 \nmid n$ there is a minimal regular surface S_n of general type with

$$K_{S_n}^2 = 2(n-3)^2, \quad p_g(S_n) = \left(\frac{n}{2} - 2\right)\left(\frac{n}{2} - 1\right),$$

such that S_n is rigid, but not infinitesimally rigid: $h^1(S_n, \Theta) = 6$.

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such that S_n is rigid, but not infinitesimally rigid: $h^1(S_n, \Theta) = 6$.

The canonical models of these surfaces have exactly 6 singular points, all nodes. The hard part is proving their rigidity, since Kuranishi's rigidity criterion fails.

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Generalization to higher dimension

Lemma

Let M, N be compact complex manifolds, such that

$$h^0(M,\Theta)h^1(N,\mathcal{O}) = h^0(N,\Theta)h^1(M,\mathcal{O}) = 0$$

Then $\mathrm{Def}(M \times N) = \mathrm{Def}(M) \times \mathrm{Def}(N)$.

Then, if M is a regular surface of general type solving the M-K Problem and N is a rigid manifold, by Künneth formula $M \times N$ is a solution too.

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Then, if M is a regular surface of general type solving the M-K Problem and N is a rigid manifold, by Künneth formula $M \times N$ is a solution too. Using some known rigid manifolds[4] we obtain

Theorem

There are rigid manifolds of dimension d and Kodaira dimension κ that are not infinitesimally rigid for all possible pairs (d, κ) with $d \geq 5$ and $\kappa \neq 0, 1, 3$ and for $(d, \kappa) = (3, -\infty), (4, -\infty), (4, 4)$.

[4] listed in Ingrid Bauer and Fabrizio Catanese, *On rigid compact complex surfaces and manifolds*, *Adv. Math.* **333**, 620–669 (2018).

We need M rigid: these are rare.

Theorem (Ingrid Bauer and Fabrizio Catanese, *On rigid compact complex surfaces and manifolds*, *Adv. Math.* **333**, 620–669 (2018).)

Let M be a smooth compact complex surface, which is rigid. Then either

1. *M is a minimal surface of general type, or*
2. *M is a Del Pezzo surface of degree $d \geq 5$*
3. *M is an Inoue surface of type S_M or $S^-_{N,p,q,r}$*

Rigid Del Pezzo and Inoue surfaces are infinitesimally rigid, so every surface solving M-K Problem is minimal with Kodaira dimension 2.

How do check rigidity when the Kuranishi criterion fails?

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Burns and Wahl[5] show how to associate to each smooth rational curve E with $E^2 = -2$ in a complex surface M a $1-$dimensional subspace $H^1_E(M)$ of $H^1(M, \Theta)$.

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Note that in particular if M is the minimal resolution of the singularities of a *nodal* surface[6], M can't be infinitesimally rigid.

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The minimal model of any rigid surface of general type whose canonical model is singular does the job.

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A necessary condition for M to be rigid is that it is obstructed along this line: $H^1_E(M) \not\subset \mathrm{Def}(M)$. A way to check it has been provided by Kas.

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Kas provides an explicit way to compute $\alpha(\eta)$ for all $\eta \in H^0\left(M, \Omega^1 \otimes \Omega^2\right)$.

[7] Arnold Kas, *Ordinary double points and obstructed surfaces*, Topology **16** (1), 51 – 64 (1977).

A criterion to prove rigidity

The main tool for the proof of the first theorem is the following rigidity criterion for *nodal* surfaces.

Theorem

Let M be the minimal res. of the sing. of a nodal surface X. Assume that

1. $H^1(X, \Theta) = 0$;
2. *the elements $\alpha_{\nu_i} \in H^2(M, \Theta)$ associated to the nodes ν_i of X as in the discussion of the Kas formula are linearly independent in $H^0(M, \Omega^1 \otimes \Omega^2)^\vee$.*

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Then M is rigid and $h^1(M, \Theta)$ equals the number of nodes of X.

Note that if X as at least a node, then M is not infinitesimally rigid.

The Fermat curves

The Fermat curve of degree n, $C := \{\sum_{j=0}^{2} x_j^n = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$ admits a natural action of the group $G \cong (\mathbb{Z}/n\mathbb{Z})^2$:

$$(a_1, a_2)(x_0 : x_1 : x_2) = (x_0 : e^{a_1 \frac{2\pi i}{n}} x_1 : e^{a_2 \frac{2\pi i}{n}} x_2).$$

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This $G-$action has only three orbits of cardinality different by n^2, all of cardinality n:

- $C \cap \{x_0 = 0\}$ with stabilizer $\langle (1,1) \rangle \cong \mathbb{Z}/n\mathbb{Z}$
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By Hurwitz formula $C/G \cong \mathbb{P}^1$.

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$$g(x, y) = \left(gx, \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} gy \right);$$

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In fact S_4 is a numerical Campedelli surface with fundamental group $(\mathbb{Z}/2\mathbb{Z})^3$: these are well known, their Kuranishi family has dimension 6.

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Open problems

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Construct rigid manifolds M with $h^1(M, \Theta) = 1$, resp. arbitrarily high.

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Problem (3)

For every fixed natural numbers $\kappa, d \in \mathbb{N}$ with $\kappa \leq d \geq 3$ are there rigid not inf. rigid manifolds of dimension d and Kodaira dimension κ?
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Problem (4)

Construct rigid surfaces to which our criterion does not apply.

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