Linear Feedback Shift Registers and Complexity A survey

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Bunny TN 3

Trento, 12 marzo 2012

Outline

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1. Binary sequences and Complexity

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2. Finite State Machines and LFSR

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3. HW/SW Complexity

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4. LFSR structures

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5. LFSR ?

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- **Simulation**

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- *Linear Complexity Profile*

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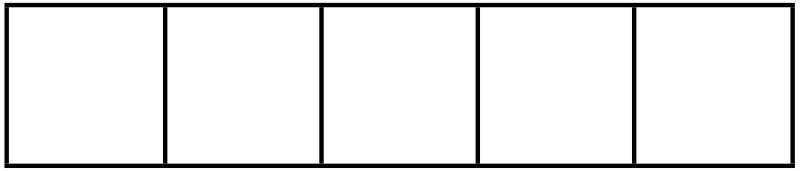
Kolmogorov: the algorithmic complexity description of an object is the length of the shortest binary computer program that describe the object

Linear registers



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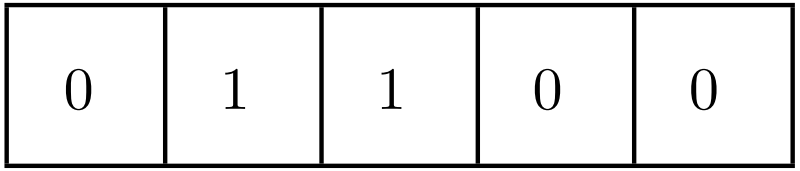

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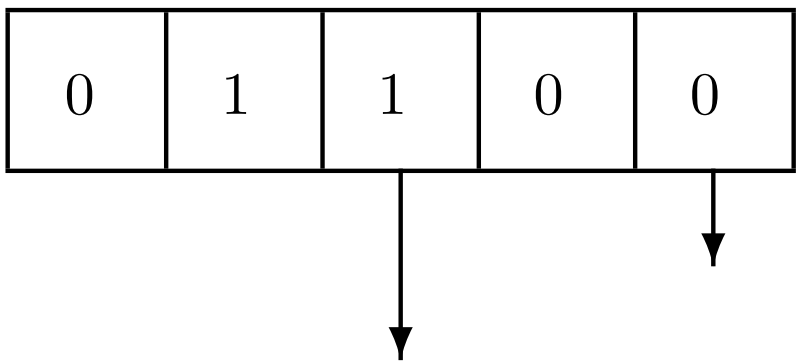

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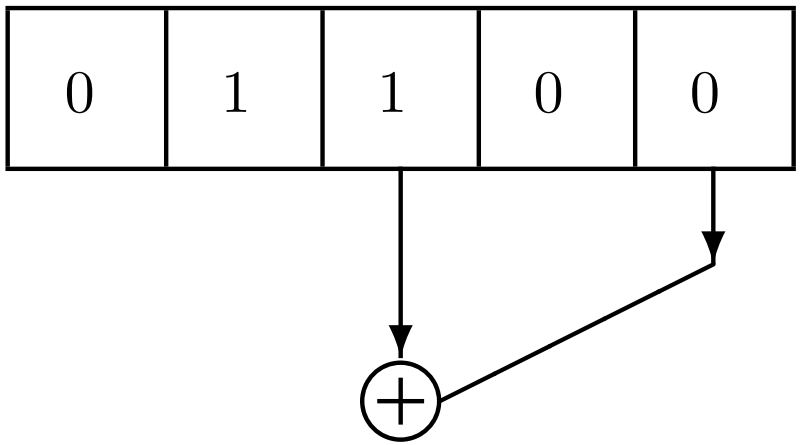

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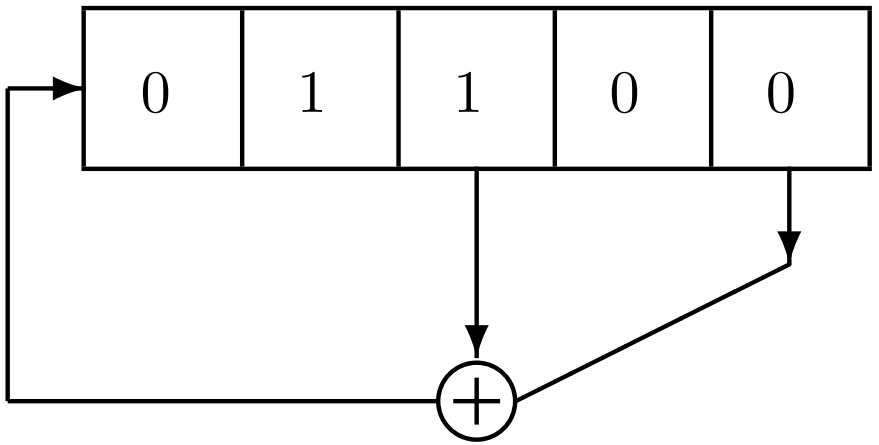

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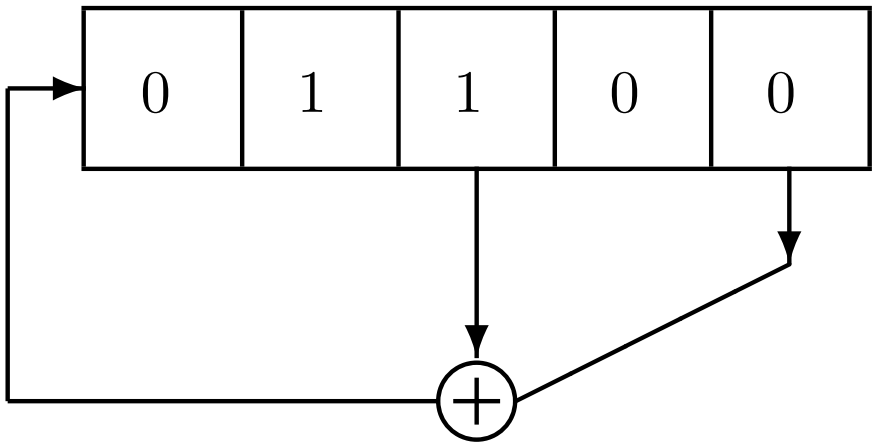

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- Using linear recurrences, i.e. recurrent equations of order m

$$x(n) = a_1 x(n-1) + a_2 x(n-2) + \cdots + a_m x(n-m)$$

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$g(Z)$ is the LFSR polynomial generator, and is also the characteristic polynomial of the transition matrix $\mathfrak{M}$.

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  - τ is the minimum integer such that $g(Z)$ divides $1 - Z^{-\tau}$.
- The maximum value of τ is $2^m - 1$, and is attained by primitive polynomial generators.
  The generated sequences are called m-sequences, in this case the period is independent of the initial state (the all-zeros state is excluded).

LFSR and Cyclic codes

The block of symbols forming a period of an m-sequence generated by a given LFSR can be considered (are) as codewords of a cyclic code which is the dual code

$$(2^m - 1, m, 2^{m-1})$$

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This interpretation is useful for computing the run distribution within a codeword.

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$$c(\delta) = \frac{1}{\tau}\sum_{i=1}^{\tau}(-1)^{x(i+\delta)+x(i)} = \begin{cases} 1 & \text{if } \delta = 0 \\ -\frac{1}{\tau} & \text{if } \delta \neq 0 \bmod \tau \end{cases}$$

Run distribution

A run of 1s of length k in a binary sequence consists of k consecutive 1s between two 0s

$$\ldots 01111110 \ldots \quad \ldots 0110 \quad \ldots 010 \quad \ldots 011111111110$$

and a run of 0s is similarly defined with the role of 0 and 1 exchanged.
Golomb derived the 0-1 run distributions, which are the same in any code word of a dual Hamming code:

- 1run of length m of '1s', and 0 runs of length m of '0s'
- 0 run of length $m-1$ of '1s', and 1 runs of length $m-1$ of '0s'
- 2^{m-k-2} runs of length k, of either '0s' or '1s',
  for $1 \leq k \leq m-2$.

(1)

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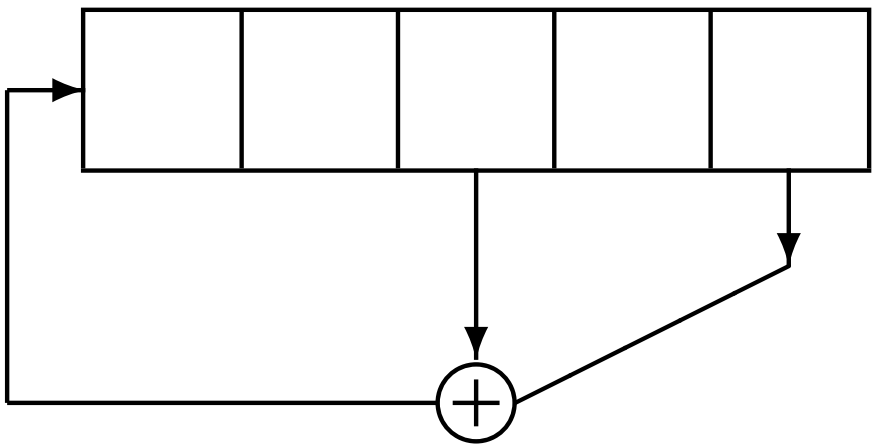
- Tridiagonal LFSR obtained from a tridiagonal matrix with 1s in the upper and lower sub-diagonals.

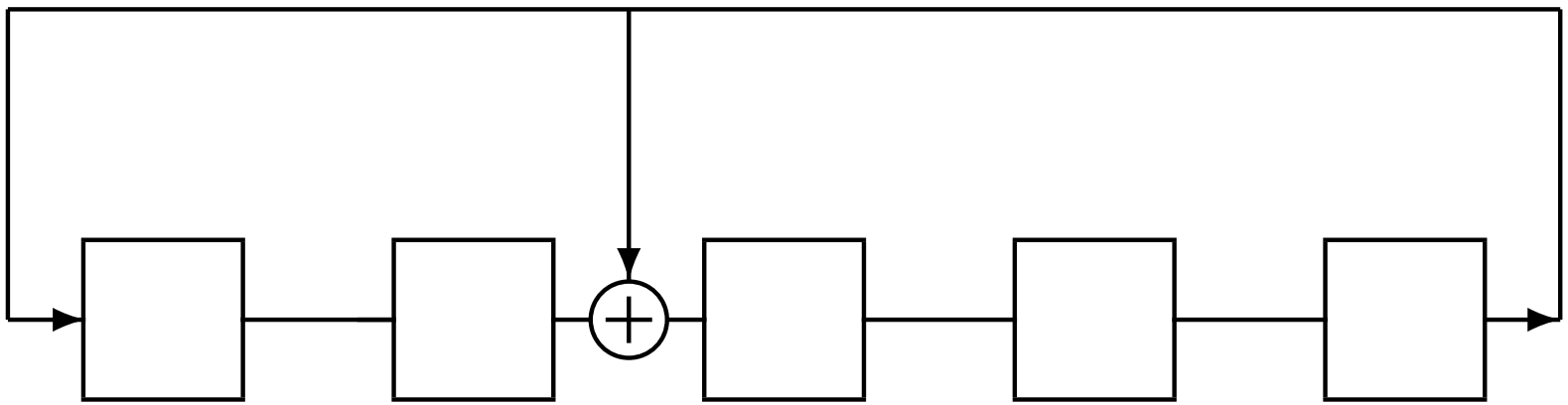
Fibonacci LFSR of order $m = 5$

$$\mathfrak{M}_F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$



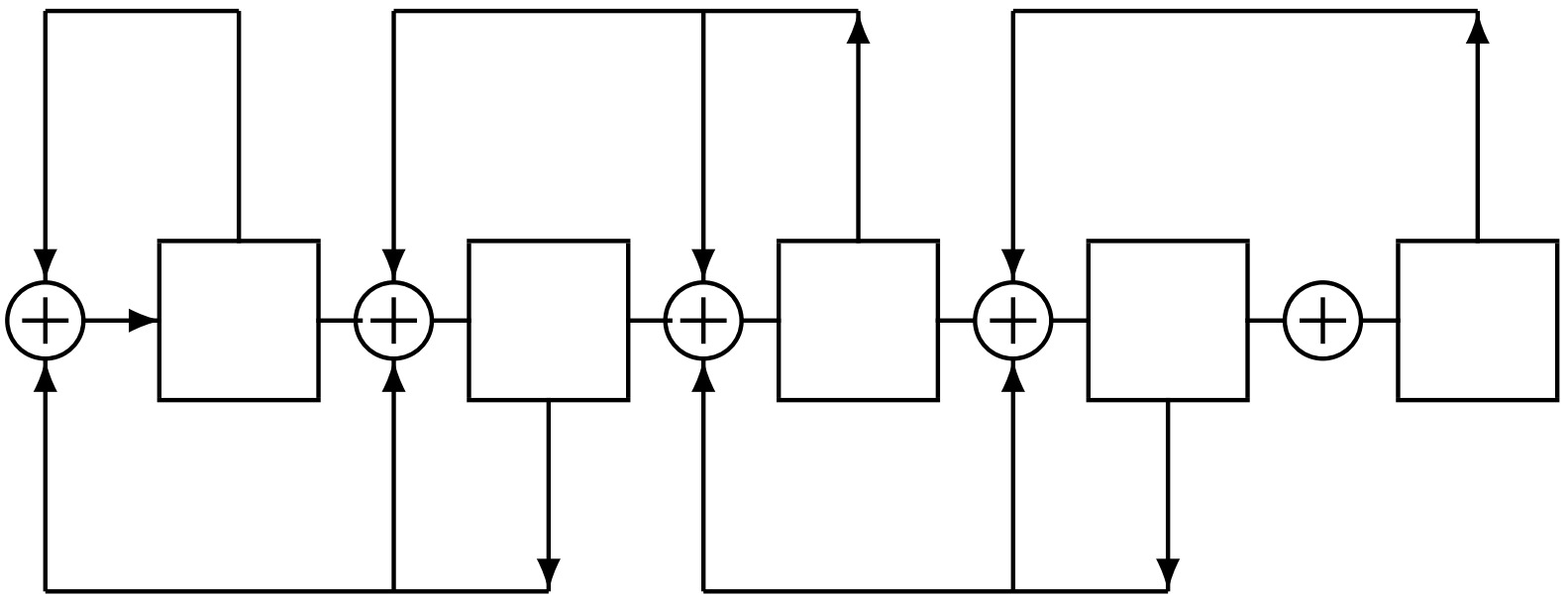
Galois LFSR of order $L = 5$

$$\mathfrak{M}_G = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



Tridiagonal LFSR of order $L = 5$

$$\mathfrak{M}_T = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



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HW/SW complexity

- Fibonacci and Galois LFSRs have the same complexity, which is upper bounded by the length L.
  Note that, interesting generator polynomials have a small number of coefficients equal to 1s, possibly 3 coefficients .
  Unfortunately binary irreducible trinomials does not exist for every L.
- Tridiagonal LFSR have a slightly larger complexity, which is upper bounded by $3L - 2$, nevertheless, in some circumstances may be preferred.

Note that not every binary polynomial of degree L is the characteristic polynomial of a tridiagonal matrix, however, it has been proved that every binary irreducible polynomial is the characteristic polynomial of a tridiagonal matrix.

Linear complexity profile

A linear sequence generated by a LFSR of length m has a period of length not greater than $2^m - 1$: that is "The linear complexity of the sequence is small with respect to its length".

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Berlekamp-Massey's algorithm

Given a sequence $\mathcal{X}$ of length N, the Berlekamp-Massey's algorithm yields the length of the shortest LFSR generating $\mathcal{X}$.

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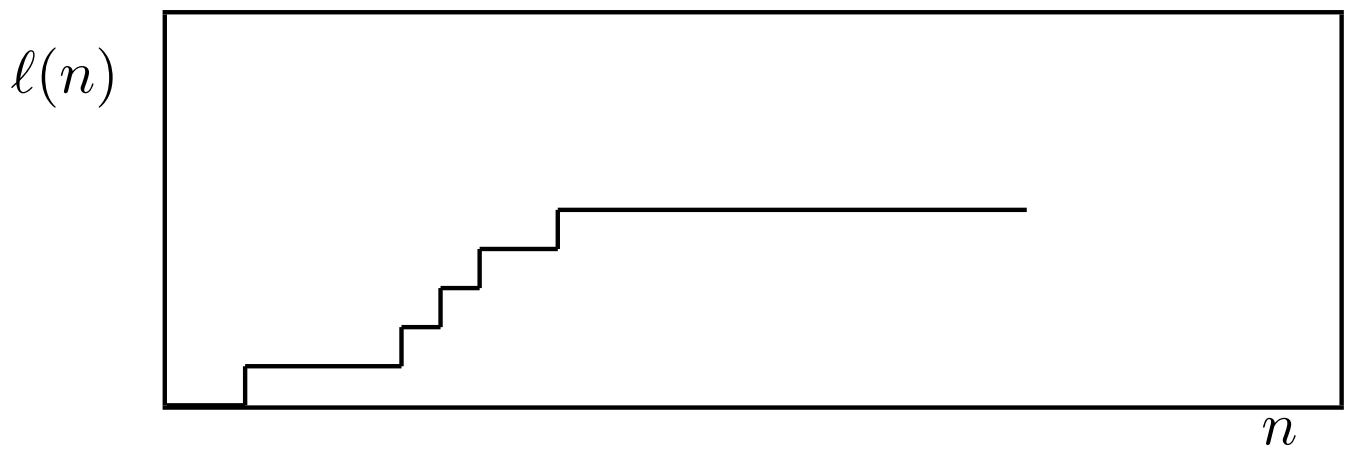

Self-Clock Controlled LFSR

A self-clock controlled LFSR is a linear feedback shift register such that some states are skipped depending on the states themselves .

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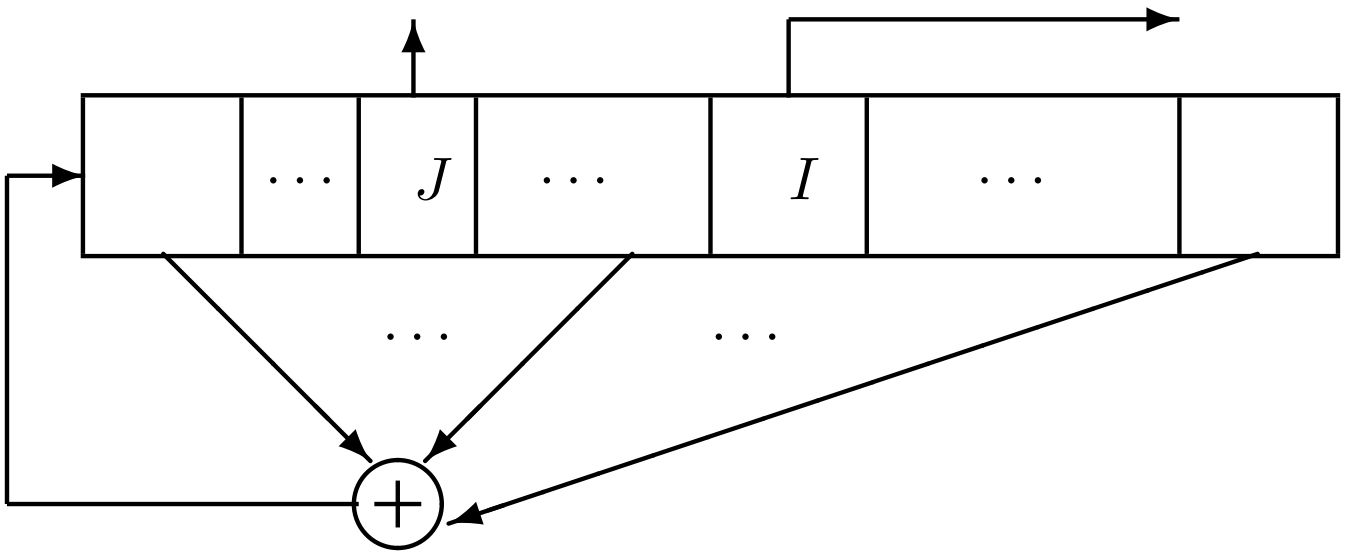

Self-Clock Controlled Fibonacci LFSR

An example is a Fibonacci LFSR in which a cell J is marked: any time that, in the transition to a new state, in cell J occurs a 1, the new state is skipped and a second transition is operated (no further transition is done).

Example

00001	
10000	
01000	*skipped state*
00100	
10010	
01001	*skipped state*
10100	
01010	*skipped state*
00101	
00010	

Period of a Clock-controlled LFSR sequence

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- The period is approximately $2/3$ of τ:

$$\tau = \frac{2}{3}(2^m - 1) - \frac{2}{3}\delta = \frac{2}{3}\left(2^m - \frac{3 - (-1)^m}{2}\right) \quad .$$

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- The generated sequences belong to a linear code: each sequence can be seen as a code word of a punctured code (the punctured symbols correspond to the skipped states).
- Different sequences produced by the same LFSR belong to different linear codes.

0-1 Distributions in clock-controlled LFSR sequences

The numbers N_{0I} and N_{1I} of '0s' and '1s', in a sequence generated by a self-clock controlled LFSR, depend on both the relative position of control and output cells, and the implementation LFSR-type, namely Fibonacci, Galois, or Tridiagonal. For the Fibonacci LFSR N_{0I} and N_{1I} can be computed in closed form.

Using the closed form of N_{0I} and N_{1I}, it is immediately seen that the clocked sequence is perfectly balanced, i.e. $N_{0I} = N_{1I}$, if and only if $I = 1$ if m is odd, and $I = 2$ if m is even.

Linear complexity profile

The Linear complexity profile of a clock controlled LFSR sequence is practically optimal as it can be theoretically shown

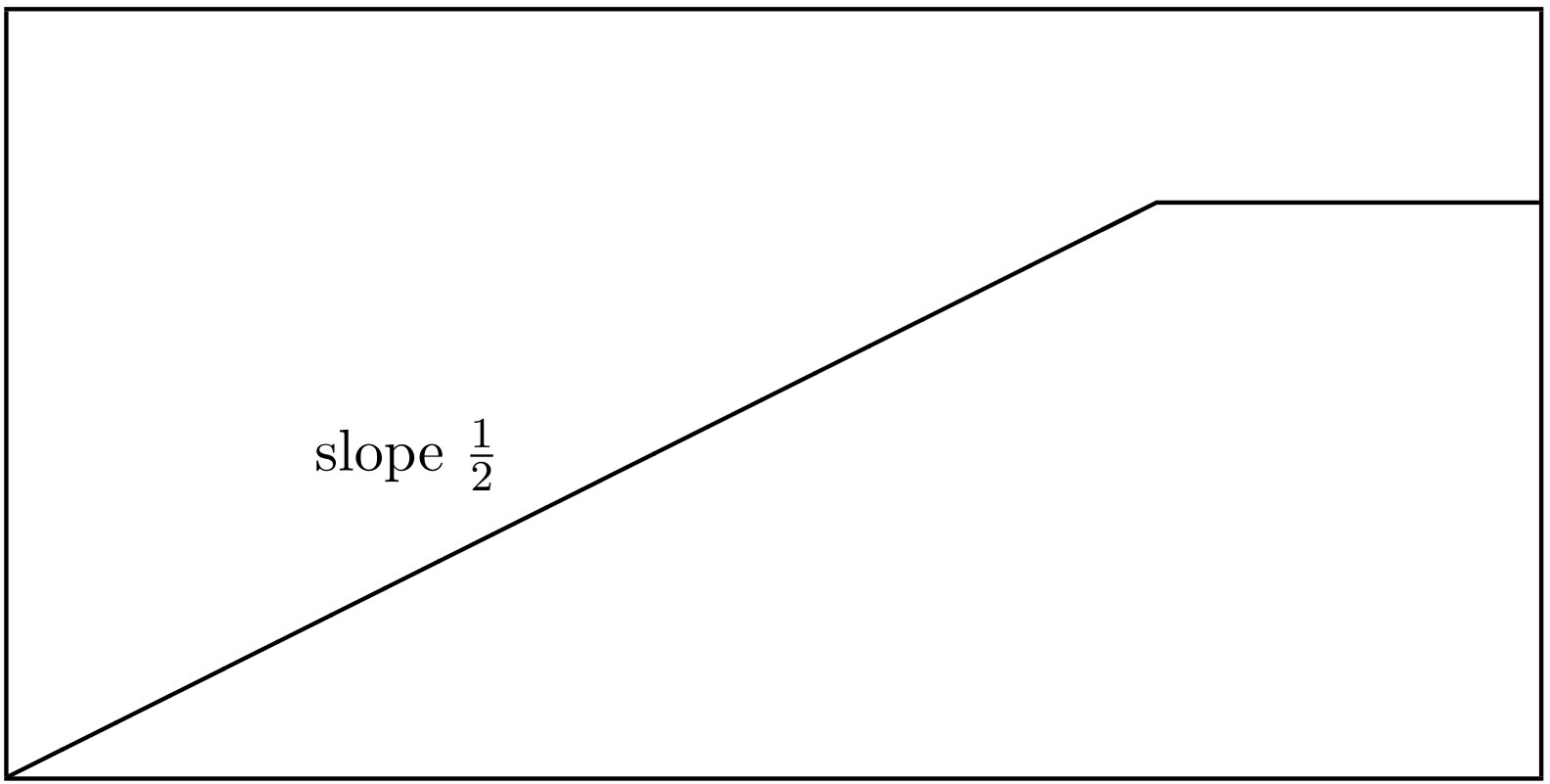

Figure: LCP for a clock controlled Fibonacci LFSR of length 22

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1) LFSR are good generators of random sequences, which are easy to implement and may work fast.
   However, in cryptography, the use of these linear sequences needs further artifices to counteract the weaknesses implicit in the linearity.
2) Self-clock controlled LFSR have optimal linear complexity profile, indistinguishable from that of genuine random sequences, thus may be (more) directly used in cryptographic applications.

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An endeavor that was and remains a source of challenging problems for engineers and mathematicians.

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