Complete permutation polynomials of monomial type

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(joint works with D. Bartoli, M. Giulietti and L. Quoos)
(based on the work of thesis of E. Franzè)

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Outline

1. Permutation polynomials: an introduction
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2. Monomial complete permutation polynomials: our results
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2. Monomial complete permutation polynomials: our results

3. Particular cases: degree 8 and 9 in characteristic 2 and 3
Some definitions

\( \mathbb{F}_\ell: \) finite field with \( \ell = p^h \) elements

Plane curve \( C: F(X, Y, T) = 0 \)

\( \mathbb{F}_\ell \)-rational point of \( C \): \( P = (x, y, z) \in PG(2, \ell) \) such that \( F(x, y, z) = 0 \)

Definition

\( f(x) \in \mathbb{F}_\ell[x] \) is a permutation polynomial (shortly, a PP) of \( \mathbb{F}_\ell \)
if \( x \mapsto f(x) \) is a bijection of \( \mathbb{F}_\ell \) (iff \( x \mapsto f(x) \) is injective over \( \mathbb{F}_\ell \))

Definition

\( f(x) \in \mathbb{F}_\ell[x] \) is a complete permutation polynomial (shortly, a CPP) of \( \mathbb{F}_\ell \)
if both \( f(x) \) and \( f(x) + x \) are PPs of \( \mathbb{F}_\ell \)

Definition

\( f(x) \in \mathbb{F}_\ell[x] \) is an exceptional polynomial over \( \mathbb{F}_\ell \)
if \( f(x) \) is a PP of an infinite number of extensions of \( \mathbb{F}_\ell \)
CPPs and Cryptography

**Definition**

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**Definition**

\( f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2 \) Boolean function is

- **bent** if \( x \mapsto f(x + a) + f(x) \) is balanced \( \forall a \in \mathbb{F}_2^n \) (\( \Leftrightarrow f \) is PNF)
- **bent-negabent** if both \( x \mapsto f(x + a) + f(x) \) and \( x \mapsto f(x + a) + f(x) + Tr(ax) \) are balanced \( \forall a \in \mathbb{F}_2^n \)

**LINK:**

any PP of \( \mathbb{F}_{2^n} \) gives rise to a bent function over \( \mathbb{F}_2^n \)

any CPP of \( \mathbb{F}_{2^n} \) gives rise to a bent-negabent function over \( \mathbb{F}_2^n \)
Link with curves

\[ f(x) \in \mathbb{F}_\ell[x] \quad \mapsto \quad C_f : \frac{f(x) - f(y)}{x - y} = 0 \]

\( f(x) \) is a PP of \( \mathbb{F}_\ell \) \( \iff \) \( C_f \) has no affine \( \mathbb{F}_\ell \)-rational points \((a, b)\) with \( a \neq b \)
Link with curves

\[ f(x) \in \mathbb{F}_\ell[x] \implies C_f : \frac{f(x) - f(y)}{x - y} = 0 \]

\( f(x) \) is a PP of \( \mathbb{F}_\ell \iff C_f \) has no affine \( \mathbb{F}_\ell \)-rational points \((a, b)\) with \( a \neq b \)

Theorem

\( C \) absolutely irreducible curve of degree \( d \) defined over \( \mathbb{F}_\ell \)

The number \( N_\ell \) of \( \mathbb{F}_\ell \)-rational points satisfies

\[ N_\ell \geq \ell + 1 - (d - 1)(d - 2)\sqrt{\ell} \]

\[ \downarrow \]

for \( \ell \) large enough:

\( f(x) \) is a PP of \( \mathbb{F}_\ell \)

\[ \downarrow \]

\( C_f \) has no \( \mathbb{F}_\ell \)-rat. abs. irr. components distinct from \( X = Y \)
Conversely:

**Theorem (Cohen 1970)**

\[ C_f \text{ contains no } \mathbb{F}_\ell\text{-rational abs. irr. component distinct from } \ X = Y \]

\[ \Downarrow \]

\[ f(x) \text{ is an exceptional polynomial over } \mathbb{F}_\ell \]
Conversely:

**Theorem (Cohen 1970)**

\[ \mathcal{C}_f \text{ contains no } \mathbb{F}_\ell\text{-rational abs. irr. component distinct from } X = Y \]

\[ \downarrow \]

\[ f(x) \text{ is an exceptional polynomial over } \mathbb{F}_\ell \]

*It is not difficult to construct PP without any prescribed structure*

**Remark**

\[ f(x) \text{ is a PP of } \mathbb{F}_\ell \iff \alpha f(\gamma x + \delta) + \beta \text{ is a PP of } \mathbb{F}_\ell \ (\alpha, \beta, \gamma, \delta \in \mathbb{F}_\ell, \alpha, \gamma \neq 0) \]

**PP-equivalence:**

\[ f(x) \approx \alpha f(\gamma x + \delta) + \beta, \quad \alpha, \beta, \gamma, \delta \in \mathbb{F}_\ell, \alpha, \gamma \neq 0 \]
The monomial case

- $b^{-1}x^d$ is a PP of $\mathbb{F}_\ell \iff (d, \ell - 1) = 1$
- $b^{-1}x^d$ is a CPP of $\mathbb{F}_\ell \iff (d, \ell - 1) = 1$ and $x^d + bx$ is a PP of $\mathbb{F}_\ell$
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$f_b(x) = b^{-1}x^{\frac{q^n-1}{q-1}+1}$ has been studied as CPP of $\mathbb{F}_{q^n}$ for $n = 2, 3, 4$ and partially for $n = 6$
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**EXPLICIT LIST** of all $b \in \mathbb{F}_{q^n}$ such that $f_b$ is a CPP of $\mathbb{F}_{q^n}$, in the cases:

- $n = 7$, for arbitrary $q$ (E. Franzè, Master Thesis)
- $n = 6$, for arbitrary $q$ (Bartoli-Giulietti-Z., FFA 2016)
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**Conjecture** (Wu-Li-Helleseth-Zhang 2015)

If $n + 1$ is prime, $n + 1 \neq p$, $\gcd(n + 1, q^2 - 1) = 1$, then:

there exist CPPs of $\mathbb{F}_{q^n}$ of type $b^{-1}x^{\frac{q^n-1}{q-1}+1}$
The monomial case

- $b^{-1}x^d$ is a PP of $\mathbb{F}_\ell$ $\iff (d, \ell - 1) = 1$
- $b^{-1}x^d$ is a CPP of $\mathbb{F}_\ell$ $\iff (d, \ell - 1) = 1$ and $x^d + bx$ is a PP of $\mathbb{F}_\ell$

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**GOAL:** to characterize for any $n$ the $b \in \mathbb{F}_{q^n}$ such that $f_b = b^{-1}x^{\frac{q^n-1}{q-1}+1}$ is a CPP of $\mathbb{F}_{q^n}$
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$f_b(x) = b^{-1}x^{\frac{q^n-1}{q-1}}+1$ has been studied as CPP of $\mathbb{F}_{q^n}$ for $n = 2, 3, 4$ and partially for $n = 6$

**GOAL**: to characterize for any $n$ the $b \in \mathbb{F}_{q^n}$ such that $f_b = b^{-1}x^{\frac{q^n-1}{q-1}}+1$ is a CPP of $\mathbb{F}_{q^n}$

**WE OBTAIN**: complete classification for $n^4 < q = p^m$ with the exception of the cases

- $n + 1 = p^r$, with $r > 1$
- $n + 1 = p^r(p^r - 1)/2$, with $p \in \{2, 3\}$, $r > 1$, $\gcd(r, 2m) = 1$
$b \in \mathbb{F}_{q^n} \iff A_i(b) := \sum_{0 \leq j_1 < j_2 < \ldots < j_i \leq n-1} b^{q^{j_1} + q^{j_2} + \ldots + q^{j_i}} \in \mathbb{F}_q$

$i$-th elementary symmetrical polynomial in $b, b^q, \ldots, b^{q^{n-1}}$
\[ b \in \mathbb{F}_{q^n} \implies A_i(b) := \sum_{0 \leq j_1 < j_2 < \ldots < j_i \leq n-1} b^{q^{j_1} + q^{j_2} + \ldots + q^{j_i}} \in \mathbb{F}_q \]

\textit{i-th elementary symmetrical polynomial in } b, b^q, \ldots, b^{q^{n-1}}

**Proposition (Wu-Li-Helleseth-Zhang 2013)**

If \( n^4 < q \), then:

\[ b^{-1} x^{\frac{q^n-1}{q-1}} + 1 \]

is a CPP of \( \mathbb{F}_{q^n} \) \iff \( x^{n+1} + A_1(b)x^n + \cdots + A_n(b)x \)

is an exceptional polynomial over \( \mathbb{F}_q \).
Let \( b \in \mathbb{F}_{q^n} \) imply \( A_i(b) := \sum_{0 \leq j_1 < j_2 < \ldots < j_i \leq n-1} b^{q^{j_1} + q^{j_2} + \ldots + q^{j_i}} \in \mathbb{F}_q \)

\( i \)-th elementary symmetrical polynomial in \( b, b^q, \ldots, b^{q^{n-1}} \)

**Proposition (Wu-Li-Helleseth-Zhang 2013)**

If \( n^4 < q \), then:

\[
\begin{align*}
&b^{-1}x \left( \frac{q^n}{q-1} \right) + 1 \\
is a CPP of \mathbb{F}_{q^n} &\iff \gcd(n + 1, q - 1) = 1, \\
&x^{n+1} + A_1(b)x^n + \cdots + A_n(b)x \\
is an exceptional polynomial over \mathbb{F}_q
\end{align*}
\]

**Remark**

\[
\begin{align*}
b^{-1}x \left( \frac{q^n}{q-1} \right) + 1 &\text{ is a CPP of } \mathbb{F}_{q^n} \iff b^{-q^i}x \left( \frac{q^n}{q-1} \right) + 1 &\text{ is a CPP of } \mathbb{F}_{q^n}
\end{align*}
\]
Proposition (Wu-Li-Helleseth-Zhang 2013)

If \( n^4 < q \), then:

\[
b^{-1} x \frac{q^n-1}{q-1} + 1 \quad \iff \quad \gcd(n + 1, q - 1) = 1,
\]

is a CPP of \( \mathbb{F}_{q^n} \) if and only if

\[
x^{n+1} + A_1(b)x^n + \cdots + A_n(b)x^n+1 + A_1(b)x^n + \cdots + A_n(b)x
\]

is an exceptional polynomial over \( \mathbb{F}_q \).

Definition

Let

\[
g(x) = x^{n+1} + \lambda_1 x^n + \cdots + \lambda_{n-1} x^2 + \lambda_n x \in \mathbb{F}_q[x], \ \lambda_n \neq 0,
\]

be a PP of \( \mathbb{F}_q \).

\( g(x) \) is good if the roots of

\[
v_g(x) := \frac{g(-x)}{-x} = x^n - \lambda_1 x^{n-1} + \cdots + (-1)^{n-1} \lambda_{n-1} x + (-1)^n \lambda_n
\]

form a unique orbit under the Frobenius map \( z \mapsto z^q \).
Proposition

If \( n^4 < q \), then:

\[
b \in \mathbb{F}_{q^n} \setminus \mathbb{F}_q \text{ is such that } b^{-1}x^{\frac{q^n-1}{q-1}} + 1 \text{ is a CPP of } \mathbb{F}_{q^n} \iff b \text{ is a root of } v_g(x) = \frac{g(-x)}{-x}
\]

for some \( g \) good exceptional pol.

of degree \( n + 1 \) over \( \mathbb{F}_q \)

with \( g(0) = 0 \) and \( g'(0) \neq 0 \)

Definition

An exceptional polynomial \( g \) is decomposable if \( g(x) = g_1(g_2(x)) \) with \( g_1, g_2 \) exceptional pol., \( \deg(g_1), \deg(g_2) > 1 \)
Proposition

If $n^4 < q$, then:

$b \in \mathbb{F}_{q^n} \setminus \mathbb{F}_q$ is such that $b^{-1} x^{\frac{q^n-1}{q-1}} + 1$ is a CPP of $\mathbb{F}_{q^n}$

$b$ is a root of $v_g(x) = \frac{g(-x)}{-x}$

for some $g$

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Definition

An exceptional polynomial $g$ is decomposable if

$g(x) = g_1(g_2(x))$ with $g_1, g_2$ exceptional pol., $\text{deg}(g_1), \text{deg}(g_2) > 1$

Proposition

$g$ good exceptional polynomial $\implies$ $g$ indecomposable
Idea

In order to classify all CPPs of type \( f(x) = b^{-1} x^{\frac{q^n-1}{q-1}} + 1 \)
take all the good indecomposable exceptional polynomials
and determine the roots of \( v_g(x) \)
Idea

In order to classify all CPPs of type $f(x) = b^{-1}x^{\frac{q^n-1}{q-1}+1}$
take all the good indecomposable exceptional polynomials
and determine the roots of $v_g(x)$

Unfortunately:

the complete classification of indecomposable exceptional polynomials
is not known!
Remark

\( f(x) \) is a good PP of \( \mathbb{F}_\ell \) \iff \( \alpha f(\gamma x) + \beta \) is a good PP of \( \mathbb{F}_\ell \) (\( \alpha, \beta, \gamma \in \mathbb{F}_\ell, \alpha, \gamma \neq 0 \))

CPP-equivalence:

\[
 f(x) \approx \alpha f(\gamma x) + \beta, \quad \alpha, \beta, \gamma \in \mathbb{F}_\ell, \alpha, \gamma \neq 0
\]
Remark

\[ f(x) \text{ is a good PP of } \mathbb{F}_\ell \iff \alpha f(\gamma x) + \beta \text{ is a good PP of } \mathbb{F}_\ell \ (\alpha, \beta, \gamma \in \mathbb{F}_\ell, \alpha, \gamma \neq 0) \]

CPP-equivalence:

\[ f(x) \approx \alpha f(\gamma x) + \beta, \quad \alpha, \beta, \gamma \in \mathbb{F}_\ell, \alpha, \gamma \neq 0 \]

\[ \downarrow \]

We use the known partial classification of indecomposable exceptional polynomial, up to CPP-equivalence.
Classification of indecomposable exceptional polynomials, up to CPP-equivalence

A) $n + 1 \nmid q - 1$ is a prime different from $p$ and

A1) \( g(t) = (t + e)^{n+1} - e^{n+1}, \ e \in \mathbb{F}_q \)

A2) \( g(t) = D_{n+1}(t + e, a) - D_{n+1}(e, a), \)
\[ a, e \in \mathbb{F}_q, \ a \neq 0, \ n + 1 \nmid q^2 - 1 \]
\[ D_{n+1}(t, a) \quad \text{Dickson polynomial of degree } n + 1 \]

B) $n + 1 = p$ and \( g(t) = (t + e) \left( (t + e)^{\frac{p-1}{r}} - a \right)^r - e \left( e^{\frac{p-1}{r}} - a \right)^r \)
\[ r \mid p - 1, \ a, e \in \mathbb{F}_q, \ a^{r(q-1)/(p-1)} \neq 1. \]

C) $n + 1 = s(s - 1)/2$
\[ p \in \{2, 3\}, \ q = p^m, \ r > 1, \ s = p^r > 3 \ and \ (r, 2m) = 1. \]

D) $n + 1 = p^r$ with $r > 1$. 
Case A1

\( n + 1 \) is prime, \( n + 1 \neq p \), \( n + 1 \) does not divide \( q - 1 \)

\( \zeta_{n+1} := (n+1)\)-th primitive root of unity

**Proposition**

Let \( e \in \mathbb{F}_q^* \). Then

\[
g(t) = (t + e)^{n+1} - e^{n+1}
\]

is good exceptional over \( \mathbb{F}_q \) \iff \( \text{ord}_{n+1}(q) = n \)

If \( \text{ord}_{n+1}(q) = n \), then for each \( e \in \mathbb{F}_q^* \) and \( i \in \{1, \ldots, n\} \)

\[
(e(\zeta_{n+1}^i - 1))^{-1} \times \frac{q^n-1}{q-1} + 1
\]

is a CPP of \( \mathbb{F}_{q^n} \)
Case A2

$n + 1$ is prime, $n + 1 \neq p$, $n + 1$ does not divide $q^2 - 1$

(Dickson polynomials)

$$D_{n+1}(t, a) = \sum_{k=0}^{n/2} \frac{n+1}{n+1-k} \binom{n+1-k}{k} (-a)^k t^{n+1-2k}$$

Proposition

$$g(x) = D_{n+1}(x + e, a) - D_{n+1}(e, a), \ e, a \in \mathbb{F}_q, \ a \neq 0, \ D_{n+1}'(e, a) \neq 0,$$

is good exceptional over $\mathbb{F}_q$ if and only if one of the following cases occurs:

i) $4 \mid n$ and $\text{ord}_{n+1}(q) = n$

ii) $4 \nmid n$ and

$$\begin{cases} e^2 - 4a \notin \square_q, & \text{ord}_{n+1}(q) = n/2 \\ e^2 - 4a \in \square_q, & \text{ord}_{n+1}(q) = n \end{cases}$$
Case B

\[ n + 1 = p \]

\[ \mathbb{N}_{\mathbb{F}_q/\mathbb{F}_p} : \text{the norm map } \mathbb{F}_q \rightarrow \mathbb{F}_p, \ x \mapsto x^{1+p+p^2+\cdots+q/p}. \]

Theorem

Let \( n^4 < q \). Then

\[ b^{-1} x^{\frac{q^n-1}{q-1}+1} \]

is a CPP of \( \mathbb{F}_{q^n} \)

\[ \Downarrow \]

for some \( r \mid n \), one of the following cases occurs:

i) \( b \in \{ \zeta_{q-1}^{i} \mid \gcd(r, i) = 1 \} \)

ii) \( b \in \{ (v_0 - \lambda u_0)^r - e \mid \lambda \in \mathbb{F}_p^*, \ e, u_0^{p-1} \in \mathbb{F}_q^*, \ u_0^r \neq 1, \quad v_0^r = e, \ \text{ord} \left( \mathbb{N}_{\mathbb{F}_q/\mathbb{F}_p} \left( \frac{u_0^{p-1}}{e^{(p-1)/r}} \right) \right) = p - 1 \} \)
\[ n + 1 = 8, \ p = 2 \]

\[ F(x) \in \mathbb{F}_q[x] \text{ monic of degree } 8 \]

**Proposition**

\( F(x) \) is good exceptional over \( \mathbb{F}_q \) if and only if

\[ F(x) = x^8 + ax^4 + bx^2 + cx \text{ is additive and} \]

\[ x^7 + ax^3 + bx + c \text{ is irreducible over } \mathbb{F}_q. \]
\[ n + 1 = 9, \quad p = 3 \]

No classification is known!

When is
\[ F(x) = x^9 + A_1x^8 + A_2x^7 + A_3x^6 + A_4x^5 + A_5x^4 + A_6x^3 + A_7x^2 + A_8x \]
good exceptional?

Theorem (Cohen 1970)
\[ C_F \text{ contains no } \mathbb{F}_\ell\text{-rational component distinct from } X = Y \]
\[ \downarrow \]
\[ F(x) \text{ is an exceptional polynomial over } \mathbb{F}_\ell \]

- Determine when
\[ C_F := \frac{F(x) - F(y)}{x - y} = 0 \]

has only non-rational components (other than \( x - y \))
- Study when the roots of \( v_F(x) \) are in a unique orbit under Frobenius
Proposition

\[ F(x) \text{ is good exceptional over } \mathbb{F}_q \text{ if and only if} \]

i) \[ F(x) = x^9 + A_6x^3 + A_8x \]
and \[ x^8 + A_6x^2 + A_8 \] irreducible over \( \mathbb{F}_q \);

ii) \[ F(x) = x^9 + A_3x^6 + A_4x^5 + A_5x^4 + \left( A_2^3 + A_3 \frac{A_5}{A_3} + \frac{A_5^2}{A_4} \right)x^3 \]
\[ + \left( 2A_3A_4 + 2\frac{A_5^3}{A_4^2} \right)x^2 + \left( 2A_3A_5 + A_4^2 + 2\frac{A_5^4}{A_3^3} \right)x, \]

1. \( A_4 \neq 0 \),
2. the polynomial \[ x^8 + 2A_3x^2 + 2A_4 \in \mathbb{F}_q[x] \] has no roots in \( \mathbb{F}_{q^4} \);

iii) \[ F(x) = x^9 + A_2x^7 + A_3x^6 + A_5x^4 + \left( A_2^3 + \frac{A_3A_5}{A_2} \right)x^3 + \]
\[ \left( 2A_2A_5 + 2\frac{A_3^3}{A_2^2} \right)x^2 + \left( A_2^4 + A_3A_5 + \frac{A_5^2}{A_2} + \frac{A_3^3}{A_2^2} \right)x, \]

1. \( 2A_2 \) is not a square in \( \mathbb{F}_q \),
2. the polynomial \( v_F(x) = F(-x)/(-x) \) is irreducible over \( \mathbb{F}_q \).
Thank you for your attention!