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Is length contraction really paradoxical?

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A thought experiment is proposed in which a moving conducting shuttle encounters a gap between two conducting rails connected at one end through a bulb and a source of steady voltage. A naive application of length contraction leads to contradictory results when the encounter is examined from the rail and the shuttle frames, viz., that the bulb should switch off in one frame but should keep glowing in the other. However the interaction between the shuttle and the gap is so arranged that it is possible to analyze the experiment quantitatively in both the Lorentz frames within the framework of elementary relativistic kinematics. It is shown that the results of such a calculation lead to an exact agreement between the observed effects in the two frames. The article includes an Appendix that contains a compact bibliography of several of the paradoxes in the theory of relativity.

I. INTRODUCTION

In this article we present a thought experiment in which we let a moving object meet a complementary object with a gap of equal rest length. As in the other length contraction paradoxes of this genre (see Appendix), the encounter results in observable effects that appear contradictory at first sight (viz. a bulb switches off in one frame but glows on in the other). But, unlike the others, the interaction we choose (which is electrical) can be made weak enough for the two objects to continue their original motion. This permits us to calculate the observed effect in detail in both the Lorentz frames using only the kinematics of a single special Lorentz transformation and see if the results tally with each other.

It is shown that a careful analysis brings about an unequivocal and quantitative resolution of the paradox. The calculation involves only the length contraction, time dilation, and velocity addition formulas, and as such lies within the scope of a first course in relativity. It is hoped that the exercise will help strengthen the ability of the freshman to switch Lorentz frames and perhaps convince him of the inner consistency of the rather unfamiliar consequences of the Lorentz transformations.

II. THE PARADOX

Let us consider two long parallel conducting rails that are open at one end but connected at the other end through a bulb and a source of steady voltage all in series (Fig. 1). One of the rails has a gap of rest length l_0 that is bridged by a parallel conducting strip AB very close to the rail itself. An H-shaped object with conducting prongs C and D connected by an insulating rod fits snugly between the guide rails and is capable of moving freely between them like a shuttle in a weaver's loom. The rest length of the shuttle is also equal to l_0.

When both prongs of the shuttle are to the left of the gap (see Fig. 1), the rails are shorted by the shuttle and the bulb glows. We wish to examine from the "rail frame" and the "shuttle frame" what happens to the glowing bulb if the shuttle moves forward uniformly at a speed βc with respect to the rails. We assume that the friction and electrical interaction are small enough that the momentum of the shuttle changes by a negligible amount during its motion.

When observed from the rail frame the moving shuttle CD suffers a length contraction and occupies the position shown in Fig. 1 at the instant at which C is losing contact with A. Thereafter, there occurs a period of time during which both prongs C and D lose contact with the upper rail and the circuit opens. The bulb should therefore stop glowing until prong D of the shuttle reestablishes contact with the rail at B and the circuit closes again. Thus in the rail frame the bulb switches off for a time T_{off} before starting to glow again.

When observed from the shuttle frame, the bulb and the entire circuit move backwards and it is the gap and the bridge AB across it that suffer the length contraction (see Fig. 2). Since the shuttle is now longer than the gap, there is no instant at which both prongs C and D of the shuttle lie within the gap AB. Therefore, one or the other of the prongs C and D is always in contact with the rails since B establishes contact with D before A loses contact with C. Hence, in the shuttle frame, the circuit never becomes open. It then appears that the bulb should never stop glowing, which conclusion is paradoxical.

The full resolution of the paradox requires that we should prove that not only does the bulb switch off in the shuttle frame as well, but that it switches off in it for a time T'_{off} that is *longer* than T_{off}. For, the "off time" T_{off} of the bulb in the rail frame is a proper time, and the nonproper time T'_{off} must be related to T_{off} by the time dilation formula

$$T'_{\text{off}} = \gamma T_{\text{off}}, \quad \text{where } \gamma = (1-\beta^2)^{-1/2}. \tag{1}$$

III. RESOLUTION

First we convince ourselves that $T'_{\text{off}} > 0$ and then we verify Eq. (1).

Referring to Fig. 1 let us call the event of C losing contact with A as $\hat{L}$ (for left event) and the event of D making contact with B as $\hat{R}$ (for right event). In the rail frame it is evident that $\hat{R}$ occurs *later* than $\hat{L}$. Turning now to Fig. 2 it is also evident that $\hat{R}$ occurs *earlier* than $\hat{L}$ in the shuttle frame. Such a reversal of the past and future can occur only if events $\hat{L}$ and $\hat{R}$ are separated by a spacelike interval.

Let us again refer to Fig. 2. In the shuttle frame we have already seen that one or the other of the prongs C and D always touches the upper rail and the circuit never becomes open at any given instant. However, this is not a sufficient

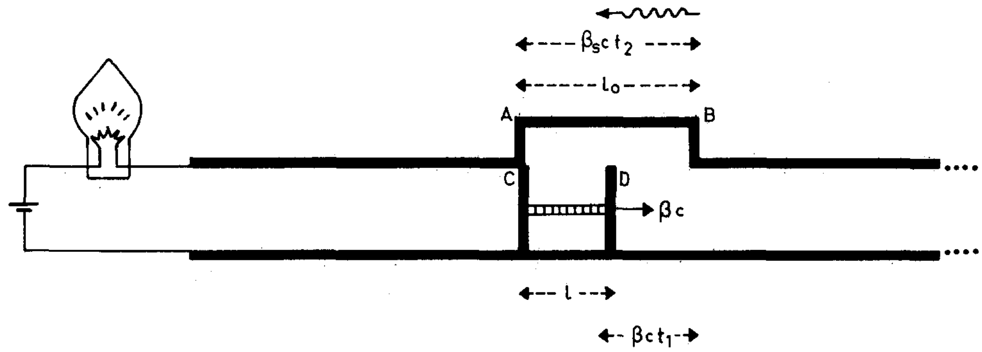

Fig. 1. Configuration of the circuit in the rail frame.

condition to keep the bulb glowing. The bulb will switch off if it does not receive current continuously, and for that it is essential that the electrical pulse generated at B, when it makes contact with D, should reach A *before* A loses contact with C. Otherwise, A will lose contact with C before it receives the current pulse from $\hat{R}$ and there will be a break in the continuity of the current crossing A towards the bulb. The bulb would therefore switch off during the interregnum. This in turn means that the electromagnetic signal sent along the transmission line from the event $\hat{R}$ should be fast enough to reach the event $\hat{L}$ if there were to be no switching off of the bulb.

We have already seen that the two events $\hat{L}$ and $\hat{R}$ have a spacelike separation, which implies that they cannot be connected by any physical signal. Hence it follows that however fast it travels along the line (which depends on the circuit parameters like the distributed inductance, capacitance, and resistance of the line), the current pulse from $\hat{R}$ will reach A only *after* A loses contact with C. This proves the assertion that the bulb will switch off for a period of time $T'_{\text{off}} > 0$ in the shuttle frame as well.

To calculate the off-time of the bulb in the rail frame let us refer to Fig. 1. It is evident that current stops flowing from A to the bulb from the event $\hat{L}$ till the pulse from $\hat{R}$ arrives at A. This time interval is just the off-time T_{off} of the bulb. (It can be checked that there is no need to consider the time delay caused by the traversal of the pulse from A to the bulb because it cancels away from the equations in both the frames.) Thus T_{off} of the bulb is equal to the sum of the time (t_1) required for D to reach B in Fig. 1 and the time (t_2) required for the pulse sent at $\hat{R}$ to reach back to A:

$$T_{\text{off}} = t_1 + t_2 = \frac{l_0 - l}{\beta c} + \frac{l_0}{\beta_s c}$$

$$= \frac{l_0}{c}\left(\frac{1}{\beta_s} + \frac{1}{\beta}\right) - \frac{l_0\sqrt{1-\beta^2}}{\beta c}, \qquad (2)$$

where $\beta_s c$ is the speed of the electromagnetic signal along the two-rail transmission line. However, we do not need an expression for β_s: *our purpose is not to calculate T_{off} but only to verify Eq.* (1).

Let us now compute the off time T'_{off} in the shuttle frame. Refer to Fig. 2. Taking, for instance, $t' = 0$ at event $\hat{R}$, let us call t'_1 as the time at which event $\hat{L}$ occurs, i.e., the instant when C beaks contact with A and stops sending current to the bulb. Let us call t'_2 as the time at which the pulse sent by B at event $\hat{R}$ reaches A on its path to the bulb. We have already shown that $t'_2 > t'_1$. Thus the off time T'_{off} of the bulb is given by

$$T'_{\text{off}} = t'_2 - t'_1. \qquad (3)$$

From Fig. 2 it is obvious that

$$t'_1 = \frac{l_0 - l}{\beta c} = \frac{l_0(1-\sqrt{1-\beta^2})}{\beta c}. \qquad (4)$$

In order to calculate t'_2, we note that it is the time at the instant when the electromagnetic pulse traveling along the *moving* transmission line overtakes the *moving* end A of the bridge AB. Let $\beta'_s c$ be the speed of electromagnetic waves along the moving transmission line. Then it is obvious from Fig. 2 that t'_2 satisfies the equation

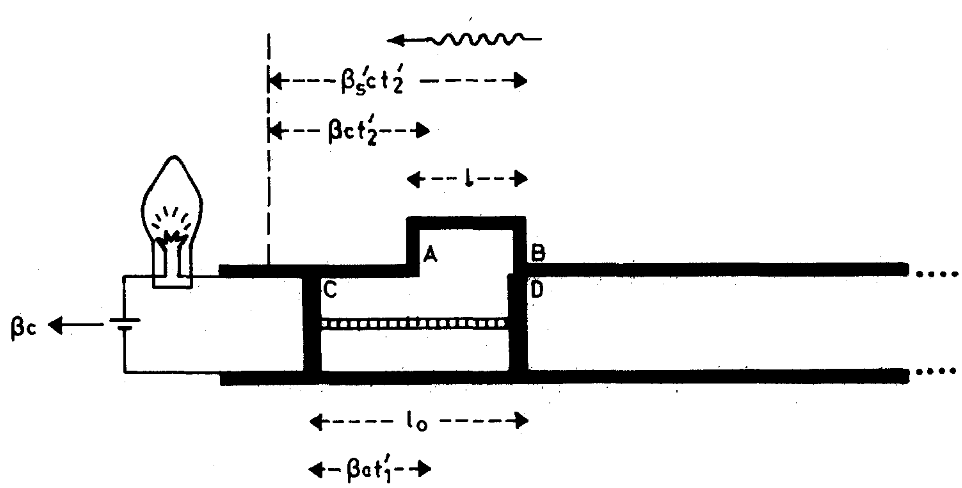

Fig. 2. Configuration of the circuit in the shuttle frame.

$$\beta'_s ct'_2 = \beta ct'_2 + l. \tag{5}$$

From Eq. (5) it follows that

$$t'_2 = l_0\sqrt{1-\beta^2}/c(\beta'_s - \beta). \tag{6}$$

Thus from Eqs. (3), (4), and (6), we get

$$T'_{\text{off}} = \frac{l_0\sqrt{1-\beta^2}}{c}\left(\frac{1}{\beta'_s - \beta} + \frac{1}{\beta}\right) - \frac{l_0}{\beta c}. \tag{7}$$

We now recall that $\beta_s c$ is the speed of the electromagnetic pulse with respect to the transmission line and βc is the speed of the transmission line with respect to the shuttle. Hence the speed $\beta'_s c$ of the electromagnetic pulse with respect to the shuttle is given by the relativistic velocity addition formula:

$$\beta'_s = (\beta_s + \beta)/(1 + \beta_s\beta). \tag{8}$$

Substituting for β'_s in Eq. (7) from Eq. (8), it is a simple matter to get

$$T'_{\text{off}} = \frac{1}{\sqrt{1-\beta^2}}\left[\frac{l_0}{c}\left(\frac{1}{\beta_s} + \frac{1}{\beta}\right) - \frac{l_0\sqrt{1-\beta^2}}{\beta c}\right]. \tag{9}$$

A comparison of Eqs. (9) and (2) reveals that

$$T'_{\text{off}} = \gamma T_{\text{off}}, \tag{10}$$

which is identical to Eq. (1) and which we set out to verify. This, therefore, completely resolves the paradox.

ACKNOWLEDGMENTS

I thank Dr. Debabrata Basu for many helpful discussions. Sumita Das presented this paper in her undergraduate curricular seminar. Many helpful comments of a referee are gratefully acknowledged.

APPENDIX

As is well known, the theory of relativity is full of unexpected results. It is said that Einstein himself was surprised by the magnitude of the Thomas precession.[1] Most of us are puzzled when we first face the paradoxes of relativity. Clearly, there is no room in physics for any real paradox—all paradoxes are due to flaws in our understanding. But it is precisely for this reason that they are such a help in clarifying our confusion.[2] We give here a brief outline of several paradoxes in relativity.

1. Length contraction paradoxes

The most popular is the pole-vaulter's paradox[3]: Can the running pole vaulter with the Lorentz-contracted pole be trapped in the small barn? The paradox is resolved when it is recalled that shock waves have a finite speed of propagation.[4] If both the doors of the barn are treated as open, there is no collision and hardly any paradox aside from a reversal of time ordering of events.[5] If, on the other hand, the rear door is treated as closed, the pole vaulter is thrown out of his inertial frame, which leads to acceleration and stress effects.

Does the sliding coin drop down the hole in the table?[6] This paradox is resolved by recalling that there are no rigid bodies in relativity. The coin "flows," as it were, down the moving hole in the coin's frame. The pull of gravity, once again, throws the coin out of its inertial frame.

Will the moving brick be able to clear the slit in the moving wall?[7] The clue lies in Wigner rotation.[8] It is thus a "two-dimensional" length contraction paradox.[9]

2. Visual appearance of moving objects

Can we see and photograph length contraction? The answer[10] depends on the distance of the moving object and is affected by an apparent rotation in perspective due to aberration.

3. Stress effects due to length contraction

When a rod accelerates and shrinks, does it experience additional stresses?[11] This question has led to a critical examination of rigidity, simultaneity, and the meaning of proper frame for accelerating objects.[12]

4. Upper limit on the proper length

When the front end of a rod accelerates and the rod suffers length contraction, the average speed of the rear end depends on the rest length of the rod. This gives an interesting upper limit on the rest length of a rod.[13]

5. The thread paradox

Can a flexible string exert a shear force? The solution of this paradox lies in the fact that force and acceleration need not be collinear at high velocities.[14]

6. The right-angle lever paradox

Originally attributed to von Laue, this deals with the strikingly paradoxical result that the equilibrium of a lever depends on the state of motion of the observer. The resolution lies in the fact that a net torque need not produce rotation but only a change in the angular momentum.[15]

7. Tachyon exchange thought experiments

What is exchanged when a tachyon is exchanged? Can tachyons be used as signals and detected? Does the existence of tachyons violate causality and laws of thermodynamics? These provocative questions help critically analyze all the concepts of relativity.[16]

8. Superluminal velocities

There are many situations when a speed exceeds that of light in vacuum. These are in all cases unphysical speeds.[17]

9. Oppenheimer paradox and related experiments

A number of little-known experiments of the unipolar induction variety involving rotating frames with induced polarizations and magnetizations are likely to assume practical importance (as has been the case with the Sagnac effect).[18]

10. The twin paradox

This is in a class by itself.[19] Everyone agrees that there is no symmetry in the twins' motions and so no paradox. But is has generated more literature and livelier debate[20] than all the rest put together. Some resolve it within special relativity using Doppler-shifted signals.[21] Others place it roundly in general relativity.[22] The issue perhaps can be settled by experiments.[23] But despite overwhelming experimental evidence of the darkness of the night sky, Olber's paradox[24] did take a century to resolve, and Zeno's absurd contentions[25] stood their ground until whole new branches of science came into being. Perhaps the last word on the twin paradox has yet to be said.

[1]G. E. Uhlenbeck, Phys. Today **29**, 43 (1976). For Thomas precession see A. C. Hirshfeld and F. Metzger, Am. J. Phys. **54**, 550 (1986) and the references therein; in particular, W. H. Furry, Am. J. Phys. **23**, 517 (1955).

[2]See Feynman's remarks in the preface to the field angular momentum paradox in R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, PA, 1964), Vol. II, Sec. 17-4. To the criticism we often hear that proposing paradoxes and then resolving them is a self-defeating exercise we may reply (with apologies to the poet Robert Graves) that in resolving a paradox we truly progress from a confusion of our understanding to an understanding of our confusion.

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SOLUTION TO THE PROBLEM ON P. 874

The equation for the orbit on a displaced circle is

$$r^2 + 2ra\epsilon\cos(\theta) - a^2(1-\epsilon^2) = 0\,, \tag{1}$$

which may be written as

$$r/a = -\epsilon\cos(\theta) + [1-\epsilon^2\sin^2(\theta)]^{1/2}\,. \tag{2}$$

For $\theta = 0$ and $\theta = \pi$ the radial distance from the origin is the same for both orbits. The largest difference occurs at $\theta = \pm\pi/2$, where

$$r/a = 1-\epsilon^2 \tag{3}$$

for the ellipse, and

$$r/a = (1-\epsilon^2)^{1/2} \tag{4}$$

for the displaced circle.

The fractional difference is

$$\Delta = (1-\epsilon^2)^{-1/2} - 1 \approx \epsilon^2/2\,. \tag{5}$$

Among the planets in the solar system $\Delta > 1\%$ only for Mercury and Pluto.

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