Parte I
Sistemi di coordinate.

Coordinate sferiche.

Trasformazioni.

$$\begin{aligned} x &= r\sin\vartheta\cos\phi \\ y &= r\sin\vartheta\sin\phi \\ z &= r\cos\vartheta \end{aligned} \qquad \begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \vartheta &= \arccos\left(\frac{z}{r}\right) = \arcsin\left(\frac{1}{r}\sqrt{x^2+y^2}\right) \\ \phi &= \arctan\left(\frac{y}{x}\right) \end{aligned}$$

$$r \in [0,\infty) \quad \vartheta \in [0,\pi] \quad \phi \in [0,2\pi)$$

Elementi di linea, superficie e volume.

$$\begin{aligned} d\mathbf{l} &= dr\,\hat{\mathbf{r}} + r\,d\vartheta\hat{\boldsymbol{\theta}} + r\sin\vartheta d\phi\hat{\boldsymbol{\phi}} \\ d\mathbf{s} &= r^2\sin\vartheta\,d\vartheta\,d\phi\hat{\mathbf{r}} \\ dV &= r^2\sin\vartheta\,d\vartheta\,d\phi\,dr \end{aligned}$$

Vettori e versori.

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\sin\vartheta\cos\phi \\ r\sin\vartheta\sin\phi \\ r\cos\vartheta \end{pmatrix} \qquad \hat{\boldsymbol{\theta}} = \frac{d\mathbf{r}/d\vartheta}{|d\mathbf{r}/d\vartheta|} = \begin{pmatrix} \cos\vartheta\cos\phi \\ \cos\vartheta\sin\phi \\ -\sin\vartheta \end{pmatrix}$$

$$\hat{\mathbf{r}} = \frac{d\mathbf{r}/dr}{|d\mathbf{r}/dr|} = \begin{pmatrix} \sin\vartheta\cos\phi \\ \sin\vartheta\sin\phi \\ \cos\vartheta \end{pmatrix} \qquad \hat{\boldsymbol{\phi}} = \frac{d\mathbf{r}/d\phi}{|d\mathbf{r}/d\phi|} = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}$$

Derivate parziali dei versori.

$$\frac{\partial\hat{\mathbf{r}}}{\partial\vartheta} = \begin{pmatrix} \cos\vartheta\cos\phi \\ \cos\vartheta\sin\phi \\ -\sin\vartheta \end{pmatrix} = \hat{\boldsymbol{\theta}} \qquad \frac{\partial\hat{\mathbf{r}}}{\partial\phi} = \begin{pmatrix} -\sin\vartheta\sin\phi \\ \sin\vartheta\cos\phi \\ 0 \end{pmatrix} = \sin\vartheta\hat{\boldsymbol{\phi}}$$

$$\frac{\partial\hat{\boldsymbol{\theta}}}{\partial\vartheta} = \begin{pmatrix} -\sin\vartheta\cos\phi \\ -\sin\vartheta\sin\phi \\ -\cos\vartheta \end{pmatrix} = -\hat{\mathbf{r}} \qquad \frac{\partial\hat{\boldsymbol{\theta}}}{\partial\phi} = \begin{pmatrix} -\cos\vartheta\sin\phi \\ \cos\vartheta\cos\phi \\ 0 \end{pmatrix} = \cos\vartheta\hat{\boldsymbol{\phi}}$$

$$\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\phi} = \begin{pmatrix} -\cos\phi \\ -\sin\phi \\ 0 \end{pmatrix} = -\sin\vartheta\,\hat{\mathbf{r}} - \cos\vartheta\hat{\boldsymbol{\theta}}$$

$$\frac{\partial\hat{\mathbf{r}}}{\partial r} = 0 \qquad \frac{\partial\hat{\boldsymbol{\theta}}}{\partial r} = 0 \qquad \frac{\partial\hat{\boldsymbol{\phi}}}{\partial r} = 0 \qquad \frac{\partial\hat{\boldsymbol{\phi}}}{\partial\vartheta} = 0$$

Derivate temporali.

$$\begin{aligned} \dot{\mathbf{r}} &= \dot{r}\,\hat{\mathbf{r}} + r\sin\vartheta\dot{\phi}\hat{\boldsymbol{\phi}} + r\dot{\vartheta}\hat{\boldsymbol{\theta}} \\ \dot{\hat{\mathbf{r}}} &= \sin\vartheta\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{\vartheta}\hat{\boldsymbol{\theta}} \\ \dot{\hat{\boldsymbol{\theta}}} &= \cos\vartheta\dot{\phi}\hat{\boldsymbol{\phi}} - \dot{\vartheta}\hat{\mathbf{r}} \\ \dot{\hat{\boldsymbol{\phi}}} &= -\left(\sin\vartheta\,\hat{\mathbf{r}} + \cos\vartheta\hat{\boldsymbol{\theta}}\right)\dot{\vartheta} \end{aligned}$$

Coordinate cilindriche.

Trasformazioni.

$$\begin{aligned} x &= r\cos\vartheta \\ y &= r\sin\vartheta \\ z &= z \end{aligned} \qquad \begin{aligned} r &= \sqrt{x^2+y^2} \\ \vartheta &= \arctan\left(\frac{y}{x}\right) \\ z &= z \end{aligned}$$

$$r \in [0,\infty) \quad \vartheta \in [0,2\pi) \quad z \in (-\infty,\infty)$$

Elementi di linea, superficie e volume.

$$\begin{aligned} d\mathbf{l} &= dr\,\hat{\mathbf{r}} + r\,d\vartheta\hat{\boldsymbol{\theta}} + dz\,\hat{\mathbf{z}} \\ d\mathbf{s} &= r\,d\vartheta\,dz\,\hat{\mathbf{r}} \\ dV &= r\,drd\vartheta dz \end{aligned}$$

Vettori e versori.

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\vartheta \\ r\sin\vartheta \\ z \end{pmatrix} \qquad \hat{\boldsymbol{\theta}} = \frac{d\mathbf{r}/d\vartheta}{|d\mathbf{r}/d\vartheta|} = \begin{pmatrix} -\sin\vartheta \\ \cos\vartheta \\ 0 \end{pmatrix}$$

$$\hat{\mathbf{r}} = \frac{d\mathbf{r}/dr}{|d\mathbf{r}/dr|} = \begin{pmatrix} \cos\vartheta \\ \sin\vartheta \\ 0 \end{pmatrix} \qquad \hat{\mathbf{z}} = \frac{d\mathbf{r}/dz}{|d\mathbf{r}/dz|} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Derivate parziali dei versori.

$$\frac{\partial\hat{\mathbf{r}}}{\partial r} = 0 \quad \frac{\partial\hat{\boldsymbol{\theta}}}{\partial r} = 0 \quad \frac{\partial\hat{\mathbf{z}}}{\partial r} = 0 \qquad \frac{\partial\hat{\mathbf{r}}}{\partial\vartheta} = \begin{pmatrix} -\sin\vartheta \\ \cos\vartheta \\ 0 \end{pmatrix} = \hat{\boldsymbol{\theta}}$$

$$\frac{\partial\hat{\mathbf{r}}}{\partial z} = 0 \quad \frac{\partial\hat{\boldsymbol{\theta}}}{\partial z} = 0 \quad \frac{\partial\hat{\mathbf{z}}}{\partial z} = 0 \qquad \frac{\partial\hat{\boldsymbol{\theta}}}{\partial\vartheta} = \begin{pmatrix} -\cos\vartheta \\ -\sin\vartheta \\ 0 \end{pmatrix} = -\hat{\mathbf{r}}$$

$$\frac{\partial\hat{\mathbf{z}}}{\partial\vartheta} = 0$$

Derivate temporali.

$$
\begin{aligned}
\dot{\mathbf{r}} &= \dot{r}\,\hat{\mathbf{r}} + r\dot{\vartheta}\hat{\boldsymbol{\theta}} + \dot{z}\hat{\mathbf{z}} \\
\dot{\hat{\mathbf{r}}} &= \dot{\vartheta}\hat{\boldsymbol{\theta}} \\
\dot{\hat{\boldsymbol{\theta}}} &= -\dot{\vartheta}\hat{\mathbf{r}} \\
\dot{\hat{\mathbf{z}}} &= 0
\end{aligned}
$$

Parte II
Operatori differenziali.

Rappresentazione in differenti sistemi di ccordinate.

Coordinate cartesiane.

$$
\begin{aligned}
\nabla f &= \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} \\
\nabla \cdot \mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\
\nabla \times \mathbf{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}} \\
&= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\
\nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\
\nabla^2 \mathbf{A} &= \triangle \mathbf{A} = \frac{\partial^2 \mathbf{A}}{\partial x^2} + \frac{\partial^2 \mathbf{A}}{\partial y^2} + \frac{\partial^2 \mathbf{A}}{\partial z^2}
\end{aligned}
$$

Coordinate sferiche.

$$
\begin{aligned}
\nabla f &= \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \vartheta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\vartheta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} \\
\nabla \cdot \mathbf{A} &= \frac{\partial A_r}{\partial r} + \frac{2A_r}{r} + \frac{1}{r}\frac{\partial A_\vartheta}{\partial \vartheta} + \frac{A_\vartheta}{r\tan\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial A_\phi}{\partial \phi} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta A_\vartheta\right) + \frac{1}{r\sin\vartheta}\frac{\partial A_\phi}{\partial \phi} \\
\nabla \times \mathbf{A} &= \left(\frac{1}{r}\frac{\partial A_\phi}{\partial \vartheta} + \frac{A_\phi}{r\tan\vartheta} - \frac{1}{r\sin\vartheta}\frac{\partial A_\vartheta}{\partial \phi}\right)\hat{\mathbf{r}} + \left(\frac{1}{r\sin\vartheta}\frac{\partial A_r}{\partial \phi} - \frac{\partial A_\phi}{\partial r} - \frac{A_\phi}{r}\right)\hat{\boldsymbol{\theta}} + \\
&\quad + \left(\frac{\partial A_\vartheta}{\partial r} + \frac{A_\vartheta}{r} - \frac{1}{r}\frac{\partial A_r}{\partial \vartheta}\right)\hat{\boldsymbol{\phi}} \\
\nabla^2 f &= \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \vartheta^2} + \frac{1}{r^2\tan\vartheta}\frac{\partial f}{\partial \vartheta} + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 f}{\partial \phi^2}
\end{aligned}
$$

Coordinate cilindriche.

$$\begin{aligned}
\nabla f &= \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \vartheta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} \\
\nabla \cdot \mathbf{A} &= \frac{\partial A_r}{\partial r} + \frac{A_r}{r} + \frac{1}{r}\frac{\partial A_\vartheta}{\partial \vartheta} + \frac{\partial A_z}{\partial z} \\
\nabla \times \mathbf{A} &= \left(\frac{1}{r}\frac{\partial A_z}{\partial \vartheta} - \frac{\partial A_\vartheta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \\
&\quad + \left(\frac{\partial A_\vartheta}{\partial r} + \frac{A_\vartheta}{r} - \frac{1}{r}\frac{\partial A_r}{\partial \vartheta}\right)\hat{\mathbf{z}} \\
\nabla^2 f &= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \vartheta^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

Identità.

Vettori. nota: $\mathbf{r} = (x, y, z)$, da non confondere con la coordinata cilindrica omonima.

$$\nabla r = \hat{\mathbf{r}} \qquad \nabla \times \mathbf{r} = 0$$

$$\nabla\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3} \quad \nabla \cdot \hat{\mathbf{r}} = \frac{2}{r} \quad \nabla \times \hat{\mathbf{r}} = 0$$

$$\nabla \cdot \mathbf{r} = 3 \quad \nabla \times \mathbf{r} = 0 \quad \nabla r = \hat{\mathbf{r}}$$

Gradiente.

$$\nabla (fg) = g\nabla f + f\nabla g$$

$$\nabla (\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) - (\nabla \times \mathbf{A}) \times \mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B}$$

Divergenza.

$$\begin{aligned}
\nabla \cdot (f\mathbf{A}) &= (\nabla f) \cdot \mathbf{A} + f(\nabla \cdot \mathbf{A}) \\
\nabla \cdot (\mathbf{A} \times \mathbf{B}) &= (\nabla \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \\
\nabla \cdot (\nabla \times \mathbf{A}) &= 0
\end{aligned}$$

Rotore.

$$\begin{aligned}
\nabla \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A}(\nabla \cdot \mathbf{B}) - (\nabla \cdot \mathbf{A})\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \\
\nabla \times (\nabla \times \mathbf{A}) &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} \\
\nabla \times (f\mathbf{A}) &= (\nabla f) \times \mathbf{A} + f(\nabla \times \mathbf{A}) \\
\nabla \times (\nabla f) &= 0
\end{aligned}$$