A Characterization of Approximately-Controllable Linear Stochastic Differential Equations*

Rainer Buckdahn[†] Marc Quincampoix[†] Gianmario Tessitore[‡]

Abstract

The aim of this paper is to give two characterizations of approximate controllability of a controlled linear stochastic differential equation. The first characterization can be formulated by saying that an ad hoc backward stochastic differential equation has only one solution which is constant equal to zero. The second criterion for approximate controllability - which is the main result of the present Note - says that the only invariant (or viable) set contained in a suitable linear space is the trivial space 0. A explicit way for checking the invariance (or viability) of a linear space is provided. We emphasize that the characterization of approximate controllability is easily computable.

Keywords: Stochastic Viability, Stochastic Control, Controllability, Backward Stochastic Differential Equation.

1 Statement of the Problem and of the Main Result

The objective of the paper is to study controllability for the following linear stochastic differential equation :

$$\begin{cases} dy(t) = \Big(Ay(t)dt + Bu(t)\Big)dt + \sum_{i=1}^{m} C_i y(t) d\beta_i(t) \\ y(0) = x, \end{cases} \tag{1.1}$$

where $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)$, $C_i \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $i = 1, ..., m$, and the processes $\{\beta_1, ..., \beta_m,$ i=1,...,m$\}$ are independent Brownian motions defined on a complete probability space $(\Omega, \mathcal{E}, \mathbb{P})$. We denote by $\{\mathcal{F}_t : t \geq 0\}$ the filtration they generate, augmented with all $\mathbb{P}$-null sets of $\mathcal{E}$.

A process $u : \Omega \times [0, +\infty[\to \mathbb{R}^d$ is said to be an *admissible control* if it is $(\mathcal{F}_t)$-predictable and such that $\mathbb{E} \int_0^T |u(s)|^2 ds < +\infty$, for all $T > 0$.

As it is well known, under the above assumptions, for all initial datum $x \in \mathbb{R}^n$ and all admissible control u, equation (1.1) (intended in Ito sense) admits a unique predictable solution

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[†]Laboratoire de Mathématiques, UMR CNRS 6205, Université de Brest 6, Avenue Victor Le Gorgeu 29200 Brest, France. (`Rainer.Buckdahn@univ-brest.fr, Marc.Quincampoix@univ-brest.fr`).

[‡]Dipartimento di Matematica, Università di Parma, via D'Azeglio 85, 43100 Parma, Italia. (`gianmario.tessitore@unipr.it`).

y with continuous trajectories. Moreover, this solution is square integrable over all compact time intervals, i.e., for all $T > 0$, $\mathbb{E}\sup_{s\in[0,T]}|y(s)|^2 < +\infty$. Such a solution (representing the state in the system) will be denoted by $y(\cdot, x, u)$.

Definition 1.1 *We say that equation (1.1) is* approximately controllable *if for all* $x \in \mathbb{R}^n$*, all* $T > 0$*, all* $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R}^n)$ *and all* $\varepsilon > 0$ *there exists an admissible control* u *such that* $\mathbb{E}|y(T, x, u) - \eta|^2 \leq \epsilon$*.*

Moreover, we say that equation (1.1) is approximately null controllable *if the above condition holds in the particular case* $\eta = 0$*.*

We also give the following definition

Definition 1.2 *Given* $m + 1$ *linear operators* $L, M_1, ..., M_m \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$*, a linear subspace* $V \subset \mathbb{R}^n$ *is said to be* $(L; M_1, ..., M_m)$*-strictly invariant if* $LV \subset \mathit{Span}\langle V, M_1V, ..., M_mV\rangle$*.*

Remark 1.1 We notice that $V \subset \mathbb{R}^n$ is $(L; M_1, ..., M_m)$-strictly invariant if and only if there exists operators $K_1, ..., K_m \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ such that $K_iV \subset V$ and $(L+M_1K_1+...+M_mK_m)V \subset V$ or, equivalently, if and only if for all $v \in V$ there exist $w_1, ..., w_m \in V$ such that $Lv + M_1w_1 + ... + M_mw_m \in V$.

Remark 1.2 Given an arbitrary linear subspace $V \subset \mathbb{R}^n$ it is easy to compute the largest linear subspace of V which is $(L; M_1, ...M_m)$-strictly invariant (see also [17] and [12] for similar computations in wider generality). Indeed, if we put

$$V_0 = V;\ V_{i+1} = \{v \in V_i : Lv \in \mathrm{Span}\langle V_i, M_1V_i, ..., M_mV_i\rangle\} = L^{-1}(\mathrm{Span}\langle V_i, M_1V_i, ..., M_mV_i\rangle) \cap V_i,$$

then V_n is the required maximal $(L; M_1, ..., M_m)$-strictly invariant subspace of V.

Let us now present the main result of the present paper:

Theorem 1.3 *The following assertions are equivalent:*

1. *Equation (1.1) is approximately controllable.*
2. *Equation (1.1) is approximately null controllable.*
3. *The largest* $(A^*; C_1^*, ..., C_m^*)$*-strictly invariant subspace of* $\mathrm{Ker}\, B^*$ *is the origin.*

Remark 1.4 Remark 1.2 implies that condition *3* is easily computable.

2 Remarks on Related Literature

In [7] (see also [9]) S.Peng has studied the 'exact controllability' and 'exact terminal controllability' of the following stochastic linear equation with control acting on the noise term as well:

$$dy(t) = \Big(Ay(t)dt + Bu(t)\Big)dt + \Big(Cy(t) + Du(t)\Big)d\beta(t). \tag{2.1}$$

In particular in [7] it is shown that equation (2.1) is exactly terminal controllable (that is for each final condition η in $L^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R}^n)$ there exists an initial datum $y(0)$ in $\mathbb{R}^n$ and an admissible control u such that $y(T) = \eta$, $\mathbb{P}$-a.s.) if and only if D has full rank. Then algebraic conditions of Kalman type under which equation (2.1) with full rank D is exactly terminal controllable (that is each final condition can be exactly replicated starting from any initial datum . Our result can bee seen as a counterpart of the ones described above. Namely here we do not allow the control to act on the noise ($D = 0$ in (2.1). Consequently we can not expect to have exact terminal controllability (nor, a fortiori, exact controllability). Nevertheless we prove that, if we weaken our request from exact to approximate controllability, then the condition can be satisfied and in particular if satisfied if and only if computable algebraic conditions on A, B, and C hold.

We also notice that Definition 1.2 is a modification of the definition of (A, B) invariant subspaces that has been introduced in [17] and then generalized in [12], [13], [14]. In particular in [16] this last concept was used to obtain, by algebraic Riccati equation methods, a characterization of stochastic linear equations admitting a feedback that stabilizes the system for all noise intensities.

Finally we notice that the property of approximate controllability treated here is tightly connected to the one of exact null controllability. See [10] relation between this last property and backward stochastic differential equations or Riccati equations, both in finite in in infinite dimensional spaces; moreover see [3] for a review on partial result in the direction of proving null controllability of specific infinite dimensional stochastic evolution equations.

3 The Dual Equation

Let us consider the following backward stochastic differential equation

$$\begin{cases} dp(t) = -\Big(A^*p(t) + \sum_{i=1}^m C_i^* q^i(t)\Big)dt + \sum_{i=1}^m q^i(t)d\beta_i(t) \\ p(T) = \eta. \end{cases} \tag{3.1}$$

It is shown in [8] (se also [1] and [2] for earlier results in the control theory framework) that for all $T > 0$ and all $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R}^n)$ there exists a unique $m+1$-tuple of $\mathbb{R}^n$-valued predictable processes $(p, q^1, ..., q^m)$ such that equation (3.1) is satisfied, p has continuous trajectories and it holds

$$\mathbb{E} \sup_{s\in[0,T]} |p(s)|^2 < +\infty, \quad \mathbb{E}\int_0^T |q^i(s)|^2 ds < +\infty, \; i = 1, ..., m.$$

The following proposition specifies the connection between the above equation and controllability of equation (1.1).

Proposition 3.1 *Equation (1.1) is approximately controllable if and only if for all $T > 0$, every solution to equation (3.1) verifying $B^*p(s) = 0$, $\mathbb{P}$-a.s., $\forall s \in [0, T]$ is trivial, (i.e., is such that $p(s) = 0$, $\mathbb{P}$-a.s. for all $s \in [0, T]$).*

Moreover, equation (1.1) is approximately null controllable if and only if for all $T > 0$, every solution to equation (3.1) satisfying $B^*p(s) = 0$, $\mathbb{P}$-a.s., $\forall s \in [0, T]$, also verifies $p(0) = 0$.

Proof. For fixed $T > 0$ we deduce from Itô's formula that

$$d\langle p(s), y(s, x, u))\rangle = \langle p(s), Bu(s)\rangle ds + \sum_{i=1}^m \big(\langle q^i(s), y(s, x, u)\rangle + \langle p(s), C_i y(s, x, u)\rangle\big)\, d\beta^i(s),$$

where $\langle\cdot\,,\cdot\rangle$ denotes the inner product in $\mathbb{R}^n$. Since

$$\mathbb{E}\left(\int_0^T \big(\langle q^i(s), y(s, x, u)\rangle + \langle p(s), C_i y(s, x, u)\rangle\big)^2 ds\right)^{1/2} < +\infty, \quad i = 1, ..., m,$$

we can compute the mean value and obtain

$$\mathbb{E}\langle p(T), y(T,x,u))\rangle - \mathbb{E}\langle p(0), x)\rangle = \mathbb{E}\int_0^T \langle p(s), Bu(s)\rangle ds. \tag{3.2}$$

Let $L^2_{\mathcal{P}}([0,T],\mathbb{R}^d)$ be the space of all predictable processes $u : \Omega \times [0,+\infty[\to \mathbb{R}^d$ satisfying $\mathbb{E}\int_0^T |u(s)|^2 ds < +\infty$, endowed with the natural norm, and define the linear operator

$$M_T : L^2_{\mathcal{P}}([0,T],\mathbb{R}^d) \to L^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R}^n), \qquad M_T u := y(T,u,0). \tag{3.3}$$

It is evident that equation (1.1) is approximately controllable if and only if, for all $T > 0$, the image of M_T is dense in $L^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R}^n)$. Moreover by relation (3.2) (with $x = 0$ and $p(T)$ being equal to an arbitrary $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R}^n)$ we deduce that $M_T^*\eta = B^*p$. The first part of the claim follows by observing that the image of M_T is dense if and only if the kernel of M_T^* is trivial and by noticing that, thanks to the uniqueness and the continuity of the solution to equation (3.1), $\eta = 0$ if and only if $p(s) = 0$ $\mathbb{P}$-a.s. for all $s \in [0,T]$.

As far as the second part of the proposition is concerned we introduce the linear operator

$$L_T : \mathbb{R}^d \to L^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R}^n), \qquad L_T x := y(T,0,x). \tag{3.4}$$

and notice that equation (1.1) is approximately null controllable if and only if, for all $T > 0$, $L_T[\mathbb{R}^n] \subset \overline{M_T[L^2_{\mathcal{P}}([0,T],\mathbb{R}^d)]}$.

Then again by relation (3.2) (now with $u = 0$) we get that $L_T^*\eta = p(0)$, and the claim follows recalling that $L_T[\mathbb{R}^n] \subset \overline{M_T[L^2_{\mathcal{P}}([0,T],\mathbb{R}^d)]}$ if and only if $\mathrm{Ker}[M_T^*] \subset \mathrm{Ker}[L_T^*]$. □

Now we interprete equation (3.1) as a forward equation. More precisely, we consider the equation:

$$\begin{cases} dp(t) = -\Big(A^*p(t) + \sum_{i=1}^m C_i^* q^i(t)\Big)dt + \sum_{i=1}^m q^i(t) d\beta_i(t) \\ p(0) = \theta. \end{cases} \tag{3.5}$$

For all $\theta \in \mathbb{R}^n$ and $q^i \in L^2_{\mathcal{P}}([0,T],\mathbb{R}^d)$, $i = 1, ..., m$, there exists a unique predictable solution p with continuous trajectories verifying, for all $T > 0$, $\mathbb{E}\sup_{s\in[0,T]} |p(s)|^2\, ds < +\infty$. We denote this solution by $p(\cdot, q^1, ..., q^m, \theta)$.

By existence and uniqueness of solutions to equation (3.1) we get that for all $T > 0$ and $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R}^n)$, there exists unique $\theta \in \mathbb{R}^n$ and unique $q^i \in L^2_{\mathcal{P}}([0,T],\mathbb{R}^d)$, $i = 1, ..., m$ such that $p(\cdot, q^1, ..., q^m, \theta) = \eta$. Thus Proposition 3.1 can be reformulated as follows:

Proposition 3.2 *Equation (1.1) is approximately controllable if and only if for all $T > 0$, all $\theta \in \mathbb{R}^n$ and all $q^i \in L^2_{\mathcal{P}}([0,T],\mathbb{R}^d)$, $i = 1, ..., m$, for which $B^*p(s, q^1, ..., q^m, \theta) = 0$, $\mathbb{P}$-a.s. $\forall s \in [0,T]$ it holds $p(s, q^1, ..., q^m, \theta) = 0$, $\mathbb{P}$-a.s. $\forall s \in [0,T]$.*

Moreover equation (1.1) is approximately null controllable if and only if for all $T > 0$, all $\theta \in \mathbb{R}^n$ and all $q^i \in L^2_{\mathcal{P}}([0,T],\mathbb{R}^d)$, $i = 1, ..., m$, such that $B^*p(s, q^1, ..., q^m, \theta) = 0$, $\mathbb{P}$-a.s. $\forall s \in [0,T]$, it holds $\theta = 0$.

4 Local in Time Viability

Proposition 3.2 justifies our interest in the following concept:

Definition 4.1 *A linear subspace $V \subset \mathbb{R}^n$ is said to be* locally in time viable *(l.i.t.v.) with respect to equation (2.5) if for all $\theta \in V$ there exists a $T > 0$ and $q^i \in L^2_{\mathcal{P}}([0,T],\mathbb{R}^d)$, $i = 1, ..., m$, such that $p(s, q^1, ..., q^m, \theta) \in V$ $\mathbb{P}$-a.s. for all $s \in [0,T]$.*

Moreover, the set of all $\theta \in V$ for which there exists a $T > 0$ and $q^i \in L^2_{\mathcal{P}}([0,T],\mathbb{R}^d)$, $i = 1, ..., m$, such that $p(s, q^1, ..., q^m, \theta) \in V$ $\mathbb{P}$-a.s. for all $s \in [0,T]$, is called the local viability kernel *of V*

Note that the above notion of local (in time) viability slightly differs from the local (in space) viability defined and studied in [6].

We recall here some basic facts on Riccati equations and linear quadratic optimal control related to equation (3.5). The reader can find proofs (in a much wider generality) for instance in [17] or [18].

For an arbitrarily fixed subspace $V \subset \mathbb{R}^n$ let Π_V denote the orthogonal projection on V. For all $N \geq 1$, we consider the following Riccati equation with values in $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$:

$$\begin{cases} P'_N(s) = -AP_N(s) - P_N(s)A^* - \sum_{i=1}^{m} P_N(s)C_i^*[I + P_N(s)]^{-1}C_iP_N(s) + N\Pi_V, \ t \geq 0, \\ P_N(0) = 0. \end{cases} \tag{4.1}$$

The above equation admits a unique continuous solution with values in the cone of linear symmetric non-negative operators in $\mathbb{R}^n$. Moreover, forall $t > 0$ the sequence $\{P_N(t) : N \in \mathbb{N}\}$ increases in N.

The following equation, satisfied for all $0 \leq t \leq T$, is known as *fundamental relation*:

$$\begin{aligned} \mathbb{E}\langle P_N(T-t)p_t, p_t\rangle = E\int_t^T \left[N|\Pi_V p_s|^2 + \sum_{i=1}^m |q^i(s)|^2 \right] ds \\ - \sum_{i=1}^m \mathbb{E}\int_t^T \left| [I + P_N(T-s)]^{1/2} \left[[I + P_N(T-s)]^{-1}C_iP(T-s)p(s) - q^i(s) \right] \right|^2 ds \end{aligned} \tag{4.2}$$

where we use the short-writing $p_s = p(s, q^1, ..., q^m, \theta)$. This relation is got by applying Itô's formula to the product of $P_N(T-t)p_t$ with p_t.

Using (4.2) one can immediately deduce that

$$\langle P_N(T)\theta, \theta\rangle \leq \mathbb{E}\int_0^T \left[N|\Pi_V p(s, q^1, ..., q^m, \theta)|^2 + \sum_{i=1}^m |q^i(s)|^2 \right] ds \tag{4.3}$$

for arbitrary $q^i \in L^2_{\mathcal{P}}([0,T],\mathbb{R}^d)$, $i = 1, ..., m$.

Moreover for all $\theta \in \mathbb{R}^n$ there exist suitable $\overline{q}^i \in L^2_{\mathcal{P}}([0,T],\mathbb{R}^d)$, $i = 1, ..., m$, for which the second term in (4.2) vanishes:

$$\langle P_N(T)\theta, \theta\rangle = \mathbb{E}\int_0^T \left[N|\Pi_V p(s, \overline{q}^1, ..., \overline{q}^m, \theta)|^2 + \sum_{i=1}^m |\overline{q}^i(s)|^2 \right] ds \tag{4.4}$$

We will call these controls $\overline{q}^i \in L^2_{\mathcal{P}}([0,T],\mathbb{R}^d)$, $i = 1, ..., m$ optimal.

Proposition 4.1 *The viability kernel of V with respect to equation (2.5) has the following representation:*

$$\left\{ \theta \in V | \exists T > 0 : \lim_{N\to\infty} \langle P_N(T)\theta, \theta\rangle < +\infty \right\}. \tag{4.5}$$

Proof. If θ is in the viability kernel and $q^i \in L^2_{\mathcal{P}}([0,T],\mathbb{R}^d)$, $i=1,....,m$ are such that $p(s,q^1,...,q^m,\theta) \in V$, $\mathbb{P}$-a.s., for all $s \in [0,T]$ then by (4.3) $\langle P_N(T)\theta,\theta\rangle \le \sum_{i=1}^m \mathbb{E}\int_0^T |q^i(s)|^2 ds$ $\forall N \in \mathbb{N}$.

Vice versa, choosing for every $N \in \mathbb{N}$ the optimal set of controls $(\overline{q}^1_N,...,\overline{q}^m_N)$ we get

$$\langle P_N(T)\theta,\theta\rangle = \mathbb{E}\int_0^T \left[N|\Pi_V p(s,\overline{q}^1,...,\overline{q}^m_N,\theta)|^2 + \sum_{i=1}^m |\overline{q}^i_N(s)|^2 \right] ds.$$

Thus the sequences $\{\overline{q}^i_N : N \in \mathbb{N}\}$, $i=1,...,m$, are bounded in $L^2_{\mathcal{P}}([0,T],\mathbb{R}^d)$ and, consequently, for a suitable subsequence of $\{(\overline{q}^1_N,\dots,\overline{q}^m_N) : N \in \mathbb{N}\}$ (that, abusing notations, will still be denoted by $\{(\overline{q}^1_N,\dots,\overline{q}^m_N) : N \in \mathbb{N}\}$) we can assume that, for some $\overline{q}^i$ in $L^2_{\mathcal{P}}([0,T],\mathbb{R}^d)$, $i = 1,...,m$, $\overline{q}^i_N \rightharpoonup \overline{q}^i$ weakly in $L^2_{\mathcal{P}}([0,T],\mathbb{R}^d)$. Moreover, since equation (3.5) is affine in q^1,...,q^m we have $p(s,\overline{q}^1_N,...,\overline{q}^m_N,\theta) \rightharpoonup p(s,\overline{q}^1,...,\overline{q}^m,\theta)$.

Consequently, by (4.4)

$$\langle P_N(T)\theta,\theta\rangle \ge N\mathbb{E}\int_0^T |\Pi_V p(s,\overline{q}^1_N,...,\overline{q}^m_N,\theta)|^2 ds$$

and we can conclude $\Pi_V p(s,\overline{q}^1_,...,\overline{q}^m_,\theta) = 0$, $\mathbb{P}$-a.s. for every $s \in [0,T]$. □

The next result that will be essential in the following, is now very easy to prove. See [11] for a different proof in a much more general nonlinear situation but with bounded control space. While in [11], the viability kernel is always closed. This is not necessarily the case in our concept (when the set is not a finite dimensional linear space).

Theorem 4.2 *The viability kernel of an arbitrary subspace $V \subset \mathbb{R}^n$ is locally in time viable.*

Proof. Fix θ in the viability kernel and let $T>0$ and $q^i \in L^2_{\mathcal{P}}([0,T],\mathbb{R}^d)$, $i=1,....,m$ such that $p(s,q^1,...,q^m,\theta) \in V$, $\mathbb{P}$-a.s. for every $s \in [0,T]$. Then, for every $t<T$, by (4.2)

$$\mathbb{E}\langle P_N(T-t)p(t,q^1,...,q^m,\theta), p(t,q^1,...,q^m,\theta)\rangle \le \sum_{i=1}^m \mathbb{E}\int_t^T |q^i(s)|^2 ds.$$

Thus, by monotone convergence, $\mathbb{E}\, lim_{N+\infty}\langle P_N(T-t)p(t,q^1,...,q^m,\theta), p(t,q^1,...,q^m,\theta)\rangle < +\infty$ and we can conclude with the help of Proposition 4.1 that $p(t,q^1,...,q^m,\theta)$ belongs, $\mathbb{P}$-a.s. to the viability kernel of V. □

Remark 4.3 In the above argument we prove something more precise then the claim of the theorem. Namely we show that for all θ in the viability kernel of V, all $T>0$, and all $q^i \in L^2_{\mathcal{P}}([0,T],\mathbb{R}^d)$, $i=1,....,m$, for which $\{p(s,q^1,...,q^m,\theta), s \in [0,T]\} \subset V$, $\mathbb{P}$-a.s. , we even have that $\{p(s,q^1,...,q^m,\theta), s \in [0,T]\}$ is $\mathbb{P}$-almost surely a subset of the viability kernel of V.

5 Proof of the Main Result

We start by showing that our problem can now be reduced to the computation of a viability kernel.

Theorem 5.1 *The following assertions are equivalent:*

1. *Equation (1.1) is approximately controllable.*

2. *Equation (1.1) is approximately null controllable.*

3. *The viability kernel of* $\mathrm{Ker}B^*$ *is trivial (i.e., it contains only the origin).*

Proof. *2* $\Leftrightarrow$ *3* is just a reformulation of Proposition 3.2. Moreover *1* $\Rightarrow$ *2* is evident by definition. Thus it remains to prove that *3* $\Rightarrow$ *1.* This will be done by exploiting Proposition 3.2.

Assume that the viability kernel of KerB* is trivial and let $q^i \in L^2_{\mathcal{P}}([0,T],\mathbb{R}^d)$, $i = 1, ..., m$, such that $p(s, q^1, ..., q^m, \theta) \in \mathrm{Ker}\, B^*$, $\mathbb{P}$-a.s. for every $s \in [0,T]$. Then we know that $\theta = 0$ but also, see Remark 4.3, that $p(s, q^1, ..., q^m, \theta)$ belongs to the viability kernel of $\mathrm{Ker}B^*$. But since this last set coincides with the origin we conclude that $p(s, q^1, ..., q^m, \theta) = 0$, $\mathbb{P}$-a.s. for all s in $[0,T]$. □

Then we only need to characterize the local in time viability in an explicit way. We refer the reader to [4] and to the appendix of [5] for other characterizations of stochastic viability for nonlinear systems with bounded control space.

Theorem 5.2 *A subspace* $V \subset \mathbb{R}^n$ *is locally in time viable if and only if it is* $(A^*; C_1^*, ..., C_m^*)$*-strictly invariant.*

Proof. We start proving that any $(A^*; C_1^*, ..., C_m)$-strictly invariant subspace is locally in time viable.

For this we just notice that there exist linear operators K_i, $i = 1, ...m$ such that $K_i V \subset V$ and $(A^* + C_1^* K_1 + ... + C_m^* K_m)V \subset V$. Thus, for any $\theta \in V$, we consider the following forward linear equation:

$$\begin{cases} d\widetilde{p}(t) = -\Big(A^* + \sum_{i=1}^m C_i^* K^i\Big)\widetilde{p}(t)dt + \sum_{i=1}^m K^i \widetilde{p}(t) d\beta_i(t), \\ \widetilde{p}(0) = \theta, \end{cases}$$

which solution $\widetilde{p}$ is clearly in V. If then we set $q^i = K^i \widetilde{p}$ then $p(t, q^1, ..., q^m, \theta) = \widetilde{p}(t) \in V$ for all $t > 0$.

Vice versa, assume now that $\{p(s, q^1, ..., q^m, \theta), s \in [0,T]\} \subset V$ $\mathbb{P}$-a.s., for suitable θ in V, $T > 0$ and q^i in $L^2_{\mathcal{P}}([0,T],\mathbb{R}^d)$, $i = 1, ..., m$.

If we multiply equation (3.5) by $(I - \Pi_V)$ and write $p_t = p(t, q^1, ..., q^m, \theta)$, we get:

$$\begin{cases} d(I - \Pi_V)p_t = -(I - \Pi_V)\Big(A^* p_t + \sum_{i=1}^m C_i^* q^i(t)\Big)dt + \sum_{i=1}^m (I - \Pi_V) q^i(t) d\beta_i(t), \\ p(0) = \theta. \end{cases} \tag{5.1}$$

Since $(I - \Pi_V)p_t = 0$, $\mathbb{P}$-a.s. for every $t > 0$, a successive computation of the quadratic variation in $[0,t]$ of the components of $(I - \Pi_V)p(t)$ yields $\int_0^t |(I - \Pi_V)q^i(s)|^2 ds = 0$, $\mathbb{P}$-a.s. for every $t \in [0,T]$, $i = 1$ ... $,m$. Thus $q^i_s \in V$, $\mathbb{P}$-a.s. for almost every $s \in [0,T]$.

Coming back to equation (5.1) we get $(I - \Pi_V)\Big(A^* p(t) + \sum_{i=1}^m C_i^* q^i(t)\Big) = 0$, $\mathbb{P}$-a.s. for almost every $t \in [0,T]$. If now W is the linear space

$$\left\{\theta \in V | \exists \xi_1, ...\xi_m \in V : A^*\theta + \sum_{i=1}^m C_i^* \xi_i \in V\right\}$$

we have $p_s \in W$ $\mathbb{P}$-a.s. for almost every $t \in [0, T]$. But since p has continuous trajectories and W is closed we get $\theta \in W$ and this completes the proof. □

Let us conclude the proof of Theorem 1.3.

By Theorem 4.2, the viability kernel of KerB^* is locally in time viable thus, by Theorem 5.2, it is $(A^*; C_1^*, ..., C_m^*)$-strictly invariant. Viceversa, again by Theorem 5.2, any $(A^*; C_1^*, ..., C_m^*)$-strictly invariant subspace of KerB^* is locally in time viable and consequently included in the viability kernel of KerB^*. So we can conclude that the viability kernel of KerB^* is the largest $(A^*; C_1^*, ..., C_m^*)$-strictly invariant subspace of KerB^*. The claim follows immediately by Theorem 5.1.

References

[1] J.-M. Bismut. Linear quadratic optimal stochastic control with random coefficients. *SIAM J. Contr. Optim.*, **14** (1976), 419–444.

[2] J.-M. Bismut. Contrôle des systèmes linéaires quadratiques: applications de l'intégrale stochastique. In *Séminaire de Probabilités, XII.* Lecture Notes in Math., 649, Springer, Berlin, 1978.

[3] V. Barbu, G. Tessitore, Considerations on the controllability of stochastic linear heat equations, in *Stochastic partial differential equations and applications (Trento, 2002)*, Lecture Notes in Pure and Appl. Math., 227, pp. 39–51, Dekker, New York, (2002).

[4] R. Buckdahn, S. Peng, M. Quincampoix, C. Rainer. Existence of stochastic control under state constraints *Comptes-Rendus de l'Académie des Sciences.* Série I. PARIS, t. 327 (1998), 17–22.

[5] R. Buckdahn, P. Cardaliaguet, M. Quincampoix. A representation formula for the mean curvature motion *SIAM J. Math. Anal.*, **33** (2001), 827–846.

[6] R. Buckdahn, M.Quincampoix, C. Rainer, A. Rascanu. Stochastic Control with Exit Time and Constraints, Application to Small Time Attainability of sets *Appl. Math. Optim.* **49** (2004), 99–112.

[7] Y. Liu, S. Peng, Infinite horizon backward stochastic differential equation and exponential convergence index assignment of stochastic control systems. *Automatica,* **38** (2002), 1417–1423.

[8] E. Pardoux, S. Peng, Adapted solution of a backward stochastic differential equation, *Systems and Control Lett.*, **14** (1990), 55–61.

[9] S. Peng, Backward stochastic differential equation and exact controllability of stochastic control systems, *Progr. Natur. Sci. (English Ed.)*, **4** (1994), 274–284.

[10] M. Sirbu and G. Tessitore, Null controllability of an infinite dimensional SDE with state and control-dependent noise, *Systems and Control Lett.*, **44** (2001), 385-394.

[11] M. Quincampoix, C. Rainer, Stochastic control and compatible subsets of contraints, *Bulletin des Sciences Mathématiques*, to appear.

[12] J.C. Willems, Almost A(mod B)-invariant subspaces, *Astérisque,* **75-76** (1980), 239–248.

[13] J.C. Willems, Almost invariant subspaces: an approach to high gain feedback design – Part I: almost controlled invariant subspaces *IEEE Trans. Automat. Control.,* **AC-26** (1981), 235–252.

[14] J.C. Willems, Almost invariant subspaces: an approach to high gain feedback design – Part II: almost controlled invariant subspaces *IEEE Trans. Automat. Control.,* **AC-27** (1982), 1071–1085.

[15] J.L. Willems and J.C. Willems, Feedback stabilizability for stochastic systems with state and control depending noise, *Automatica*, **12** (1976), 277–283.

[16] J.L. Willems and J.C. Willems, Robust stabilization of uncertain systems *SIAM J. Contr. Optim.*, **21** (1983), 342–372.

[17] W.M. Wonham, *Linear Multivariable Control: A Geometric Approach*,2nd ed., Springer Verlag, Berlin, New York, 1979.

[18] J. Yong, X. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations.* Springer Verlag, Berlin, New York, 1999.

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