$$\begin{aligned} dX_t &= b\,X_t\,dt + \sigma_1\,dW_1(t) + \sigma_2\,dW_2(t), \qquad && X_0 \\ dY_t &= \alpha\,X_t\,dt + \sigma\,dW_1(t), && Y_0 = 0 \end{aligned}$$

$\mathscr{G}_t = \sigma(Y_s, 0 \le s \le t)$

$$\hat{X}_t = \mathbb{E}(X_t \mid \mathscr{G}_t)$$

$$a = \sigma_1^2 + \sigma_2^2$$

$$b, \quad \sigma_1, \quad \sigma, \quad \alpha$$

Equazione di tipo Riccati

$$R_t' = a + 2\,b\,R_t - \Big(\sigma_1 + \frac{\alpha}{\sigma}\,R_t\Big)^2, \qquad R_0 = \mathsf{Var}(X_0)$$

$$d\hat{X}_t = \Big(b - \alpha\frac{\sigma_1}{\sigma} - \frac{\alpha^2}{\sigma^2}\,R_t\Big)\,\hat{X}_t\,dt + \frac{\alpha\,R_t + \sigma\,\sigma_1}{\sigma^2}\,dY_t, \qquad \hat{X}_0 = \mathbb{E}(X_0)$$

$$\varphi(t) := b - \alpha\frac{\sigma_1}{\sigma} - \frac{\alpha^2}{\sigma^2}\,R_t$$

$$\psi(t) := \frac{\alpha\,R_t + \sigma\,\sigma_1}{\sigma^2}$$

$$\hat{X}_t = e^{\int_0^t \varphi(r)\,dr}\hat{X}_0 + \int_0^t e^{\int_s^t \varphi(r)\,dr}\psi(s)\,dY_s$$

$$= e^{\int_0^t \varphi(r)\,dr}\hat{X}_0 + \psi(t)\,Y_t - \int_0^t Y_s\,(\psi'(s) - \varphi(s)\psi(s))\,e^{\int_s^t \varphi(r)\,dr}\,ds$$