# Differential Equations of Haemodynamics. Lecture Notes on Computational Haemodynamics

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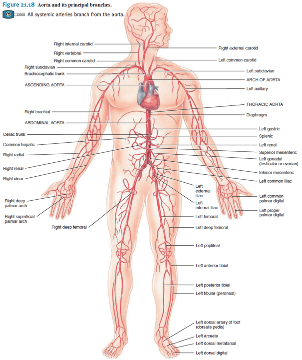
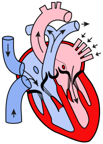
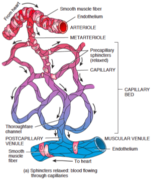
December 5, 2014

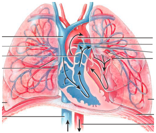
# Multiscale methods in haemodynamics

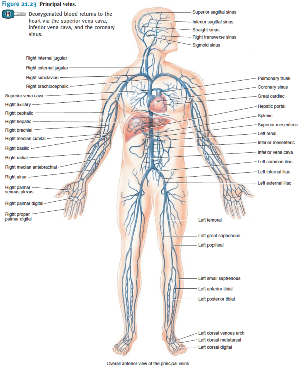
# Components of the human circulation








Figures taken from Tortora and Derrickson, 2009

# Fluid-Structure Interaction (FSI) models

- 3D Navier-Stokes equations for blood flow in compliant vessels
- 3D equations for the dynamics of the structure (vessel solid wall)
- Interaction (coupling) of these systems
- Advantages/disadvantages
    - Can resolve local details on rheology, vascular wall dynamics, flow velocity in 3D, etc
    - Desing of numerical algorithms is challenging
    - High computational cost
    - FSI models can only be used locally, perhaps coupled to other, simpler models

# One-dimensonal models

- The unknowns are integral averages of flow quantities at each section of the vessel:
  - cross-sectional area (not shape)
  - velocity
  - flow
  - pressure

  These quantities are time-dependent and space-dependent along the length of the vessel
- Equations are a non-linear system two equations with source terms
- A closure condition is needed: the tube law
- Wave propagation phenomena in the vascular network is resolved here
- Much lower computational cost than for full FSI models
- These models may be interpreted as representing a telegraphic network along which signals are transmitted: nutrients, biological signals, waste removal

# Zero-dimensional models

- Also called lumped parameter or compartmental models
- Analogy with Kirchhoff laws for hydraulic networks is used, or electric analogues
- These models use Ordinary Differential equations (ODEs), or more precisely, Differential Algebraic Equations (DAEs)
- There is no space resolution, just time variation of quantities
- No resolution of wave propagation phenomena
- These are the cheaper models and can still account for physiological effect such as regulatory mechanisms
- Suitable for the capillary bed and the heart, for example

# Geometrical multi-scale approach

- Accounting for blood flow in the complete network of vessels (millions), with very different dimensions (spatial scales), is an impossible computational task today
- A feasible approach is the *Geometrical multi-scale approach:* use 1-D models in all major vessels, 0-D models for the microvasculature and FSI models for special districts (organs) of interest for higher definition
- Linking (coupling or matching conditions) these models correctly is challenging, both from the mathematical and numerical points of view
- Here we shall adopt the geometric multi-scale models involving 1-D and 0-D submodels

# The incompressible Navier-Stokes equations

# The incompressible Navier-Stokes equations

In full, the three-dimensional, time-dependent incompressible Navier-Stokes equations are

$$\left.\begin{array}{rcl} \partial_x u + \partial_y v + w\partial_z w & = & 0 \\ \partial_t u + u\partial_x u + v\partial_y u + w\partial_z u + \frac{1}{\hat{\rho}}\partial_x p & = & g_{(x)} + \frac{\mu}{\hat{\rho}}(\partial_x^2 u + \partial_y^2 u + \partial_z^2 u) \\ \partial_t v + u\partial_x v + v\partial_y v + w\partial_z v + \frac{1}{\hat{\rho}}\partial_y p & = & g_{(y)} + \frac{\mu}{\hat{\rho}}(\partial_x^2 v + \partial_y^2 v + \partial_z^2 v) \\ \partial_t w + u\partial_x w + v\partial_y w + w\partial_z w + \frac{1}{\hat{\rho}}\partial_z p & = & g_{(z)} + \frac{\mu}{\hat{\rho}}(\partial_x^2 w + \partial_y^2 w + \partial_z^2 w) \end{array}\right\} \tag{1}$$

Here the four unknowns are

$$\left.\begin{array}{l} \mathbf{U}(x,y,z,t) = (u,v,w)\text{: Velocity vector ,} \\ p(x,y,z,t)\text{: Pressure .} \end{array}\right\} \tag{2}$$

Parameters of the equations include:

$$\left.\begin{array}{l}\hat{\rho}\text{: constant density },\mu\text{: constant viscosity },\\[2ex] \mathbf{G}=(g_{(x)},g_{(y)},g_{(z)})\text{: external force vector}\end{array}\right\} \tag{3}$$

The ratio

$$\nu = \frac{\mu}{\hat{\rho}} \tag{4}$$

is termed the kinematic viscosity.

The equations (1) can also be written more succinctly as

$$\left.\begin{array}{ll}\nabla\cdot\mathbf{U}=0 & \text{: continuity}\\[2ex] \partial_t\mathbf{U}+(\mathbf{U}\cdot\nabla)\mathbf{U}+\frac{1}{\hat{\rho}}\nabla p=\frac{1}{\hat{\rho}}\Delta\mathbf{U}+\mathbf{G} & \text{: momentum}\end{array}\right\} \tag{5}$$

where $\nabla\cdot\mathbf{U}=div\mathbf{U}$, $\nabla p$ is the gradient vector and $\Delta\mathbf{U}$ is the Laplacian operator applied to $\mathbf{U}$.

# Formulations of the equations

We study three mathematical formulations of the governing equations in Cartesian coordinates and restrict our attention to the two-dimensional case, namely

$$\partial_x u + \partial_x v = 0 \, , \tag{6}$$

$$\partial_t u + u\partial_x u + v\partial_y u + \frac{1}{\hat{\rho}}\partial_x p = \nu \left[\partial_x^2 u + \partial_y^2 u\right] \, , \tag{7}$$

$$\partial_t v + u\partial_x v + v\partial_y v + \frac{1}{\hat{\rho}}\partial_y p = \nu \left[\partial_x^2 v + \partial_y^2 v\right] \, . \tag{8}$$

We have a set of three equations (6)-(8) for the three unknowns $u, v, p$. In principle, given a domain along with initial and boundary conditions for the equations one should be able to solve this problem.

# The stream function-vorticity formulation

This formulation is attractive for the two-dimensional case but not so much in three dimensions, in which the role of a stream function is replaced by that of a *vector potential*. The magnitude of the vorticity vector can be written as

$$\zeta = \partial_x v - \partial_y u \; . \tag{9}$$

Introducing a stream function $\psi$ we have

$$u = \partial_y \psi \; , \; v = -\partial_x \psi \; . \tag{10}$$

By combining equations (7) and (8), so as to eliminate pressure $p$, and using (9) we obtain

$$\partial_t \zeta + u \partial_x \zeta + v \partial_y \zeta = \nu \left[ \partial_x^2 \zeta + \partial_y^2 \zeta \right] \; , \tag{11}$$

which is called the *vorticity transport equation* (advection-diffusion, parabolic type).

In order to solve (11) one requires the solution for the stream function $\psi$, which is in turn related to the vorticity $\zeta$ via

$$\partial_x^2 \psi + \partial_y^2 \psi = -\zeta \; . \tag{12}$$

This is called the *Poisson equation* and is of *elliptic type*. There are numerical schemes to solve (11)-(12) using the *apparent* decoupling of the parabolic-elliptic problem (6)-(8) to transform it into the parabolic equation (11) and the elliptic equation (12). A relevant observation, from the numerical point of view, is that the advection terms of the left hand side of equation (11) can be written in conservative form and hence we have

$$\partial_t \zeta + \partial_x (u\zeta) + \partial_y (v\zeta) = \nu \left[ \partial_x^2 \zeta + \partial_y^2 \zeta \right] \; . \tag{13}$$

This follows from the fact that $\partial_x u + \partial_y v = 0$, which was also used to obtain (11) from (7)-(8).

# The Artificial Compressibility Equations

This is yet another approach to formulate the incompressible Navier-Stokes equations and was originally put forward by Chorin [1] for the steady case. See also [3].

Consider the two-dimensional equations (6)-(8) in non-dimensional form

$$\partial_x u + \partial_y v = 0 \; , \tag{14}$$

$$\partial_t u + u\partial_x u + v\partial_y u + \partial_x p = \alpha \left[\partial_x^2 u + \partial_y^2 u\right] \; , \tag{15}$$

$$\partial_t v + u\partial_x v + v\partial_y v + \partial_y p = \alpha \left[\partial_x^2 v + \partial_y^2 v\right] \; , \tag{16}$$

where the following non-dimensionalisation has been used.

$$\left.\begin{array}{lll} u \leftarrow u/V_\infty \; , & v \leftarrow v/V_\infty \; , & p \leftarrow \frac{p}{\rho_\infty V_\infty^2} \; , \\ x \leftarrow x/L \; , & y \leftarrow y/L \; , & t \leftarrow tV_\infty/L \; , \\ \alpha = 1/R_{\mathrm{eL}} \; , & R_{\mathrm{eL}} = \frac{V_\infty L}{\nu_\infty} \; . & \end{array}\right\} \tag{17}$$

Multiplying (14) by the non-zero parameter $c^2$ and adding an *artificial compressibility* term $\partial_t p$ the first equation reads

$$\partial_t p + \partial_x(uc^2) + \partial_y(vc^2) = 0 \ .$$

By using equation (14) the advective terms in (15)-(16) can be written in *conservative form*, so that the modified system becomes

$$\left.\begin{aligned}
&\partial_t p + \partial_x(uc^2) + \partial_y(vc^2) = 0 \ , \\
&\partial_t u + \partial_x(u^2 + p) + \partial_y(uv) = \alpha \left[\partial_x^2 u + \partial_y^2 u\right] \ , \\
&\partial_t v + \partial_x(uv) + \partial_y(v^2 + p) = \alpha \left[\partial_x^2 v + \partial_y^2 v\right] \ .
\end{aligned}\right\} \tag{18}$$

These equations can be written in compact form as

$$\partial_t \mathbf{Q} + \partial_x \mathbf{F}(\mathbf{Q}) + \partial_y \mathbf{G}(\mathbf{Q}) = \mathbf{D} \ , \tag{19}$$

where

$$\left.\begin{array}{c}
\mathbf{Q} = \begin{bmatrix} p \\ u \\ v \end{bmatrix} \ , \quad \mathbf{F} = \begin{bmatrix} c^2 u \\ u^2 + p \\ uv \end{bmatrix} \ , \\[2em]
\mathbf{G} = \begin{bmatrix} c^2 v \\ uv \\ v^2 + p \end{bmatrix} \ , \quad \mathbf{D} = \begin{bmatrix} 0 \\ \alpha\left(\partial_x^2 u + \partial_y^2 u\right) \\ \alpha\left(\partial_x^2 v + \partial_y^2 v\right) \end{bmatrix} \ .
\end{array}\right\} \tag{20}$$

Equations (19)-(20) are called the *artificial compressibility equations*. Here $c^2$ is the *artificial compressibility factor*, a constant parameter. The source term vector in this case is a function of second derivatives. Note that the modified equations are equivalent to the original equations in the *steady state* limit.

- The left-hand side of (19)-(20) form a non-linear hyperbolic system
- The Riemann problem can be defined and solved exactly or approximately. See [2]
- Once a Riemann solver is available one can deploy Godunov-type numerical methods to solve the equations

# The one-dimensional equations

# Reynolds theorem applied to a blood vessel

Consider a generic blood vessel with axial coordinate $x$, as shown in Fig. 1

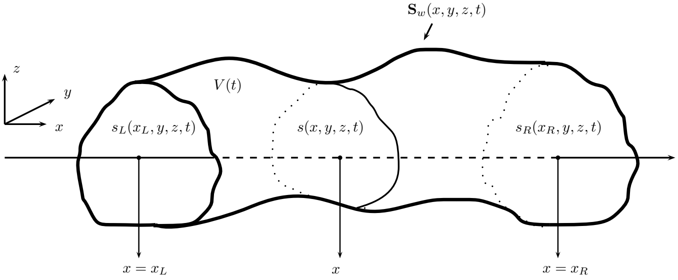



Fig. 1. Sketch of section of blood vessel in three dimensions defining control volume $V(t)$ with boundary $S(t)$.

- $s(x, y, z, t)$: generic cross section at $x$
- $s_L$: left cross section at $x = x_L$, fixed, normal to direction $x$
- $s_R$: right cross section at $x = x_R$, fixed, normal to direction $x$

- Boundary $S(t)$ of $V(t)$ is decomposed as

$$S(t) = S_w \cup \{s_L, s_R\} \tag{21}$$

- $S_w = S_w(x, y, z, t)$: vessel wall

The Reynolds transport theorem states

$$\frac{d}{dt} \int_{V(t)} \mathcal{F}(\mathbf{x}, t) dV = \int_{V(t)} \partial_t \mathcal{F}(\mathbf{x}, t) dV + \int_{S(t)} \mathcal{F} \mathbf{U}_b \cdot \mathbf{n} d\sigma \tag{22}$$

- $\mathbf{U}_b$: velocity of boundary $S(t)$
- $\mathcal{F}(\mathbf{x}, t)$: a continuous function
- $\mathbf{x} = (x, y, z)$
- $\mathbf{n}$: outward unit normal vector to $S(t)$

For our problem the Reynolds theorem becomes

$$\frac{d}{dt}\int_{V(t)} \mathcal{F}(\mathbf{x},t)dV = \int_{V(t)} \partial_t \mathcal{F}(\mathbf{x},t)dV + \int_{S_w} \mathcal{F}\mathbf{U}_w \cdot \mathbf{n} d\sigma \qquad (23)$$

- $\mathbf{U}_w$: velocity of the vessel wall $S_w$
- $\mathbf{U}_b \cdot \mathbf{n} = 0$ at $x = x_L$ and $x = x_R$

Assume the general case in which there may be fluid filtration across the vessel wall (permeable wall). Then the *relative velocity* is

$$\mathbf{U}_r = \mathbf{U}_w - \mathbf{U} \qquad (24)$$

where $\mathbf{U} = (u, v, w)$ is the blood velocity.

It is convenient to define the cross-sectional average of a quantity $a(\mathbf{x}, t)$ at section $s$ of area $A$ as

$$\overline{a} = \frac{1}{A}\int_s a(\mathbf{x},t)d\sigma \qquad \text{Note: } A = \int_s 1 d\sigma \qquad (25)$$

Obviously

$$\int_{V(t)} a(\mathbf{x},t)dV = \int_{x_L}^{x_R}\left[\int_s a(\mathbf{x},t)d\sigma\right]dx \qquad (26)$$

which by virtue of (25) becomes

$$\int_{V(t)} a(\mathbf{x},t)dV = \int_{x_L}^{x_R} A\overline{a}(\mathbf{x},t)dx \tag{27}$$

Because $x_L$ and $x_R$ are independent of time

$$\frac{d}{dt}\int_{V(t)} \mathcal{F}(\mathbf{x},t)dV = \frac{d}{dt}\int_{x_L}^{x_R} A\overline{\mathcal{F}}(\mathbf{x},t)dx = \int_{x_L}^{x_R} \partial_t(A\overline{\mathcal{F}}(\mathbf{x},t))dx \tag{28}$$

The second term on the right-hand side of (23), in view of (24), becomes

$$\int_{S_w} \mathcal{F}\mathbf{U}_w \cdot \mathbf{n}d\sigma = \int_{S_w} \mathcal{F}\mathbf{U}_r \cdot \mathbf{n}d\sigma + \int_{S_w} \mathcal{F}\mathbf{U} \cdot \mathbf{n}d\sigma \tag{29}$$

- For an impermeable wall, the first term on RHS of (29) is zero
- In view of (21), the second term on RHS of (29) becomes

$$\int_{S_w} \mathcal{F}\mathbf{U} \cdot \mathbf{n}d\sigma = \int_{S(t)} \mathcal{F}\mathbf{U} \cdot \mathbf{n}d\sigma - \int_{S_L} \mathcal{F}\mathbf{U} \cdot \mathbf{n}d\sigma - \int_{S_R} \mathcal{F}\mathbf{U} \cdot \mathbf{n}d\sigma \tag{30}$$

Since $u$ is the $x$ component of velocity of $\mathbf{U}$, then (30) becomes

$$\int_{S_w} \mathcal{F}\mathbf{U}\cdot\mathbf{n}d\sigma = \int_{S(t)} \mathcal{F}\mathbf{U}\cdot\mathbf{n}d\sigma + \int_{s_L} \mathcal{F}u d\sigma - \int_{s_R} \mathcal{F}u d\sigma \tag{31}$$

The first term on the RHS of (31), in view of Gauss' theorem, becomes

$$\int_{S(t)} \mathcal{F}\mathbf{U}\cdot\mathbf{n}d\sigma = \int_{V(t)} \nabla\cdot(\mathcal{F}\mathbf{U})dV \tag{32}$$

and thus (31) becomes

$$\int_{S_w} \mathcal{F}\mathbf{U}\cdot\mathbf{n}d\sigma = \int_{V(t)} \nabla\cdot(\mathcal{F}\mathbf{U})dV + \int_{s_L} \mathcal{F}u d\sigma - \int_{s_R} \mathcal{F}u d\sigma \tag{33}$$

From (25) we may write the last two terms in (33) as

$$\int_{s_L} \mathcal{F} u d\sigma - \int_{s_R} \mathcal{F} u d\sigma = -\int_{x_L}^{x_R} \frac{\partial}{\partial x} \left( A\overline{\mathcal{F}u} \right) dx \tag{34}$$

and thus (33) becomes

$$\left.\begin{aligned} \int_{S_w} \mathcal{F}\mathbf{U}_w \cdot \mathbf{n} d\sigma &= \int_{S_w} \mathcal{F}\mathbf{U}_r \cdot \mathbf{n} d\sigma + \int_{V(t)} \nabla \cdot (\mathcal{F}\mathbf{U}) dV \\ &\quad - \int_{x_L}^{x_R} \frac{\partial}{\partial x} \left( A\overline{\mathcal{F}u} \right) dx \end{aligned}\right\} \tag{35}$$

Inserting (28) and (35) into (23) and using (26) yields

$$\left.\begin{aligned} \int_{x_L}^{x_R} \frac{\partial}{\partial t} \left( A\overline{\mathcal{F}} \right) dx &= \int_{V(t)} \partial_t \mathcal{F} dV + \int_{V(t)} \nabla \cdot (\mathcal{F}\mathbf{U}) dV \\ &\quad + \int_{S_w} \mathcal{F}\mathbf{U}_r \cdot \mathbf{n} d\sigma - \int_{x_L}^{x_R} \frac{\partial}{\partial x} \left( A\overline{\mathcal{F}u} \right) dx \end{aligned}\right\} \tag{36}$$

That is

$$\left.\begin{aligned}\int_{x_L}^{x_R} \frac{\partial}{\partial t}\left(A\overline{\mathcal{F}}\right)dx = \quad & \int_{x_L}^{x_R}\left(\int_s \frac{\partial}{\partial t}\mathcal{F}d\sigma\right)dx + \int_{x_L}^{x_R}\left(\int_s \nabla\cdot(\mathcal{F}\mathbf{U})d\sigma\right)dx \\ & + \int_{x_L}^{x_R}\left(\int_{\partial s}\mathcal{F}\mathbf{U}_r\cdot\mathbf{n}d\xi\right)dx - \int_{x_L}^{x_R}\frac{\partial}{\partial x}\left(A\overline{\mathcal{F}u}\right)dx\end{aligned}\right\} \tag{37}$$

which after rearranging becomes

$$\frac{\partial}{\partial t}(A\overline{\mathcal{F}}) + \frac{\partial}{\partial x}(A\overline{\mathcal{F}u}) = \int_s \left(\frac{\partial}{\partial t}\mathcal{F} + \nabla\cdot(\mathcal{F}\mathbf{U})\right)d\sigma + \int_{\partial s}\mathcal{F}\mathbf{U}_r\cdot\mathbf{n}d\xi \tag{38}$$

- The function $\mathcal{F}$ in (38) is open to choice
- Appropriate choices of $\mathcal{F}$ will give the governing blood flow equations
- For an impermeable wall the last term on RHS vanishes

# Governing 1D equations: conservation of mass

We use the area-averaged version (38) of Reynolds transport theorem, with reference to Fig. 1.

By setting $\mathcal{F} = 1$ and noting that for an incompressible fluid $\nabla \cdot \mathbf{U} = 0$, equation (38) gives the *equation of continuity* or *mass conservation equation*

$$\frac{\partial}{\partial t} A + \frac{\partial}{\partial x}(A\overline{u}) = \int_{\partial s} \mathbf{U}_r \cdot \mathbf{n} d\xi \tag{39}$$

which for an impermeable wall becomes homogenous (no source term)

$$\frac{\partial}{\partial t} A + \frac{\partial}{\partial x}(A\overline{u}) = 0 \tag{40}$$

- $A(x, t)$ is the unknown cross-sectional area
- $\overline{u}(x, t)$ is the unknown cross-section averaged velocity

# Governing 1D equations: balance of momentum

By setting $\mathcal{F} = u$, noting that $\nabla \cdot (u\mathbf{U}) = u\nabla \cdot \mathbf{U} + \mathbf{U} \cdot \nabla u$, that for an incompressible fluid $\nabla \cdot \mathbf{U} = 0$ and that

$$\frac{\partial}{\partial t} u + \mathbf{U} \cdot \nabla u = \frac{Du}{Dt} \tag{41}$$

equation (38) becomes

$$\frac{\partial}{\partial t}(A\overline{u}) + \frac{\partial}{\partial x}(A\overline{u^2}) = \int_s \frac{Du}{Dt} d\sigma + \int_{\partial s} u\mathbf{U}_r \cdot \mathbf{n} d\xi \tag{42}$$

Now impose the equation for momentum balance

$$\int_{V(t)} \frac{D(\rho\mathbf{U})}{Dt} dV = \int_{V(t)} \rho\mathbf{B} dV + \int_{S(t)} \mathbf{T}\mathbf{n} d\sigma \tag{43}$$

- $\rho$: density (constant here)
- $\mathbf{B}$: body force
- $\mathbf{T}$: Cauchy stress tensor

Applying the divergence theorem to last term on RHS of (43) we obtain

$$\int_{V(t)} \frac{D\mathbf{U}}{Dt} dV = \int_{V(t)} \mathbf{B} dV + \frac{1}{\rho} \int_{V(t)} \nabla \cdot \mathbf{T} dV \tag{44}$$

From the constitutive law for a fluid

$$\mathbf{T} = -p\mathbf{I} + \mathbf{D} \tag{45}$$

- $p$: pressure
- $\mathbf{I}$: identity tensor
- $\mathbf{D}$: tensor of deviatoric stresses

Equation (44) can now be written thus

$$\int_{x_L}^{x_R} \left( \int_s \frac{D\mathbf{U}}{Dt} dS \right) dx = \int_{x_L}^{x_R} \left( \int_s \left[ \mathbf{B} + \frac{1}{\rho} (-\nabla p + \nabla \cdot \mathbf{D} \right] d\sigma \right) dx \tag{46}$$

For the $x$-component, equation (46), with obvious notation for $b$ and $d$, becomes

$$\int_s \frac{Du}{Dt} d\sigma = \int_s [b + \frac{1}{\rho}(-\frac{\partial}{\partial x} p + d)] d\sigma \tag{47}$$

Returning to (42) we have

$$\frac{\partial}{\partial t}(A\overline{u}) + \frac{\partial}{\partial x}(A\overline{u^2}) = \int_s [b + \frac{1}{\rho}(-\frac{\partial}{\partial x} p + d)] d\sigma + \int_{\partial s} u \mathbf{U}_r \cdot \mathbf{n} d\xi \tag{48}$$

In terms of area-averages we have

$$\frac{\partial}{\partial t}(A\overline{u}) + \frac{\partial}{\partial x}(A\overline{u^2}) = \frac{A}{\rho}[\rho\overline{b} - \frac{\partial}{\partial x}\overline{p} + \overline{d}] + \int_{\partial s} u \mathbf{U}_r \cdot \mathbf{n} d\xi \tag{49}$$

The Coriolis coefficient $\alpha$ is introduced via

$$\alpha(\overline{u})^2 = \overline{u^2} = \frac{1}{A}\int_s u^2 d\sigma \tag{50}$$

- $\alpha$ depends on the assumed velocity profile
- $\alpha = 1$ for a flat profile and $\alpha = 4/3$ for a parabolic velocity profile

*Viscous forces* are represented by $\overline{d}$ and here we assume the linear relation

$$\frac{A}{\rho}\overline{d} = -R\overline{u} \tag{51}$$

$R > 0$ represents *viscous resistance* per unit length of tube.

Finally, the momentum equation becomes

$$\frac{\partial}{\partial t}(A\overline{u}) + \frac{\partial}{\partial x}(\alpha A(\overline{u})^2) + \frac{A}{\rho}\frac{\partial}{\partial x}\overline{p} = A\bar{b} - R\overline{u} + \int_{\partial s} u\mathbf{U}_r \cdot \mathbf{n} d\xi \tag{52}$$

The governing equations are (39) and (52) and the unknowns are:

- $A(x,t)$: cross-sectional area
- $\overline{u}(x,t)$: area-averaged velocity
- $\overline{p}(x,t)$: area averaged pressure

There are more unknowns that equations and hence a closure condition is needed, a *tube law*. This will come from mechanical considerations.

# Simplified equations

We assume an impermeable wall, zero *body forces* and drop bars from unknowns. Then the continuity and momentum equations become

$$\left.\begin{aligned} &\partial_t A + \partial_x(Au) = 0 \\ &\partial_t(Au) + \partial_x\left(\alpha A u^2\right) + \tfrac{A}{\rho}\partial_x p = -Ru \end{aligned}\right\} \tag{53}$$

The unknowns:

- $A(x,t)$: cross-sectional area of vessel
- $u(x,t)$: cross-sectional averaged velocity;
- $p(x,t)$: cross-sectional averaged internal pressure

Flow rate:

$$q = Au$$

Parameters:

- $\rho$: density of blood
- $\alpha$: Coriolis coefficient. Here we choose $\alpha = 1$

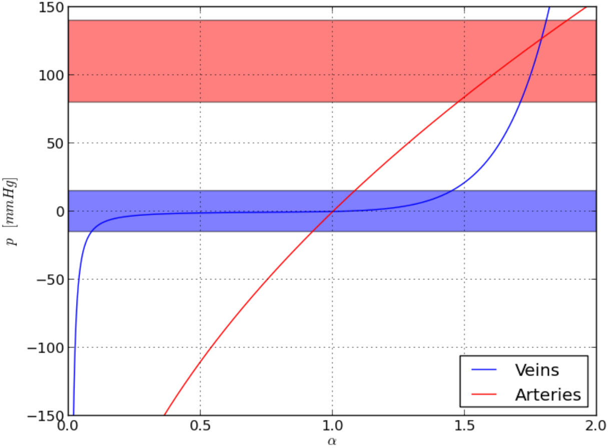
# A closure condition: the tube law

$$p(x,t) = p_e(x,t) + K(x)\left[\left(\frac{A}{A_0}\right)^m - \left(\frac{A}{A_0}\right)^n\right] \tag{54}$$

with

$$K(x) = \begin{cases} K_a = \sqrt{\dfrac{\pi}{A_0}}\left(\dfrac{h_0 E}{1-\nu^2}\right) & m = 1/2\,,\quad n = 0 \qquad \text{for arteries} \\[2ex] K_v = \dfrac{E}{12(1-\nu^2)}\left(\dfrac{h_0}{r_0}\right)^3 & m \approx 10\,,\quad n = -3/2 \quad \text{for veins} \end{cases} \tag{55}$$

- $A_0(x)$: vessel cross-sectional area at equilibrium
- $r_0(x)$: vessel radius at equilibrium
- $h_0(x)$: vessel wall thickness
- $E(x)$: Young's modulus
- $\nu$: Poisson'a ratio, taken as $\nu = 1/2$




**Tube law: arteries versus veins**. Behaviour of pressure as function of non-dimensional cross-sectional area, for arteries and veins.

# Simplified equations for arteries

Assuming $h_0$, $A_0$, $E$, $p_{ext}$ to be constant, then the term involving the pressure gradient in (53) becomes

$$\frac{A}{\rho}\partial_x p = \partial_x \left[\frac{1}{3}\frac{K_a}{\rho\sqrt{A_0}}A^{3/2}\right] \tag{56}$$

and thus the momentum equation equation in (53) can be written in conservative form as

$$\partial_t(Au) + \partial_x\left(Au^2 + \gamma A^{3/2}\right) = -Ru \tag{57}$$

where

$$\gamma = \frac{1}{3}\frac{K_a}{\rho\sqrt{A_0}} \quad \text{: constant} \tag{58}$$

Therefore the full system of governing equations (53) can be written in conservative-law form, as

$$\partial_t \mathbf{Q} + \partial_x \mathbf{F}(\mathbf{Q}) = \mathbf{S}(\mathbf{Q}) \tag{59}$$

where

$$\mathbf{Q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \equiv \begin{bmatrix} A \\ Au \end{bmatrix}, \quad \mathbf{S}(\mathbf{Q}) = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ -Ru \end{bmatrix} \tag{60}$$

and the flux vector is

$$\mathbf{F}(\mathbf{Q}) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \equiv \begin{bmatrix} Au \\ Au^2 + \gamma A^{3/2} \end{bmatrix} \tag{61}$$

Note that in terms of the conserved variables the flux vector becomes

$$\mathbf{F}(\mathbf{Q}) = \begin{bmatrix} f_1(q_1, q_2) \\ f_2(q_1, q_2) \end{bmatrix} \equiv \begin{bmatrix} q_2 \\ \dfrac{q_2^2}{q_1} + \gamma q_1^{3/2} \end{bmatrix} \tag{62}$$

# Equations for vessels with variable properties

Recall the one-dimensional equations with $\alpha = 1$ read

$$\left.\begin{array}{l} \partial_t A + \partial_x (Au) = 0 \\ \\ \partial_t (Au) + \partial_x \left(Au^2\right) + \frac{A}{\rho} \partial_x p = -Ru \end{array}\right\} \quad (63)$$

along with the tube law written as

$$p(x,t) = p_e(x,t) + K(x)\psi(A) \quad (64)$$

The non-conservative term in the momentum equation can be re-written as

$$\partial_t (Au) + \partial_x \left(Au^2\right) + \partial_x \frac{pA}{\rho} - \frac{p}{\rho} \partial_x A = -Ru \quad (65)$$

But note the following identity (see Elad etal. 1991)

$$-\frac{p}{\rho} \partial_x A = -\frac{1}{\rho} \left\{ p_e \partial_x A + \partial_x \left[ K \int \psi(A) dA \right] - \partial_x K \int \psi(A) dA \right\} \quad (66)$$

Use of (66) into (65), followed by manipulations yields a conservative form for the momentum equation and the whole system can be written as

$$\left.\begin{array}{rcl} \partial_t A + \partial_x (uA) & = & 0 \\ \partial_t (Au) + \partial_x \left[ A \left( u^2 + \frac{p - p_e}{\rho} \right) - \frac{K}{\rho} \int \psi(A) dA \right] & = & s_M \end{array}\right\} \quad (67)$$

with the momentum source term given as

$$s_M = -Ru - \frac{A}{\rho} \partial_x p_e - \frac{1}{\rho} \partial_x K \int \psi(A) dA \quad (68)$$

- The source term $s_M$ includes spatial gradients of parameters of the problem: $p_e(x,t)$, $K(x)$
- These geometric-type source terms are difficult to treat numerically, specially for large gradients, or even discontinuities

# Zero-Dimensional Equations (Lumped-Parameter Models, or Compartmental Models)

# The idea of 0-D models

- The concept of *compartment* is introduced to refer to a particular *district* of the body
- The full body may be *modelled* (represented) by a finite set of compartments, linked in an appropriate way
- Within each compartment, haemodynamics variables are constant in space and vary only in time (spatial homogeneity)
- In a 0-D model one studies the behaviour of a compartment in relation to others

# Recalling the 1D blood flow equations

The continuity and momentum equations, plus the tube law are

$$\left.\begin{aligned} &\partial_t A + \partial_x Q = 0 \; , \\[2ex] &\partial_t Q + \partial_x \left(\alpha A u^2\right) + \tfrac{A}{\rho}\partial_x p = -K_R u \\[2ex] &p(x,t) = p_e(x,t) + K\left[\left(\tfrac{A}{A_0}\right)^m - \left(\tfrac{A}{A_0}\right)^n\right] \end{aligned}\right\} \tag{69}$$

The unknowns are:

- $A(x,t)$: cross-sectional area of vessel
- $u(x,t)$: averaged velocity; $Q = Au$: flow rate
- $p(x,t)$: internal pressure

Parameters:

- $K_R$: viscous resistance of the flow per unit length of the tube
- $\rho$: density of blood
- $\alpha$: Coriolis coefficient. Here we choose $\alpha = 1$

# Define spatial integral averages

Consider a segment $[x_1, x_2]$ of length $\Delta s = x_2 - x_1$ of a vessel and define integral averages

$$\left.\begin{aligned}
\hat{Q}(t) &= \frac{1}{\Delta s}\int_{x_1}^{x_2} Q(x,t)dx \\
\hat{p}(t) &= \frac{1}{\Delta s}\int_{x_1}^{x_2} p(x,t)dx \\
\hat{A}(t) &= \frac{1}{\Delta s}\int_{x_1}^{x_2} A(x,t)dx
\end{aligned}\right\} \tag{70}$$

# The continuity equation

Integrate in space the continuity equation in the interval $[x_1, x_2]$, the first of equations (69), namely

$$\partial_t A + \partial_x Q = 0 \; . \tag{71}$$

We obtain

$$\partial_t \left( \frac{1}{\Delta s} \int_{x_1}^{x_2} A(x,t) dx \right) + \frac{1}{\Delta s} (Q(x_2, t) - Q(x_1, t)) = 0 \; . \tag{72}$$

This is an ordinary differential equation (ODE) in time for the averaged cross-sectional area, namely

$$\Delta s \frac{d}{dt} \hat{A}(t) + Q(x_2, t) - Q(x_1, t) = 0 \; . \tag{73}$$

# The momentum equation

In deriving the averaged momentum equation we make the following four assumptions:

- We neglect the term $\partial_x \left(\alpha A u^2\right)$ in the second of equations (69)
- We assume that variations of the coefficient $\beta(x)$ in the tube law are small and can thus be replaced by an average $\beta$
- We assume that variations of area $A(x,t)$ around the equilibrium area $A_0(x)$ are small
- It is assumed that $A_0(x)$ is constant and denoted by $A_0$

The resulting momentum equation reads

$$\partial_t Q + \frac{A}{\rho}\partial_x p + K_R \frac{Q}{A} = 0 \; . \tag{74}$$

Integrating (74) term by term we obtain

$$\left.\begin{aligned}
&\frac{1}{\Delta s}\int_{x_1}^{x_2} \partial_t Q(x,t)dx = \frac{d}{dt}\hat{Q}(t) \\
&\frac{1}{\Delta s}\int_{x_1}^{x_2} \frac{A(x,t)}{\rho}\partial_x p(x,t)dx = \frac{A_0}{\Delta s\rho}[p(x_2,t) - p(x_1,t)] \\
&\frac{1}{\Delta s}\int_{x_1}^{x_2} K_R\frac{Q(x,t)}{A(x,t)}dx = \frac{K_R}{A_0}\hat{Q}(t)\ .
\end{aligned}\right\} \tag{75}$$

Thus the averaged momentum equation (74) becomes

$$\frac{\Delta s\rho}{A_0}\frac{d}{dt}\hat{Q}(t) + \frac{\Delta s\rho K_R}{A_0^2}\hat{Q}(t) + p(x_2,t) - p(x_1,t) = 0\ . \tag{76}$$

# Use of the tube law in the mass equation

As an example, consider the simple tube law for arteries

$$p = p_{ext} + \beta(x)\left(\sqrt{A} - \sqrt{A_0}\right)\ . \tag{77}$$

Differentiating (77) with respect to time we obtain

$$\partial_t p = \partial_t p_{ext} + \beta(x)\frac{1}{2\sqrt{A}}\partial_t A \tag{78}$$

Integration of this equation with respect to $x$ gives

$$\left.\begin{aligned} \partial_t\left(\frac{1}{\Delta s}\int_{x_1}^{x_2} p(x,t)dx\right) \quad = \quad & \partial_t\left(\frac{1}{\Delta s}\int_{x_1}^{x_2} p_{ext}(x,t)dx\right) \\ & + \frac{\beta}{2\sqrt{A_0}}\partial_t\left(\frac{1}{\Delta s}\int_{x_1}^{x_2} A(x,t)dx\right) \end{aligned}\right\} \tag{79}$$

Further manipulations of (79) yield a relationship between $\hat{A}(t)$ and pressure $\hat{p}(t)$, including $\hat{p}_{ext}(t)$, namely

$$\frac{d}{dt}\hat{A}(t) = \frac{2\sqrt{A_0}}{\beta}\frac{d}{dt}\hat{p}(t) - \frac{2\sqrt{A_0}}{\beta}\frac{d}{dt}\hat{p}_{ext}(t) \ . \tag{80}$$

Substitution of (80) into (73) yields the continuity equation in the form

$$\frac{2\sqrt{A_0}\Delta s}{\beta}\frac{d}{dt}\hat{p}(t) + Q(x_2, t) - Q(x_1, t) = \frac{2\sqrt{A_0}\Delta s}{\beta}\frac{d}{dt}\hat{p}_{ext}(t) \tag{81}$$

# The zero-dimensional equations

Collecting the equations (81) and (76) we obtain the system

$$\left.\begin{aligned} C\frac{d}{dt}\hat{p}(t) + Q(x_2,t) - Q(x_1,t) &= C\frac{d}{dt}\hat{p}_{ext}(t) \\ L\frac{d}{dt}\hat{Q}(t) + R\hat{Q}(t) + p(x_2,t) - p(x_1,t) &= 0 \end{aligned}\right\} \tag{82}$$

where the coefficients $C$, $L$ and $R$ are defined as

$$\left.\begin{aligned} C &= \frac{2\sqrt{A_0}\Delta s}{\beta} && \text{Capacitance. Mass storage term due to compliance} \\ L &= \frac{\rho\Delta s}{A_0} && \text{Inductance. Inertia term in the momentum equation} \\ R &= \frac{\rho\Delta s K_R}{A_0^2} && \text{Resistance (due to viscosity and radii of vessels)} \end{aligned}\right\} \tag{83}$$

# Recalling electric circuits

- An electric circuit: a source of electric energy (e.g. a generator) and a resistors or an inductor or a capacitor.
- Kirchhoff's second law for the flow of current: *In a closed circuit, the impressed voltage $E(t)$ is equal to the sum of the voltage drops $E_*$ in the rest of the circuits*
- Experiments give

$$\left.\begin{array}{lcl} \textbf{Resistor}: & \text{voltage drop across the resistor} & E_R = RI \\ \textbf{Inductor}: & \text{voltage drop across the inductor} & E_L = L\frac{d}{dt}I \\ \textbf{Capacitor}: & \text{voltage drop across the capacitor} & E_C = \frac{1}{C}Q \end{array}\right\} \tag{84}$$

Here

$$\left.\begin{array}{ll} I(t) & \text{current (ampere)} \\ R & \text{resistance (ohms)} \\ L & \text{inductance (henrys)} \\ C & \text{capacitance (farads)} \end{array}\right\} \tag{85}$$

# Analogy between blood flow and electric circuits

| Haemodynamics | Electric circuit |
|---|---|
| Pressure p | Voltage E |
| Flow Q | Current I |
| Blood viscosity | Resistance R |
| Blood inertia | Inductance L |
| Wall compliance | Capacitance C |

Table 1. Analogy between 0-Dimensional haemodynamics and electric circuits.

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