A Short Introduction to Two-Scale Modelling

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Abstract. We define and characterize weak and strong *two-scale convergence* in L^p via a transformation of variable, extending Nguetseng's definition. We derive several properties, including weak and strong two-scale compactness. We then deal with the two-scale convergence of basic first-order differential operators.

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Prelude

Physicists and other applied scientists have always been reading the world in terms of scales, but for a long time mathematics missed a precise representation of that notion. This was somehow surrogated by asymptotic expansions, that were extensively used by engineers. But this tool does not seem to fit well the approach of functional analysis; asymptotic expansions did not exhibit the necessary rigour, and remained a heuristic tool.

Eventually in 1989, in the seminal paper [9] Gabriel Nguetseng introduced the notion of two-scale convergence, and proved some results (in particular concerning compactness and convergence of derivatives) that showed that this notion is also satisfactory for the purposes of functional analysis. Another fundamental contribution was provided by Gregoire Allaire in [1], and then by several others.

In these lecture notes we review those results, and outline some more recent developments.

An Apocryphal Representation. Let us consider the following sequence of functions, indexed by the vanishing parameter ε:

$$u_{1\varepsilon}(x) := \sin(2\pi x/\varepsilon) \qquad \forall x \in \mathbf{R}. \tag{1}$$

Thus $u_{1\varepsilon} \rightharpoonup 0$ in $L^p_{loc}(\mathbf{R})$ $(1 \le p < +\infty)$, so that any information concerning the form of $u_{1\varepsilon}$ is lost in the limit. In order to represent the high oscillations of these functions, it seems natural to introduce the variable $y = x/\varepsilon$, so that

$$u_{1\varepsilon}(x) \simeq \sin(2\pi y) =: u_1(y). \tag{2}$$

The symbol $\simeq$ is here used as a pale surrogate of equivalence, and of course wants a precise definition. Similarly

$$u_{2\varepsilon}(x) := x\sin(2\pi x/\varepsilon) \simeq x\sin(2\pi y) =: u_2(x, y), \tag{3}$$

and more generally, for a large class of functions w of two variables,

$$u_{3\varepsilon}(x) := w(x, x/\varepsilon) \simeq w(x, y). \tag{4}$$

A natural extension of this order of ideas involves convergence instead of equivalence. Whenever $w_\varepsilon(x, y) \to w(x, y)$ (in a sense to be specified),

$$u_{4\varepsilon}(x) := w_\varepsilon(x, x/\varepsilon) \simeq w_\varepsilon(x, y) \to w(x, y). \tag{5}$$

One may thus be tempted to define a new sort of convergence:

$$\begin{aligned} &\text{“}u_\varepsilon(x) \to w(x, y)\text{”} \quad \Leftrightarrow \\ &\exists\{w_\varepsilon\} : u_\varepsilon(x) = w_\varepsilon(x, x/\varepsilon) \ \ \forall x, \forall \varepsilon, \quad w_\varepsilon(x, y) \to w(x, y) \end{aligned} \tag{6}$$

(still without specifying the topology of the convergence). But for a generic sequence $u_\varepsilon = u_\varepsilon(x)$ it is not obvious how a sequence $\{w_\varepsilon\} = \{w_\varepsilon(x, y)\}$ as in (6) might be devised.

Enters Nguetseng. Let Ω be a domain of $\mathbf{R}^N$, and set $Y := [0, 1[^N$. In the seminal work [9], Nguetseng introduced the following concept.

A bounded sequence $\{u_\varepsilon\}$ of $L^2(\Omega)$ is said (weakly) *two-scale convergent* to $u \in L^2(\Omega \times Y)$ iff

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)\ \psi\Big(x, \frac{x}{\varepsilon}\Big)\, dx = \iint_{\Omega \times Y} u(x, y)\ \psi(x, y)\, dxdy, \tag{7}$$

for any smooth function $\psi : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ that is Y-periodic w.r.t. the second argument.

The reader will notice the idea of using the functions of the form (4) as test functions, and will appreciate the originality of conceiving the convergence of a sequence of functions of a single variable to a function of two variables.

Periodicity here plays a major role. Nguetseng also extended two-scale convergence beyond periodicity, see [10].

1. Two-Scale Convergence via Two-Scale Decomposition

We shall denote by $\mathcal{Y}$ the set $Y := [0, 1[^N$ equipped with the topology of the N-dimensional torus, and identify any function on $\mathcal{Y}$ with its Y-periodic extension to $\mathbf{R}^N$. (E.g. $\mathcal{D}(\mathcal{Y}) \neq \mathcal{D}(Y)$, whereas $L^p(\mathcal{Y}) = L^p(Y)$ for any $p \in [1, +\infty]$.) We shall deal with real-valued functions. The extension to complex or vector-valued functions would be trivial.

Two-Scale Decomposition. For any $\varepsilon > 0$, we decompose real numbers and real vectors as follows:

$$\begin{aligned} &\hat{n}(x) := \max\{n \in \mathbf{Z} : n \leq x\}, \quad \hat{r}(x) := x - \hat{n}(x)\ (\in [0, 1[) \qquad \forall x \in \mathbf{R}, \\ &\mathcal{N}(x) := (\hat{n}(x_1), ..., \hat{n}(x_N)) \in \mathbf{Z}^N, \quad \mathcal{R}(x) := x - \mathcal{N}(x) \in \mathcal{Y} \quad \forall x \in \mathbf{R}^N. \end{aligned} \tag{1.1}$$

Thus $x = \varepsilon[\mathcal{N}(x/\varepsilon) + \mathcal{R}(x/\varepsilon)]$ for any $x \in \mathbf{R}^N$.

Besides the above two-scale decomposition, we define a *two-scale composition* function:

$$S_\varepsilon(x, y) := \varepsilon\mathcal{N}(x/\varepsilon) + \varepsilon y \qquad \forall (x, y) \in \mathbf{R}^N \times \mathcal{Y}, \forall \varepsilon > 0. \tag{1.2}$$

As $S_\varepsilon(x, y) = x + \varepsilon[y - \mathcal{R}(x/\varepsilon)]$,

$$S_\varepsilon(x, y) \to x \quad \text{ uniformly in } \mathbf{R}^N \times \mathcal{Y}, \text{ as } \varepsilon \to 0. \tag{1.3}$$

Notice that then

$$\|u \circ S_\varepsilon - u\|_{L^p(\mathbf{R}^N \times \mathcal{Y})} \to 0 \qquad \forall u \in L^p(\mathbf{R}^N \times \mathcal{Y}). \tag{1.3'}$$

This is easily checked for any smooth u, and is extended by density to any $u \in L^p(\mathbf{R}^N \times \mathcal{Y})$.

Two-Scale-Integration Lemma. The next result is at the basis of our approach to two-scale convergence. Let us first denote by $\mathcal{L}(\mathbf{R}^N)$ ($\mathcal{B}(\mathbf{R}^N)$, resp.) the σ-algebra of the Lebesgue- (Borel-, resp.) measurable subsets of $\mathbf{R}^N$, and by $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes ...$ the σ-algebra generated by any finite family $\mathcal{A}_1, \mathcal{A}_2, ...$ of σ-algebras. Let us also set

$$\mathcal{F} := \big\{ f : \mathbf{R}^N \times \mathcal{Y} \to \mathbf{R} \text{ measurable w.r.t. to either } \mathcal{B}(\mathbf{R}^N) \otimes \mathcal{L}(\mathcal{Y}) \text{ or } \mathcal{L}(\mathbf{R}^N) \otimes \mathcal{B}(\mathcal{Y}) \big\}. \tag{1.4}$$

This class includes all Caratheodory functions.

Lemma 1.1 *(Fundamental Integration Lemma) Assume that*

$$\begin{aligned} &f \in \mathcal{F} \textit{ and} \\ &\textit{either } f \in L^1(\mathcal{Y}; L^\infty(\mathbf{R}^N)) \textit{ and } f \textit{ has compact support, or } f \in L^1(\mathbf{R}^N; L^\infty(\mathcal{Y})). \end{aligned} \tag{1.5}$$

For any $\varepsilon > 0$ the functions

$$\mathbf{R}^N \to \mathbf{R} \colon x \mapsto f(x, x/\varepsilon), \qquad \mathbf{R}^N \times \mathcal{Y} \to \mathbf{R} \colon (x,y) \mapsto f(S_\varepsilon(x,y), y)$$

are then integrable, and

$$\int_{\mathbf{R}^N} f(x, x/\varepsilon)\, dx = \iint_{\mathbf{R}^N \times Y} f(S_\varepsilon(x,y), y)\, dxdy \qquad \forall \varepsilon > 0. \tag{1.6}$$

(As it was anticipated, the function f is extended with periodicity w.r.t. the second argument.)
For any $p \in [1, +\infty]$ and any $\varepsilon > 0$, the operator $g \mapsto g \circ S_\varepsilon$ is then a (nonsurjective) linear isometry $L^p(\mathbf{R}^N) \to L^p(\mathbf{R}^N \times \mathcal{Y})$.

Proof of (1.6). As $\mathbf{R}^N = \bigcup_{m \in \mathbf{Z}^N} (\varepsilon m + \varepsilon \mathcal{Y})$ and $\mathcal{N}(x/\varepsilon) = m$ for any $x \in \varepsilon m + \varepsilon \mathcal{Y}$, we have

$$\begin{aligned} &\int_{\mathbf{R}^N} f(x, x/\varepsilon)\, dx = \sum_{m \in \mathbf{Z}^N} \int_{\varepsilon m + \varepsilon Y} f(x, x/\varepsilon)\, dx = \varepsilon^N \sum_{m \in \mathbf{Z}^N} \int_Y f(\varepsilon[m+y], y)\, dy \\ &= \sum_{m \in \mathbf{Z}^N} \int_{\varepsilon m + \varepsilon Y} dx \int_Y f(\varepsilon[\mathcal{N}(x/\varepsilon) + y], y)\, dy = \int_{\mathbf{R}^N} dx \int_Y f(S_\varepsilon(x,y), y)\, dy \end{aligned} \tag{1.7}$$

(these sums are absolutely convergent). $\square$

Two-Scale Convergence in L^p. We shall represent an arbitrary but prescribed, positive and vanishing sequence of real numbers by ε; e.g., $\varepsilon = \{1, 1/2, ..., 1/n, ...\}$. Our results will not depend on the specific choice of this sequence, that however will be kept fixed throughout. For any sequence of measurable functions, $u_\varepsilon : \mathbf{R}^N \to B$, and any measurable function, $u : \mathbf{R}^N \times \mathcal{Y} \to \mathbf{R}$,

> we say that u_ε *two-scale converges* to u in some specific sense,
> whenever $u_\varepsilon \circ S_\varepsilon \to u$ in the corresponding standard sense.

In this way we define strong and weak (weak star for $p = \infty$) two-scale convergence ($1 \le p \le +\infty$), that we denote by $u_\varepsilon \underset{2}{\to} u$, $u_\varepsilon \underset{2}{\rightharpoonup} u$, $u_\varepsilon \overset{*}{\underset{2}{\rightharpoonup}} u$ (resp.):

$$u_\varepsilon \underset{2}{\to} u \text{ in } L^p(\mathbf{R}^N \times \mathcal{Y}) \quad \Leftrightarrow \quad u_\varepsilon \circ S_\varepsilon \to u \text{ in } L^p(\mathbf{R}^N \times \mathcal{Y}), \quad \forall p \in [1, +\infty]; \tag{1.8}$$

$$u_\varepsilon \underset{2}{\rightharpoonup} u \text{ in } L^p(\mathbf{R}^N \times \mathcal{Y}) \quad \Leftrightarrow \quad u_\varepsilon \circ S_\varepsilon \rightharpoonup u \text{ in } L^p(\mathbf{R}^N \times \mathcal{Y}), \quad \forall p \in [1, +\infty[; \tag{1.9}$$

$$u_\varepsilon \overset{*}{\underset{2}{\rightharpoonup}} u \text{ in } L^\infty(\mathbf{R}^N \times \mathcal{Y}) \quad \Leftrightarrow \quad u_\varepsilon \circ S_\varepsilon \overset{*}{\rightharpoonup} u \text{ in } L^\infty(\mathbf{R}^N \times \mathcal{Y}). \tag{1.10}$$

For any domain $\Omega \subset \mathbf{R}^N$, we then define two-scale convergence in $L^p(\Omega \times \mathcal{Y})$ by extending functions to $\mathbf{R}^N \setminus \Omega$ with vanishing value. We similarly define a.e. two-scale convergence, quasi-uniform two-scale convergence, two-scale convergence in measure, and so on. These definitions are trivially extended to vector-valued functions ranging in Banach spaces. In all of these cases the limit is obviously unique. We refer to the usual convergence over $\mathbf{R}^N$ as *one-scale* convergence.

For instance,

$$u_\varepsilon(x) := \psi(x, x/\varepsilon) \underset{2}{\to} \psi(x,y) \qquad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y})(1 \le p \le +\infty), \forall \psi \in \mathcal{D}(\mathbf{R}^N \times \mathcal{Y}).$$

By the next result, weak and strong two-scale convergence may be regarded as intermediate properties between the usual (one-scale) weak and strong convergence.

Proposition 1.2 *Let $p \in [1, +\infty[$, $\{u_\varepsilon\}$ be a sequence in $L^p(\mathbf{R}^N)$ and $u \in L^p(\mathbf{R}^N \times \mathcal{Y})$. Then*

$$\begin{aligned} &u_\varepsilon \to u \ \ \text{in } L^p(\mathbf{R}^N) \ \Rightarrow \ u_\varepsilon \underset{2}{\to} u \ \ \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}), \\ &\text{whenever } u \text{ is independent of } y \text{ the converse also holds,} \end{aligned} \tag{1.11}$$

$$u_\varepsilon \underset{2}{\to} u \ \ \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}) \ \Rightarrow \ u_\varepsilon \underset{2}{\rightharpoonup} u \ \ \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}), \tag{1.12}$$

$$u_\varepsilon \underset{2}{\rightharpoonup} u \ \ \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}) \ \Rightarrow \ u_\varepsilon \rightharpoonup \int_Y u(\cdot, y)\, dy \ \ \text{in } L^p(\mathbf{R}^N). \tag{1.13}$$

For any measurable function $f : \mathbf{R} \times \mathbf{R}^N \times \mathcal{Y} \to \mathbf{R}$ such that

$$\begin{aligned} &f(\cdot, x, y) \text{ is Lipschitz-continuous, uniformly w.r.t. } (x,y) \in \mathbf{R}^N \times \mathcal{Y}, \\ &f(\xi, \cdot, \cdot) \in \mathcal{F} \ (\text{cf. } (1.4)) \text{ for any } \xi \in \mathbf{R}, \end{aligned} \tag{1.14}$$

one has

$$u_\varepsilon \underset{2}{\to} u \ \ \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}) \ \Rightarrow \ f(u_\varepsilon(x), x, x/\varepsilon) \underset{2}{\to} f(u(x,y), x, y) \ \ \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}). \tag{1.15}$$

Proof. For any $u \in L^p(\mathbf{R}^N)$,

$$\int_{\mathbf{R}^N} |u_\varepsilon(x) - u(x)|^p \, dx \overset{(1.6)}{=} \iint_{\mathbf{R}^N \times \mathcal{Y}} |u_\varepsilon(S_\varepsilon(x,y)) - u(S_\varepsilon(x,y))|^p \, dxdy. \tag{1.16}$$

Hence

$$\begin{aligned} &\big| \|u_\varepsilon \circ S_\varepsilon - u\|_{L^p(\mathbf{R}^N \times \mathcal{Y})} - \|u_\varepsilon - u\|_{L^p(\mathbf{R}^N)} \big| \\ &\overset{(1.16)}{=} \big| \|u_\varepsilon \circ S_\varepsilon - u\|_{L^p(\mathbf{R}^N \times \mathcal{Y})} - \|u_\varepsilon \circ S_\varepsilon - u \circ S_\varepsilon\|_{L^p(\mathbf{R}^N \times \mathcal{Y})} \big| \\ &\le \|u - u \circ S_\varepsilon\|_{L^p(\mathbf{R}^N \times \mathcal{Y})} \to 0 \quad (\text{by } (1.3')). \end{aligned}$$

The statement (1.11) is thus established. (1.12) and (1.15) are straightforward consequences of the definition of two-scale convergence.

In view of proving (1.13), let us assume that $u_\varepsilon \underset{2}{\rightharpoonup} u$ in $L^p(\mathbf{R}^N \times \mathcal{Y})$, and fix any (bounded if $p > 1$) Lebesgue measurable set $A \subset \mathbf{R}^N$. Applying Lemma 1.1 and noticing that $u_\varepsilon \chi_A \underset{2}{\rightharpoonup} u \chi_A$ in $L^1(\mathbf{R}^N \times \mathcal{Y})$, we then have

$$\int_A u_\varepsilon(x)\, dx = \iint_{\mathbf{R}^N \times Y} (u_\varepsilon \chi_A)(S_\varepsilon(x,y))\, dxdy \rightharpoonup \iint_{\mathbf{R}^N \times Y} u(x,y) \chi_A(x)\, dxdy = \int_A dx \int_Y u(x,y)\, dx.$$

As the finite linear combinations of indicator functions χ_A are dense in $L^{p'}(\mathbf{R}^N)$, we conclude that $u_\varepsilon \rightharpoonup \int_Y u(\cdot, y)\, dy$ in $L^p(\mathbf{R}^N)$. □

Remark. For $p=\infty$ the implication (1.11) may fail. As a counterexample it suffices to select any real a that is no integral multiple of ε_n for any n, and set $u_{\varepsilon_n} = \chi_{[a,+\infty[}$ for any $n \in \mathbf{N}$. This constant sequence does not two-scale converge in $L^\infty(\mathbf{R}^N \times \mathcal{Y})$, as $u_{\varepsilon_n} \circ S_{\varepsilon_n}$ is constant w.r.t. x in a small neighbourhood of a for any n. This shows that strong two-scale convergence in $L^\infty(\mathbf{R}^N \times \mathcal{Y})$ to discontinuous functions is a rather restrictive property. On the other hand it is easy to see that for $p=\infty$ (1.13) holds with $\overset{*}{\rightharpoonup}$ ($\underset{2}{\overset{*}{\rightharpoonup}}$, resp.) in place of $\rightharpoonup$ ($\underset{2}{\rightharpoonup}$, resp.).

Two-scale convergence may also be introduced in the space of continuous functions, see [12]. □

Note the asymptotic two-scale decomposition, for any $u \in L^p_{loc}(\mathbf{R}^N \times \mathcal{Y})$),

$$\begin{aligned} &u(x,y) = \hat{u}(x) + \tilde{u}(x,y) \qquad \text{for a.a. } (x,y) \in \mathbf{R}^N \times \mathcal{Y}, \\ &\text{with} \int_Y \tilde{u}(x,y)\, dy = 0 \qquad \text{for a.a. } x \in \mathbf{R}^N. \end{aligned} \tag{1.17}$$

For $p=2$ this decomposition is orthogonal:

$$\|u\|^2_{L^2(\mathbf{R}^N \times \mathcal{Y})} = \|\hat{u}\|^2_{L^2(\mathbf{R}^N)} + \|\tilde{u}\|^2_{L^2(\mathbf{R}^N \times \mathcal{Y})}; \tag{1.18}$$

hence

$$\iint_{\mathbf{R}^N \times Y} uw\, dxdy = \int_{\mathbf{R}^N} \hat{u}\hat{w}\, dx + \iint_{\mathbf{R}^N \times Y} \tilde{u}\tilde{w}\, dxdy \qquad \forall u, w \in L^2(\mathbf{R}^N \times \mathcal{Y}). \tag{1.18$'$}$$

Two-Scale Convergence of Distributions. Let us denote by $\langle \cdot, \cdot \rangle$ ($\langle\langle \cdot, \cdot \rangle\rangle$, resp.) the duality pairing between $\mathcal{D}(\mathbf{R}^N)$ ($\mathcal{D}(\mathbf{R}^N \times \mathcal{Y})$, resp.) and its dual space. For any sequence $\{u_\varepsilon\}$ in $\mathcal{D}'(\mathbf{R}^N)$ and any $u \in \mathcal{D}'(\mathbf{R}^N \times \mathcal{Y})$, we say that u_ε two-scale converges to u in $\mathcal{D}'(\mathbf{R}^N \times \mathcal{Y})$ iff

$$\langle u_\varepsilon(x), \psi(x, x/\varepsilon) \rangle \to \langle\langle u(x,y), \psi(x,y) \rangle\rangle \qquad \forall \psi \in \mathcal{D}(\mathbf{R}^N \times \mathcal{Y}). \tag{1.19}$$

By Proposition 2.6 ahead, this extends the weak two-scale convergence of $L^p(\mathbf{R}^N \times \mathcal{Y})$. Two-scale convergence in the space of measures is similarly defined.

For instance, let us fix any $y_0 \in \mathcal{Y}$ and a sequence $\{\theta_\varepsilon\}$ in $L^1(\mathcal{Y})$ such that $\theta_\varepsilon \rightharpoonup \delta_{y_0}$ (the Dirac measure concentrated at y_0) in $\mathcal{D}'(\mathcal{Y})$. After extending θ_ε to $\mathbf{R}$ by Y-periodicity, it is easy to see that e.g.

$$\begin{aligned} &|x|\theta_\varepsilon(x/\varepsilon) \rightharpoonup |x| \qquad &&\text{in } \mathcal{D}'(\mathbf{R}), \\ &|x|\theta_\varepsilon(x/\varepsilon) \underset{2}{\rightharpoonup} |x|\delta_{y_0}(y) \qquad &&\text{in } \mathcal{D}'(\mathbf{R}^N \times \mathcal{Y}). \end{aligned} \tag{1.20}$$

Parameters and Scales. So far we dealt with sequences indexed by a parameter ε, that we assumed to coincide with the ratio between two scales. But this coincidence is not really needed: we illustrate this issue dealing with two sequences of parameters. First let us define the set of all *scale sequences,* $\mathcal{E}$, namely the set of all positive vanishing sequences. Let us fix any $\hat{\varepsilon} := \{\varepsilon_1, ..., \varepsilon_n, ...\}$, $\hat{\varepsilon}' := \{\varepsilon'_1, ..., \varepsilon'_n, ...\} \in \mathcal{E}$. We say that $u_\varepsilon \underset{2}{\to} u$ in $L^p(\mathbf{R}^N \times \mathcal{Y})$ w.r.t. ε', and write $u_\varepsilon \underset{2}{\overset{\varepsilon'}{\to}} u$, iff $u_{\varepsilon_n} \circ S_{\varepsilon'_n} \to u$ in $L^p(\mathbf{R}^N \times \mathcal{Y})$ as $n \to \infty$. For instance, if $\varepsilon'_n := \varepsilon_n^2$ for any n, as $n \to \infty$ we have

$$\begin{aligned} \cos(2\pi S_{\varepsilon_n}(x,y)/\varepsilon_n) &= \cos(2\pi[\mathcal{N}(x/\varepsilon_n) + y]) = \cos(2\pi y) \\ \cos(2\pi S_{\varepsilon_n^2}(x,y)/\varepsilon_n) &= \cos(2\pi\varepsilon_n[\mathcal{N}(x/\varepsilon_n^2) + y]) = \cos(2\pi[x/\varepsilon_n + O(\varepsilon_n)]) \rightharpoonup 0 \\ \cos(2\pi S_{\varepsilon_n^2}(x,y)/\varepsilon_n^2) &= \cos(2\pi[\mathcal{N}(x/\varepsilon_n^2) + y]) = \cos(2\pi y) \\ \cos(2\pi S_{\varepsilon_n}(x,y)/\varepsilon_n^2) &= \cos(2\pi[\mathcal{N}(x/\varepsilon_n) + y]/\varepsilon_n) \rightharpoonup 0 \end{aligned}$$

in $L^p_{loc}(\mathbf{R}^N \times \mathcal{Y})$ for any $p \in [1, +\infty[$. Hence

$$\left.\begin{aligned} &\cos(2\pi x/\varepsilon) \underset{2}{\overset{\varepsilon}{\to}} \cos(2\pi y), \quad \cos(2\pi x/\varepsilon) \underset{2}{\overset{\varepsilon^2}{\to}} 0 \\ &\cos(2\pi x/\varepsilon^2) \underset{2}{\overset{\varepsilon^2}{\to}} \cos(2\pi y), \quad \cos(2\pi x/\varepsilon^2) \underset{2}{\overset{\varepsilon}{\to}} 0 \end{aligned}\right. \quad \text{in } L^p_{loc}(\mathbf{R}^N \times \mathcal{Y}), \forall p < +\infty. \tag{1.21}$$

Two-scale convergence arises in problems of *homogenization,* where the scale is determined by the data.

Multiscaling has also been considered, see [2].

2. Some Properties of Two-Scale Convergence

Let us fix any $u \in L^p(\mathbf{R}^N \times \mathcal{Y})$ ($p \in [1, +\infty[$), and set $u_{(\varepsilon)}(x) := u(x, x/\varepsilon)$ for a.e. $x \in \mathbf{R}^N$. For any sequence $\{u_\varepsilon\}$, we wonder whether any relation may be established between

$$u_\varepsilon \underset{2}{\to} u \ \text{ and } \ u_\varepsilon - u_{(\varepsilon)} \underset{2}{\to} 0 \quad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}),$$

and similarly for weak two-scale convergence.

First we notice that if u is just an element of $L^p(\mathbf{R}^N \times \mathcal{Y})$, $u_{(\varepsilon)}$ need not be measurable, see [1]. After [4,5], we then define the *coarse-scale averaging* operator M_ε:

$$(M_\varepsilon u)(x, y) := \int_Y u(\varepsilon \mathcal{N}(x/\varepsilon) + \varepsilon\xi, y)\, d\xi \qquad \text{for a.a. } (x, y) \in \mathbf{R}^N \times \mathcal{Y}. \tag{2.1}$$

Note that $M_\varepsilon u$ is measurable w.r.t. (x, y), and $(M_\varepsilon u)(x, x/\varepsilon)$ is measurable as well.

Proposition 2.1 *Let $p \in [1, +\infty[$. The operator M_ε is linear and continuous in $L^p(\mathbf{R}^N \times \mathcal{Y})$, and for any $u \in L^p(\mathbf{R}^N \times \mathcal{Y})$*

$$(M_\varepsilon u)(x, x/\varepsilon) \underset{2}{\to} u(x, y) \qquad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}). \tag{2.2}$$

If $u \in \mathcal{F}$ (cf. (1.4)) then the operator M_ε may be dropped.

Any function of $L^p(\mathbf{R}^N \times \mathcal{Y})$ ($p \in [1, +\infty[$) is thus a two-scale limit.

Proposition 2.2 *Let $p \in [1, +\infty[$. For any sequence $\{u_\varepsilon\}$ in $L^p(\mathbf{R}^N)$ and any $u \in L^p(\mathbf{R}^N \times \mathcal{Y})$,*

$$u_\varepsilon \underset{2}{\to} u \quad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}) \quad \Leftrightarrow \quad u_\varepsilon(x) - (M_\varepsilon u)(x, x/\varepsilon) \underset{2}{\to} 0 \quad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}), \tag{2.3}$$

$$u_\varepsilon \underset{2}{\rightharpoonup} u \quad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}) \quad \Leftrightarrow \quad u_\varepsilon(x) - (M_\varepsilon u)(x, x/\varepsilon) \underset{2}{\rightharpoonup} 0 \quad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}), \tag{2.4}$$

$$u_\varepsilon \underset{2}{\to} u \quad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}) \quad \Leftrightarrow \quad u_\varepsilon(x) - (M_\varepsilon u)(x, x/\varepsilon) \to 0 \quad \text{in } L^p(\mathbf{R}^N), \tag{2.5}$$

$$u_\varepsilon \underset{2}{\rightharpoonup} u \quad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}) \quad \underset{\not\Leftarrow}{\Rightarrow} \quad u_\varepsilon(x) - (M_\varepsilon u)(x, x/\varepsilon) \rightharpoonup 0 \quad \text{in } L^p(\mathbf{R}^N). \tag{2.6}$$

If $u \in \mathcal{F}$ (cf. (1.4)), then the operator M_ε may be dropped.

A Counterexample. The next counterexample shows that the converse implication of (2.6) fails. Let us set

$$u_\varepsilon(x) = e^{-x^2} \sin(2\pi x/\varepsilon) \quad \forall x \in \mathbf{R}^N, \qquad u(x, y) = 0 \quad \forall (x, y) \in \mathbf{R}^N \times [0, 1[.$$

Then $u_\varepsilon(x) - (M_\varepsilon u)(x, x/\varepsilon) = u_\varepsilon(x) \rightharpoonup 0$ in $L^p(\mathbf{R})$ for any $p \in [1, +\infty[$, but $u_\varepsilon \underset{2}{\rightharpoonup} e^{-x^2} \sin(2\pi y)$ in $L^p(\mathbf{R} \times \mathcal{Y})$. □

Characterization of Two-Scale Convergence. Next we retrieve the original definitions of weak and strong two-scale convergence of Nguetseng [9] and Allaire [1], for any $p \neq \infty$.

Proposition 2.3 *Let $p \in [1, +\infty[$. For any bounded sequence $\{u_\varepsilon\}$ in $L^p(\mathbf{R}^N)$ and any $u \in L^p(\mathbf{R}^N \times \mathcal{Y})$, $u_\varepsilon \underset{2}{\rightharpoonup} u$ in $L^p(\mathbf{R}^N \times \mathcal{Y})$ iff*

$$\int_{\mathbf{R}^N} u_\varepsilon(x) \psi(x, x/\varepsilon)\, dx \to \iint_{\mathbf{R}^N \times Y} u(x, y) \psi(x, y)\, dx dy \qquad \forall \psi \in \mathcal{D}(\mathbf{R}^N \times \mathcal{Y}). \tag{2.7}$$

As the tensor product $\mathcal{D}(\mathbf{R}^N)\otimes\mathcal{D}(\mathcal{Y})$ is dense in $\mathcal{D}(\mathbf{R}^N\times\mathcal{Y})$, (2.7) is equivalent to

$$\int_{\mathbf{R}^N} u_\varepsilon(x)\psi(x)\varphi(x/\varepsilon)\,dx \to \iint_{\mathbf{R}^N\times Y} u(x,y)\psi(x)\varphi(y)\,dxdy \qquad \forall\psi\in\mathcal{D}(\mathbf{R}^N),\forall\varphi\in\mathcal{D}(\mathcal{Y}), \quad (2.8)$$

which in turn is equivalent to each of the two following statements

$$\int_{\mathbf{R}^N} u_\varepsilon(S_\varepsilon(x,y))\psi(x)\,dx \rightharpoonup \int_{\mathbf{R}^N} u(x,y)\psi(x)\,dx \qquad \text{in } L^p(\mathcal{Y}),\forall\psi\in\mathcal{D}(\mathbf{R}^N), \qquad (2.8')$$

$$\int_{\mathcal{Y}} u_\varepsilon(S_\varepsilon(x,y))\varphi(y)\,dy \rightharpoonup \int_{\mathcal{Y}} u(x,y)\varphi(y)\,dy \qquad \text{in } L^p(\mathbf{R}^N),\forall\varphi\in\mathcal{D}(\mathcal{Y}). \qquad (2.8'')$$

Proposition 2.4 *(Norm Semicontinuity and Continuity) Let $p\in[1,+\infty[$ and $\{u_\varepsilon\}$ be a sequence in $L^p(\mathbf{R}^N)$. Then*

$$\begin{aligned} &u_\varepsilon \underset{2}{\rightharpoonup} u \ \text{ in } L^p(\mathbf{R}^N\times\mathcal{Y}) \quad \Rightarrow \\ &\liminf_{\varepsilon\to 0}\ \|u_\varepsilon\|_{L^p(\mathbf{R}^N)} \geq \|u\|_{L^p(\mathbf{R}^N\times\mathcal{Y})} \Big(\geq \Big\|\int_Y u(\cdot,y)\,dy\Big\|_{L^p(\mathbf{R}^N)}\Big), \end{aligned} \qquad (2.9)$$

$$u_\varepsilon \underset{2}{\to} u \ \text{ in } L^p(\mathbf{R}^N\times\mathcal{Y}) \quad \Rightarrow \quad \begin{cases} u_\varepsilon \underset{2}{\rightharpoonup} u & \text{in } L^p(\mathbf{R}^N\times\mathcal{Y}) \\ \|u_\varepsilon\|_{L^p(\mathbf{R}^N)} \to \|u\|_{L^p(\mathbf{R}^N\times\mathcal{Y})}. \end{cases} \qquad (2.10)$$

For $p>1$ the latter implication can be inverted.

Proposition 2.5 *Let $p\in[1,+\infty[$ and $\{u_\varepsilon\}$ be a bounded sequence in $L^p(\mathbf{R}^N)$.*
$u_\varepsilon \underset{2}{\to} u$ in $L^p(\mathbf{R}^N\times\mathcal{Y})$ iff

$$\begin{aligned} &\forall\{v_\varepsilon\}\subset L^{p'}(\mathbf{R}^N), \text{if } v_\varepsilon \underset{2}{\rightharpoonup} v \ \text{ in } L^{p'}(\mathbf{R}^N\times\mathcal{Y}) \ \ (v_\varepsilon \overset{*}{\underset{2}{\rightharpoonup}} v \ \text{ if } p'=\infty) \\ &\text{then } \int_{\mathbf{R}^N} u_\varepsilon(x)v_\varepsilon(x)\,dx \to \iint_{\mathbf{R}^N\times Y} u(x,y)v(x,y)\,dxdy. \end{aligned} \qquad (2.11)$$

An analogous characterization holds for *weak* two-scale convergence, and generalizes Proposition 2.5.

Proposition 2.6 *Let $p\in]1,+\infty[$ and $\{u_\varepsilon\}$ be a bounded sequence in $L^p(\mathbf{R}^N)$. Then $u_\varepsilon \underset{2}{\rightharpoonup} u$ in $L^p(\mathbf{R}^N\times\mathcal{Y})$ iff*

$$\begin{aligned} &\forall\{v_\varepsilon\}\subset L^{p'}(\mathbf{R}^N), \text{if } v_\varepsilon \underset{2}{\to} v \ \text{ in } L^{p'}(\mathbf{R}^N\times\mathcal{Y}), \text{ then} \\ &\int_{\mathbf{R}^N} u_\varepsilon(x)v_\varepsilon(x)\,dx \to \iint_{\mathbf{R}^N\times Y} u(x,y)v(x,y)\,dxdy. \end{aligned} \qquad (2.12)$$

Fourier Analysis of Two-Scale Convergence. Next set $\phi_n(y) := \exp\{2\pi i n y\}$ for any $y\in$ and $n\in\mathbf{Z}$, denote by ℓ^2 the complex Hilbert space of square-summable sequences $\mathbf{Z}\to\mathbf{C}$, and use the star to denote the complex conjugate.

Theorem 2.7 *(Fourier Expansion w.r.t. y and Two-Scale Plancherel Theorem)*
Let $\{u_\varepsilon\}$ be a sequence in $L^2(\mathbf{R}^N;\mathbf{C})$, $u\in L^2(\mathbf{R}^N\times\mathcal{Y};\mathbf{C})$, define S_ε as in (1.2), and set

$$\begin{aligned} a_{n,\varepsilon}(x) &:= \int_Y u_\varepsilon(S_\varepsilon(x,y))\,\phi_n(y)^*\,dy \\ a_n(x) &:= \int_Y u(x,y)\,\phi_n(y)^*\,dy \end{aligned} \qquad \text{for a.a. } x\in\mathbf{R}^N,\forall n\in\mathbf{Z},\forall\varepsilon. \qquad (2.13)$$

Then

$$u_\varepsilon \underset{2}{\rightharpoonup} u \quad \text{in } L^2(\mathbf{R}^N \times \mathcal{Y}; \mathbf{C}) \quad \Leftrightarrow \quad \{a_{n,\varepsilon}\} \rightharpoonup \{a_n\} \quad \text{in } L^2(\mathbf{R}^N; \ell^2), \tag{2.14}$$

$$u_\varepsilon \underset{2}{\to} u \quad \text{in } L^2(\mathbf{R}^N \times \mathcal{Y}; \mathbf{C}) \quad \Leftrightarrow \quad \{a_{n,\varepsilon}\} \to \{a_n\} \quad \text{in } L^2(\mathbf{R}^N; \ell^2). \tag{2.15}$$

The examples of (1.21) might be interpreted within this framework.

The statements (2.14) and (2.15) might be reformulated in terms of the Fourier expansion of $a_{n,\varepsilon}$ and a_n as functions of x, thus achieving the global Fourier expansion of $u_\varepsilon \circ S_\varepsilon$ and u w.r.t. (x, y).

Proof. The $a_{n,\varepsilon}$'s (the a_n's, resp.) are the coefficients of the partial Fourier expansion of $u_\varepsilon \circ S_\varepsilon$ (u, resp.) w.r.t. y, in the sense that

$$\begin{aligned} &u_\varepsilon(S_\varepsilon(x,y)) = \sum_{n\in\mathbf{Z}} a_{n,\varepsilon}(x)\, \phi_n(y) \qquad \text{in } L^2(\mathbf{R}^N \times \mathcal{Y}; \mathbf{C}), \forall\varepsilon, \\ &u(x,y) = \sum_{n\in\mathbf{Z}} a_n(x)\, \phi_n(y) \qquad \text{in } L^2(\mathbf{R}^N \times \mathcal{Y}; \mathbf{C}). \end{aligned} \tag{2.16}$$

By (2.8'), $u_\varepsilon \underset{2}{\rightharpoonup} u$ in $L^2(\mathbf{R}^N \times \mathcal{Y}; \mathbf{C})$ iff

$$\int_{\mathbf{R}^N} u_\varepsilon(S_\varepsilon(x,y))\, g(x)^*\, dx \rightharpoonup \int_{\mathbf{R}^N} u(x,y)\, g(x)^*\, dx \qquad \text{in } L^2(\mathcal{Y}; \mathbf{C}), \forall g \in L^2(\mathbf{R}^N; \mathbf{C}). \tag{2.17}$$

Setting

$$b_{g,n,\varepsilon} := \int_{\mathbf{R}^N} a_{n,\varepsilon}(x)\, g(x)^*\, dx, \quad b_{g,n} := \int_{\mathbf{R}^N} a_n(x)\, g(x)^*\, dx \qquad \forall n, \varepsilon,$$

by (2.16), (2.17) also reads

$$\sum_{n\in\mathbf{Z}} b_{g,n,\varepsilon}\, \phi_n(y) \rightharpoonup \sum_{n\in\mathbf{Z}} b_{g,n}\, \phi_n(y) \qquad \text{in } L^2(\mathcal{Y}; \mathbf{C}), \forall g \in L^2(\mathbf{R}^N; \mathbf{C}).$$

By testing on any $\phi_m \in L^2(\mathcal{Y}; \mathbf{C})$), we then have

$$\{b_{g,n,\varepsilon}\} \rightharpoonup \{b_{g,n}\} \qquad \text{in } \ell^2, \forall g \in L^2(\mathbf{R}^N; \mathbf{C}).$$

As g is any element of $L^2(\mathbf{R}^N; \mathbf{C})$, this means that $\{a_{n,\varepsilon}(x)\} \rightharpoonup \{a_n(x)\}$ in $L^2(\mathbf{R}^N; \ell^2)$, and (2.14) is thus established.

Let us now come to strong convergence. By (2.16),

$$\|u_\varepsilon \circ S_\varepsilon\|_{L^2(\mathbf{R}^N \times \mathcal{Y}; \mathbf{C})} = \|\{a_{n,\varepsilon}\}\|_{L^2(\mathbf{R}^N; \ell^2)} \quad \forall\varepsilon, \quad \|u\|_{L^2(\mathbf{R}^N \times \mathcal{Y}; \mathbf{C})} = \|\{a_n\}\|_{L^2(\mathbf{R}^N; \ell^2)};$$

thus

$$\|u_\varepsilon \circ S_\varepsilon\|_{L^2(\mathbf{R}^N \times \mathcal{Y}; \mathbf{C})} \to \|u\|_{L^2(\mathbf{R}^N \times \mathcal{Y}; \mathbf{C})} \quad \Leftrightarrow \quad \|\{a_{n,\varepsilon}\}\|_{L^2(\mathbf{R}^N; \ell^2)} \to \|\{a_n\}\|_{L^2(\mathbf{R}^N; \ell^2)}. \tag{2.18}$$

By (2.10), this statement and (2.14) entail (2.15). $\square$

3. Exercises

Exercise 1. Prove the Propositions of the last section. (See also [12])

Exercise 2. This statement extends Proposition 2.1. Define M_ε as in (2.1), and prove that, for any sequence $\{u_\varepsilon\}$ in $L^p(\mathbf{R}^N \times \mathcal{Y})$ ($p \in [1, +\infty[$),

$$u_\varepsilon \to u \quad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}) \quad \Rightarrow \quad (M_\varepsilon u_\varepsilon)(x, x/\varepsilon) \underset{2}{\to} u(x,y) \quad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}). \tag{3.1}$$

Does the converse statement hold? Does the analogous statement for *weak* convergence hold true?

Exercise 3. Let $p, q, r \in [1, +\infty[$ be such that $1/p + 1/q = 1/r$. Let $\{v_\varepsilon\}$ and $\{w_\varepsilon\}$ be sequences in $L^p(\mathbf{R}^N)$ and $L^q(\mathcal{Y})$, resp.. Show that

$$\begin{aligned} &v_\varepsilon \to v \ \text{ in } L^p(\mathbf{R}^N) \ \text{ and } \ w_\varepsilon \rightharpoonup w \ \text{ in } L^q(\mathcal{Y}) \\ &\Rightarrow \ v_\varepsilon(x) w_\varepsilon(x/\varepsilon) \underset{2}{\rightharpoonup} v(x)w(y) \ \text{ in } L^r(\mathbf{R}^N \times \mathcal{Y}); \end{aligned} \tag{3.2}$$

$$\begin{aligned} &v_\varepsilon \to v \ \text{ in } L^p(\mathbf{R}^N) \ \text{ and } \ w_\varepsilon \to w \ \text{ in } L^q(\mathcal{Y}) \\ &\Rightarrow \ v_\varepsilon(x) w_\varepsilon(x/\varepsilon) \underset{2}{\to} v(x)w(y) \ \text{ in } L^r(\mathbf{R}^N \times \mathcal{Y}), \end{aligned} \tag{3.3}$$

but

$$\begin{aligned} &v_\varepsilon \rightharpoonup v \ \text{ in } L^p(\mathbf{R}^N) \ \text{ and } \ w_\varepsilon \to w \ \text{ in } L^q(\mathcal{Y}) \\ &\not\Rightarrow \ v_\varepsilon(x) w_\varepsilon(x/\varepsilon) \underset{2}{\rightharpoonup} v(x)w(y) \ \text{ in } L^r(\mathbf{R}^N \times \mathcal{Y}). \end{aligned} \tag{3.4}$$

Exercise 4 (Two-Scale Convolution). Let $p \in [1, +\infty[$, $\{u_\varepsilon\}$ be a sequence of $L^p(\Omega)$ and $\{w_\varepsilon\}$ be a sequence of $L^1(\mathbf{R}^N)$ such that

$$u_\varepsilon \underset{2}{\rightharpoonup} u \quad \text{in } L^p(\Omega \times \mathcal{Y}), \qquad w_\varepsilon \underset{2}{\to} w \quad \text{in } L^1(\mathbf{R}^N \times \mathcal{Y}). \tag{3.5}$$

Show that

$$\begin{aligned} &(u_\varepsilon * w_\varepsilon)(x) := \int_{\mathbf{R}^N} u_\varepsilon(\xi) w_\varepsilon(x - \xi)\, d\xi \ \underset{2}{\rightharpoonup} \\ &(u ** w)(x,y) := \iint_{\mathbf{R}^N \times Y} u(\xi, \eta) w(x - \xi, y - \eta)\, d\xi d\eta \qquad \text{in } L^p(\Omega \times \mathcal{Y}). \end{aligned} \tag{3.6}$$

If moreover $u_\varepsilon \underset{2}{\to} u$ in $L^p(\Omega \times \mathcal{Y})$ show that then $u_\varepsilon * w_\varepsilon \underset{2}{\to} u ** w$ in $L^p(\Omega \times \mathcal{Y})$.

Exercise 5. Prove the next statement.

Proposition 3.1 *(Two-Scale Lower Semicontinuity of Convex Integrals) If*

$$\begin{aligned} &\varphi : \mathbf{R}^N \times \mathcal{Y} \to \mathbf{R} \textit{ is measurable w.r.t. } \mathcal{B}(\mathbf{R}^N) \otimes \mathcal{L}(\mathcal{Y}), \\ &\varphi(\cdot, y) \textit{ is convex (whence continuous) for a.e. } y, \end{aligned} \tag{3.7}$$

$$\exists c > 0, \exists f \in L^1(\mathcal{Y}) : \forall \xi \in \mathbf{R}^N, \textit{ for a.e. } y, \ \ c|\xi|^p - f(y) \le \varphi(\xi, y), \tag{3.8}$$

Then

$$\begin{aligned} &u_\varepsilon \underset{2}{\rightharpoonup} u \quad \textit{in } L^p(\Omega \times \mathcal{Y})^N \quad \Rightarrow \\ &\liminf_{\varepsilon \to 0} \int_\Omega \varphi(u_\varepsilon(x), x/\varepsilon)\, dx \ge \iint_{\Omega \times \mathcal{Y}} \varphi(u(x,y), y)\, dx dy. \end{aligned} \tag{3.9}$$

Exercise 5. Establish the two scale limit of the sequence $\{\cos(2\pi x/\sqrt{\varepsilon})\}$.

4. Two-Scale Compactness

In this section we extend some classical compactness theorems to two-scale convergence in L^p. We shall say that a sequence $\{u_\varepsilon\}$ is relatively compact iff it is possible to extract a convergent subsequence from any of its subsequences.

4.1 Integrable Functions. Proposition 1.2 yields the next result.

Proposition 4.1 *Let $p \in [1,+\infty[$. For any sequence $\{u_\varepsilon\}$ in $L^p(\mathbf{R}^N)$,*

$$\begin{aligned}&\{u_\varepsilon\} \text{ is strongly one-scale relatively compact in } L^p(\mathbf{R}^N) \Rightarrow\\ &\{u_\varepsilon\} \text{ is strongly two-scale relatively compact in } L^p(\mathbf{R}^N\times\mathcal{Y}) \Rightarrow\\ &\{u_\varepsilon\} \text{ is weakly two-scale relatively compact in } L^p(\mathbf{R}^N\times\mathcal{Y}) \Rightarrow\\ &\{u_\varepsilon\} \text{ is weakly one-scale relatively compact in } L^p(\mathbf{R}^N).\end{aligned} \tag{4.1}$$

Proposition 4.2 *(Weak Two-Scale Compactness in L^p)*

(i) Let $p \in]1,+\infty]$. Any sequence $\{u_\varepsilon\}$ of $L^p(\mathbf{R}^N)$ is weakly star two-scale relatively compact in $L^p(\mathbf{R}^N\times\mathcal{Y})$ iff it is bounded, hence iff it is weakly (weakly star if $p=\infty$) one-scale relatively compact in $L^p(\mathbf{R}^N)$.

(ii) Similarly, any sequence of $L^1(\mathbf{R}^N)$ is weakly star two-scale relatively compact in $C^0_c(\mathbf{R}^N\times\mathcal{Y})'$ iff it is bounded, hence iff it is weakly star one-scale relatively compact in $C^0_c(\mathbf{R}^N)'$.

(iii) Any sequence of $L^1(\mathbf{R}^N)$ is weakly two-scale relatively compact in $L^1(\mathbf{R}^N\times\mathcal{Y})$ iff it is weakly one-scale relatively compact in $L^1(\mathbf{R}^N)$.

Proof. For any $p \in [1,+\infty]$, by Lemma 1.1, $\{u_\varepsilon\}$ is bounded in $L^p(\mathbf{R}^N)$ iff $\{u_\varepsilon \circ S_\varepsilon\}$ is bounded in $L^p(\mathbf{R}^N\times\mathcal{Y})$. Parts (i) and (ii) then follow from the classical Banach-Alaoglu theorem. Part (iii) may easily be proved via the classical de la Vallée Poussin criterion. □

Proposition 4.2' *(Two-Scale Biting Lemma) Let $\{u_\varepsilon\}$ be a bounded sequence in $L^1(\mathbf{R}^N)$. There exist $u \in L^1(\mathbf{R}^N\times\mathcal{Y})$, a subsequence $\{u_{\tilde\varepsilon}\}$, and a nondecreasing sequence $\{\Omega_k\}$ of measurable subsets of $\mathbf{R}^N$ such that, denoting by $|\cdot|_N$ the N-dimensional Lebesgue measure,*

$$\begin{cases} |\mathbf{R}^N\setminus\Omega_k|_N \to 0 & \text{as } k\to\infty,\\ u_{\tilde\varepsilon}|_{\Omega_k} \underset{2}{\rightharpoonup} u|_{\Omega_k\times Y} & \text{in } L^1(\Omega_k\times\mathcal{Y}), \text{ as } \tilde\varepsilon\to 0, \forall k\in\mathbf{N}.\end{cases} \tag{4.1'}$$

Proposition 4.3 *(Two-Scale Vitali's Theorem) Let $p \in [1,+\infty[$, $\{u_\varepsilon\}$ be a sequence in $L^p(\mathbf{R}^N)$, such that*

$$\sup_\varepsilon \int_{\mathbf{R}^N\setminus B(0,R)} |u_\varepsilon(x)|^p\,dx \to 0 \qquad \text{as } R\to+\infty, \tag{4.2}$$

and $u_\varepsilon \underset{2}{\to} u$ a.e. in $\mathbf{R}^N\times Y$. Then

$$u \in L^p(\mathbf{R}^N\times\mathcal{Y}), \qquad u_\varepsilon \underset{2}{\to} u \quad \text{in } L^p(\mathbf{R}^N\times\mathcal{Y}) \tag{4.3}$$

iff $\{|u_\varepsilon|^p\}$ is equi-integrable, in the sense that, for any sequence $\{A_n\}$ of measurable subsets of $\mathbf{R}^N$,

$$\sup_\varepsilon \int_{A_n} |u_\varepsilon(x)|^p\,dx \to 0 \qquad \text{as } |A_n|_N \to 0. \tag{4.4}$$

(By ε we still denote the running parameter of a vanishing sequence.)

4.2 Differentiable Functions. Let $\{u_\varepsilon\}$ be a bounded sequence of $W^{1,p}(\mathbf{R}^N)$ ($p \in]1,+\infty[$). By Proposition 4.2 there exist $u \in W^{1,p}(\mathbf{R}^N)$ and $w \in L^p(\mathbf{R}^N)^N$ such that, up to subsequences,

$$\begin{aligned}&u_\varepsilon \rightharpoonup u \quad \text{in } W^{1,p}(\mathbf{R}^N) \quad (\text{whence } u_\varepsilon \underset{2}{\rightharpoonup} u \text{ in } L^p(\mathbf{R}^N\times\mathcal{Y})),\\ &\nabla u_\varepsilon \underset{2}{\rightharpoonup} w \quad \text{in } L^p(\mathbf{R}^N\times\mathcal{Y})^N.\end{aligned} \tag{4.5}$$

Even if $\nabla u_\varepsilon \underset{2}{\rightharpoonup} w$, this is far from entailing that $w = \nabla u$. Here is a counterexample for $N = 1$:

$$\begin{aligned} &u_\varepsilon(x) := \varepsilon \sin(2\pi x/\varepsilon) \rightharpoonup 0 =: u && \text{in } W^{1,p}(0,1), \\ &Du_\varepsilon = \cos(2\pi x/\varepsilon) \underset{2}{\rightharpoonup} \cos(2\pi y) \neq Du && \text{in } L^p(]0,1[\times\mathcal{Y}). \end{aligned} \tag{4.6}$$

Dealing with functions of x, y we shall denote the gradient operator w.r.t. x (y, resp.) by ∇_x (∇_y, resp.). Let us recall the decomposition (1.17):

$$\hat{v}(x) := \int_{\mathcal{Y}} v(x,y)\,dy, \qquad \tilde{v}(x,y) := v(x,y) - \hat{v}(x) \qquad \text{for a.e. } (x,y). \tag{4.7}$$

Theorem 4.4 *[1,4,9] Let $p \in]1,+\infty[$, and a sequence $\{u_\varepsilon\}$ of $W^{1,p}(\mathbf{R}^N)$ be such that $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\mathbf{R}^N)$. Then there exists $u_\sharp \in L^p\big(\mathbf{R}^N; W^{1,p}(\mathcal{Y})\big)$ such that $\widehat{u_\sharp} = 0$ a.e. in $\mathbf{R}^N$ and, as $\varepsilon \to 0$ along a suitable subsequence,*

$$\nabla u_\varepsilon \underset{2}{\rightharpoonup} \nabla_x u + \nabla_y u_\sharp \qquad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y})^N. \tag{4.8}$$

Conversely, for any $u \in W^{1,p}(\mathbf{R}^N)$ and any $u_\sharp \in L^p\big(\mathbf{R}^N; W^{1,p}(\mathcal{Y})\big)$, there exists a sequence $\{u_\varepsilon\}$ of $W^{1,p}(\mathbf{R}^N)$ such that

$$u_\varepsilon \to u \quad \text{in } L^p(\mathbf{R}^N), \qquad \nabla u_\varepsilon \underset{2}{\to} \nabla u + \nabla_y u_\sharp \quad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y})^N. \tag{4.9}$$

Next we confine ourselves to $N = 3, p = 2$, set

$$\begin{aligned} L^2_{\mathrm{rot}}(\mathbf{R}^3)^3 &:= \big\{ v \in L^2(\mathbf{R}^3)^3 : \nabla \times v \in L^2(\mathbf{R}^3)^3 \big\}, \\ L^2_{\mathrm{div}}(\mathbf{R}^3)^3 &:= \big\{ v \in L^2(\mathbf{R}^3)^3 : \nabla \cdot v \in L^2(\mathbf{R}^3) \big\}, \end{aligned}$$

and define $L^2_{\mathrm{rot}}(\mathcal{Y})^3, L^2_{\mathrm{div}}(\mathcal{Y})^3$ similarly. These are Hilbert spaces equipped with the respective graph norm.

Theorem 4.5 *[12] Let $\{u_\varepsilon\}$ be a bounded sequence of $L^2_{\mathrm{div}}(\mathbf{R}^3)^3$ such that $u_\varepsilon \underset{2}{\rightharpoonup} u$ in $L^2(\mathbf{R}^3 \times \mathcal{Y})^3$. Then $\hat{u} \in L^2_{\mathrm{div}}(\mathbf{R}^3)^3$ and $\nabla_y \cdot u = 0$ in $\mathcal{D}'(\mathbf{R}^3 \times \mathcal{Y})$. Moreover, there exists $u_\sharp \in L^2\big(\mathbf{R}^3; H^1(\mathcal{Y})^3\big)$ such that $\widehat{u_\sharp} = 0$ a.e. in $\mathbf{R}^N$, $\nabla_y \times u_\sharp = 0$ a.e. in $\mathbf{R}^3 \times \mathcal{Y}$, and, as $\varepsilon \to 0$ along a suitable subsequence,*

$$\nabla \cdot u_\varepsilon \underset{2}{\rightharpoonup} \nabla \cdot \hat{u} + \nabla_y \cdot u_\sharp \qquad \text{in } L^2(\mathbf{R}^3 \times \mathcal{Y}). \tag{4.10}$$

Conversely, for any $u \in L^2_{\mathrm{div}}(\mathbf{R}^3)^3$ and any $u_\sharp \in L^2\big(\mathbf{R}^3; L^2_{\mathrm{div}}(\mathcal{Y})^3\big)$ there exists a sequence $\{u_\varepsilon\}$ of $L^2_{\mathrm{div}}(\mathbf{R}^3)^3$ such that

$$u_\varepsilon \to u \quad \text{in } L^2(\mathbf{R}^3)^3, \qquad \nabla \cdot u_\varepsilon \underset{2}{\to} \nabla \cdot u + \nabla_y \cdot u_\sharp \quad \text{in } L^2(\mathbf{R}^3 \times \mathcal{Y}). \tag{4.11}$$

Theorem 4.6 *[12] Let $\{u_\varepsilon\}$ be a bounded sequence of $L^2_{\mathrm{rot}}(\mathbf{R}^3)^3$ such that $u_\varepsilon \underset{2}{\rightharpoonup} u$ in $L^2(\mathbf{R}^3 \times \mathcal{Y})^3$. Then $\hat{u} \in L^2_{\mathrm{rot}}(\mathbf{R}^3)^3$ and $\nabla_y \times u = 0$ in $\mathcal{D}'(\mathbf{R}^3 \times \mathcal{Y})^3$. Moreover, there exists $u_\sharp \in L^2\big(\mathbf{R}^3; H^1(\mathcal{Y})^3\big)$ such that such that $\widehat{u_\sharp} = 0$ a.e. in $\mathbf{R}^N$, $\nabla_y \cdot u_\sharp = 0$ a.e. in $\mathbf{R}^3 \times \mathcal{Y}$, and, as $\varepsilon \to 0$ along a suitable subsequence,*

$$\nabla \times u_\varepsilon \underset{2}{\rightharpoonup} \nabla \times \hat{u} + \nabla_y \times u_\sharp \qquad \text{in } L^2(\mathbf{R}^3 \times \mathcal{Y})^3. \tag{4.12}$$

Conversely, for any $u \in L^2_{\mathrm{rot}}(\mathbf{R}^3)^3$ and any $u_\sharp \in L^2\big(\mathbf{R}^3; L^2_{\mathrm{rot}}(\mathcal{Y})^3\big)$ there exists a sequence $\{u_\varepsilon\}$ of $L^2_{\mathrm{rot}}(\mathbf{R}^3)^3$ such that

$$u_\varepsilon \to u \quad \text{in } L^2(\mathbf{R}^3)^3, \qquad \nabla \times u_\varepsilon \underset{2}{\to} \nabla \times u + \nabla_y \times u_\sharp \quad \text{in } L^2(\mathbf{R}^3 \times \mathcal{Y})^3. \tag{4.13}$$

The reader will notice that in the last two theorems no information is inferred about the regularity of u w.r.t. x.

5. Two-Scale Compensated Compactness

In this section we shall see how Murat and Tartar's theory of *compensated compactness,* see e.g. [8], applies to two-scale convergence.

Theorem 5.1 *[8] (Single-Scale Div-Curl Lemma) Let $\{u_\varepsilon\}$ and $\{w_\varepsilon\}$ be two bounded sequences of $L^2_{\rm rot}(\mathbf{R}^3)^3$ and $L^2_{\rm div}(\mathbf{R}^3)^3$, resp.. If*

$$u_\varepsilon \rightharpoonup u, \quad w_\varepsilon \rightharpoonup w \qquad \text{in } L^2(\mathbf{R}^3)^3, \tag{5.1}$$

then $u_\varepsilon \cdot w_\varepsilon \to u \cdot w$ in $\mathcal{D}'(\mathbf{R}^3)$, that is,

$$\int_{\mathbf{R}^3} u_\varepsilon(x)\cdot w_\varepsilon(x)\, \theta(x)\, dx \to \int_{\mathbf{R}^3} u(x)\cdot w(x)\, \theta(x)\, dx \qquad \forall \theta \in \mathcal{D}(\mathbf{R}^3). \tag{5.2}$$

We extend this result to two-scale convergence via the next two lemmata.

Henceforth by appending the index $*$ to a space of functions over $\mathcal{Y}$ we shall denote the subspace of functions having vanishing mean: e.g.

$$L^2_*(\mathcal{Y}) := \big\{v \in L^2(\mathcal{Y}) : \hat{v} = 0\big\}, \qquad H^1_*(\mathcal{Y}) := \big\{v \in H^1(\mathcal{Y}) : \hat{v} = 0\big\}.$$

Lemma 5.2 *For any $u \in L^2_*(\mathcal{Y})^3$, if $\nabla \times u = 0$ in $\mathcal{D}'(\mathcal{Y})^3$ then there exists a scalar potential $\eta \in H^1(\mathcal{Y})$ such that $u = \nabla\eta$ a.e. in $\mathcal{Y}$, and conversely.*

Lemma 5.3 *If $u, w \in L^2(\mathbf{R}^3 \times \mathcal{Y})^3$ are such that*

$$\nabla_y \times u = 0 \quad \text{in } \mathcal{D}'(\mathbf{R}^3 \times \mathcal{Y})^3, \qquad \nabla_y \cdot w = 0 \quad \text{in } \mathcal{D}'(\mathbf{R}^3 \times \mathcal{Y}), \tag{5.3}$$

then

$$\iint_{\mathbf{R}^3 \times \mathcal{Y}} u(x,y)\cdot w(x,y)\theta(x)\, dxdy = \int_{\mathbf{R}^3} \hat{u}(x)\cdot \hat{w}(x)\theta(x)\, dx \qquad \forall \theta \in L^\infty(\mathbf{R}^3). \tag{5.4}$$

Proof. Let us set $\tilde{u}(x,y) := u(x,y) - \hat{u}(x)$ for a.a. $(x,y) \in \mathbf{R}^N \times \mathcal{Y}$ and define $\tilde{w}(x,y)$ similarly. We have

$$\tilde{u} \in L^2\big(\mathbf{R}^N; L^2_*(\mathcal{Y})^N\big), \qquad \nabla_y \times \tilde{u}(x,\cdot) = 0 \quad \text{in } \mathcal{D}'(\mathbf{R}^N \times \mathcal{Y})^{N^2}, \text{ for a.a. } x \in \mathbf{R}^N.$$

After Lemma 5.2 then there exists a potential $\eta \in L^2\big(\mathbf{R}^N; H^1_*(\mathcal{Y})\big)$ such that $\tilde{u} = \nabla_y \eta$ a.e. in $\mathbf{R}^N \times \mathcal{Y}$. Therefore

$$\begin{aligned} &\iint_{\mathbf{R}^N \times \mathcal{Y}} u(x,y)\cdot w(x,y)\theta(x)\, dxdy - \int_{\mathbf{R}^N} \hat{u}(x)\cdot \hat{w}(x)\theta(x)\, dx \\ &= \iint_{\mathbf{R}^N \times \mathcal{Y}} \tilde{u}(x,y)\cdot \tilde{w}(x,y)\theta(x)\, dxdy = \int_{\mathbf{R}^N} dx\ \theta(x) \int_{\mathcal{Y}} \nabla_y \eta(x,y)\cdot \tilde{w}(x,y)\, dy \\ &= \int_{\mathcal{Y}} \eta(x,y) \nabla_y \cdot \tilde{w}(x,y)\, dy = 0. \end{aligned}$$

□

We are now able to state a two-scale version of Theorem 5.1.

Theorem 5.4 *(Two-Scale Div-Curl Lemma) Let $\{u_\varepsilon\}$ and $\{w_\varepsilon\}$ be two bounded sequences of $L^2_{\rm rot}(\mathbf{R}^3)^3$ and $L^2_{\rm div}(\mathbf{R}^3)^3$, resp.. If*

$$u_\varepsilon \underset{2}{\rightharpoonup} u, \quad w_\varepsilon \underset{2}{\rightharpoonup} w \qquad \text{in } L^2(\mathbf{R}^3 \times \mathcal{Y})^3, \tag{5.5}$$

then

$$\begin{aligned}\int_{\mathbf{R}^3} u_\varepsilon(x)\cdot w_\varepsilon(x)\; \theta(x)\, dx &\to \int_{\mathbf{R}^3} \hat{u}(x)\cdot\hat{w}(x)\; \theta(x)\, dx\\ &= \iint_{\mathbf{R}^3\times\mathcal{Y}} u(x,y)\cdot w(x,y)\; \theta(x)\, dxdy \qquad \forall\theta\in\mathcal{D}(\mathbf{R}^3).\end{aligned} \tag{5.6}$$

Proof. By (5.5), $u_\varepsilon \rightharpoonup \hat{u}$ and $w_\varepsilon \rightharpoonup \hat{w}$ in $L^2(\mathbf{R}^3)^3$. Hence by Theorem 5.1

$$\int_{\mathbf{R}^3} u_\varepsilon(x)\cdot w_\varepsilon(x)\; \theta(x)\, dx \to \int_{\mathbf{R}^3} \hat{u}(x)\cdot\hat{w}(x)\; \theta(x)\, dx \qquad \forall\theta\in\mathcal{D}(\mathbf{R}^3).$$

By the boundedness hypotheses and by Theorems 4.5 and 4.6, (5.3) is fulfilled. This yields (5.4), whence (5.6) follows. □

Variants of the div-curl lemma involve other pairs of operators.

A Negative Result. The next statement somehow limits the possibilities offered by two-scale compensated compactness.

Proposition 5.5 *[3] Assume that the hypotheses of Theorem 5.4 are fulfilled and that*

$$u_\varepsilon \underset{2}{\rightharpoonup} u, \quad w_\varepsilon \underset{2}{\rightharpoonup} w \qquad \text{in } L^2(\mathbf{R}^3\times\mathcal{Y})^3. \tag{5.7}$$

This does not entail $u_\varepsilon\cdot w_\varepsilon \underset{2}{\rightharpoonup} u\cdot w$ *in* $\mathcal{D}'(\mathbf{R}^3\times\mathcal{Y})$*; that is, this does not guarantee that*

$$\int_{\mathbf{R}^3} u_\varepsilon(x)\cdot w_\varepsilon(x)\; \theta(x,x/\varepsilon)\, dx \to \iint_{\mathbf{R}^3\times\mathcal{Y}} u(x,y)\cdot w(x,y)\; \theta(x,y)\, dxdy \quad \forall\theta\in\mathcal{D}(\mathbf{R}^3\times\mathcal{Y}). \tag{5.8}$$

In [3] Briane and Casado-Díaz exhibited a counterexample based on the lack of control on oscillations of period $k\varepsilon$ for $k\in\mathbf{N}^3$ with $|k|>1$. Here we construct a different counterexample, which involves oscillations of period $\sqrt{\varepsilon}$. Loosely speaking this failure of compactness might be ascribed to the lack of control on the x-derivatives of the two-scale limit functions u and w.

Proof. Let us fix any $v\in\mathcal{D}(\mathbf{R}^3)$ ($v\not\equiv 0$), any $c\in\mathbf{R}^3\setminus\{0\}$, and set

$$f_\varepsilon(x) := \sin(|x|/\sqrt{\varepsilon})v(x), \qquad g_\varepsilon := c\sin(|x|/\sqrt{\varepsilon})v(x) \qquad \forall x\in\mathbf{R}^3, \forall\varepsilon>0.$$

Thus

$$\begin{gathered} f_\varepsilon \underset{2}{\rightharpoonup} 0 \quad \text{in } L^2(\mathbf{R}^3\times\mathcal{Y}), \qquad g_\varepsilon \underset{2}{\rightharpoonup} 0 \quad \text{in } L^2(\mathbf{R}^3\times\mathcal{Y})^3,\\ f_\varepsilon(x)g_\varepsilon(x) = c\,[\sin(|x|/\sqrt{\varepsilon})]^2 v(x)^2 \rightharpoonup \tfrac{1}{2}cv(x)^2 \not\equiv 0 \quad \text{in } L^1(\mathbf{R}^3)^3. \end{gathered} \tag{5.9}$$

Let us also fix any $\varphi,\psi\in H^1(\mathcal{Y})$ ($\varphi,\psi\not\equiv 0$) and set

$$u_\varepsilon(x) := \varepsilon\nabla[f_\varepsilon(x)\varphi(x/\varepsilon)], \qquad w_\varepsilon(x) := \varepsilon\nabla\times[g_\varepsilon(x)\psi(x/\varepsilon)] \qquad \text{for a.a. } x\in\mathbf{R}^3.$$

Notice that

$$\begin{aligned} u_\varepsilon(x) &= \varepsilon[\nabla f_\varepsilon(x)]\varphi(x/\varepsilon) + f_\varepsilon(x)(\nabla\varphi)(x/\varepsilon) \underset{2}{\rightharpoonup} 0 \quad \text{in } L^2(\mathcal{Y})^3,\\ w_\varepsilon(x) &= \varepsilon[\nabla\times g_\varepsilon(x)]\psi(x/\varepsilon) + g_\varepsilon(x)\times(\nabla\psi)(x/\varepsilon) \underset{2}{\rightharpoonup} 0 \quad \text{in } L^2(\mathcal{Y})^3. \end{aligned} \tag{5.10}$$

The hypotheses of Theorem 5.4 are thus fulfilled. But by (5.9) and (5.10)

$$u_\varepsilon(x)\cdot w_\varepsilon(x) \underset{2}{\rightharpoonup} \tfrac{1}{2}v(x)^2\nabla\varphi(y)\cdot c\times\nabla\psi(y) \qquad \text{in } L^1(\mathbf{R}^3\times\mathcal{Y})^3,$$

and in general the latter function does not vanish identically. Thus

$$u_\varepsilon(x)\cdot w_\varepsilon(x) \underset{2}{\not\rightharpoonup} 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^3\times\mathcal{Y})^3,$$

at variance with (5.8). □

Remark. Although we proved this negative result just in the setting of Theorems 5.4, an analogous counterexample could be constructed in the framework of other examples of compensated compactness.

6. Scale-Transformations

We are concerned with scale-transformations of stationary nonlinear problems. In particular, in presence of appropriate orthogonality conditions, cyclically maximal monotone relations are proved to be stable by homogenization. These results are here applied to a simple model of magnetostatics, and are compared with Γ-convergence.

An Example. A fundamental method for proving existence of a solution of a boundary-value problem for a nonlinear P.D.E. consists in
(i) approximating the problem,
(ii) deriving a priori estimates,
(iii) passing to the limit along a suitable sequence.

The main difficulties typically arise in passing to the limit in the nonlinear terms; see e.g. [6]. In the framework of homogenization, as a simple example let us consider a nonlinear relation of the form

$$w_\varepsilon(x) = \alpha(u_\varepsilon(x), x, x/\varepsilon) \qquad \text{for a.e. } x \in \Omega, \forall \varepsilon > 0, \tag{6.1}$$

for a prescribed Caratheodory function α which is periodic w.r.t. its last argument (Ω being a Euclidean domain). The dependence on the second and third argument of α respectively represent macroscopic and mesoscopic nonhomogeneity.

Let us assume that u_ε and w_ε are uniformly bounded in $L^2(\Omega)^3$. By Proposition 3.2 there exist $u = u(x, y)$ and $w = w(x, y)$ such that, as ε vanishes along a suitable subsequence,

$$u_\varepsilon \underset{2}{\rightharpoonup} u, \;\; w_\varepsilon \underset{2}{\rightharpoonup} w \qquad \text{in } L^2(\Omega \times \mathcal{Y})^3. \tag{6.2}$$

Note that

the dependence on $x \in \Omega$ accounts for a *macrooscopic structure,*
the dependence on $y \in \mathcal{Y}$ accounts for a *mesoscopic structure.*

This representation of mesoscopic structures is alternative to that based on Young's *parametrized measures,* see e.g. [11].

Whenever the function α is nonlinear, further information are needed in order to pass to the limit in (6.1). Along with the classical theory, we distinguish two cases.

Compactness. (i) *Full Compactness.* If ∇u_ε is uniformly bounded in $L^2(\Omega)^9$, then

$$u \;\text{ is independent of } y, \qquad u_\varepsilon \to u \qquad \text{in } L^2(\Omega)^3. \tag{6.3}$$

Thus here u does not exhibit any fine-scale structure. If $\alpha(\cdot, x, y)$ is either continuous or monotone, (1) by passing to the two-scale limit in (6.1) we then get the two-scale relation

$$w(x, y) = \alpha(u(x), x, y) \qquad \text{for a.e. } (x, y) \in \Omega \times \mathcal{Y}. \tag{6.4}$$

Defining the average $\hat{v}$ and the fluctuation $\tilde{v}$ as in (4.7), we thus obtain the coarse-scale relation

$$\hat{w}(x) = \int_{\mathcal{Y}} \alpha(u(x), x, y)\, dy =: \bar{\alpha}(u(x), x) \qquad \text{for a.e. } x \in \Omega. \tag{6.5}$$

(ii) *Compensated Compactness.* A less trivial setting is obtained if the a priori estimates include the uniform L^2-boundedness only of certain derivatives of u_ε and of other (suitably selected) derivatives of w_ε. This is the framework of compensated compactness. Let us consider the basic div-curl example:

$$\nabla \cdot u_\varepsilon = r \quad (\nabla \cdot := \text{div}), \qquad \nabla \times w_\varepsilon = z \quad (\nabla \times := \text{curl}), \tag{6.6}$$

(1) More precisely, rather than with a single-valued monotone function $\alpha(\cdot, x, y)$ we might deal with a maximal monotone multi-valued function.

for prescribed fields $r \in L^2(\Omega)$ and $z \in L^2(\Omega)^3$. This entails that both u and w may also depend on y and

$$\nabla_y \cdot u = 0, \quad \nabla_y \times w = 0 \qquad \text{in } \Omega \times \mathcal{Y}, \tag{6.7}$$

$$\nabla \cdot \hat{u} = \hat{r}, \quad \nabla \times \hat{w} = \hat{z} \qquad \text{in } \Omega, \tag{6.8}$$

in the sense of distributions. Note that (6.7) entails the following orthogonality property: (using the notation (4.7))

$$\int_{\mathcal{Y}} \tilde{u}(x,y) \cdot \tilde{w}(x,y)\, dy = 0 \qquad \text{for a.e. } x \in \Omega. \tag{6.9}$$

Homogeneous differential equations in the variable y like (6.7) may be the outcome of two-scale convergence, see Theorems 4.5 and 4.6. By (5.6), for any $\theta \in \mathcal{D}(\Omega)$

$$\int_\Omega u_\varepsilon(x) \cdot w_\varepsilon(x) \theta(x)\, dx \ \rightarrow \int_\Omega \hat{u}(x) \cdot \hat{w}(x) \theta(x)\, dx = \iint_{\Omega \times \mathcal{Y}} u(x,y) \cdot w(x,y) \theta(x)\, dxdy.$$

If the mapping $\alpha(\cdot, x, y)$ is *maximal monotone,* it is then easy to see that

$$w(x,y) = \alpha(u(x,y), x, y) \qquad \text{for a.e. } (x,y) \in \Omega \times \mathcal{Y}. \tag{6.10}$$

As a conclusion of this brief discussion about compactness, we remark that, loosely speaking:

the occurrence of a mesoscopic structure in a composite material is treatable
because of the *splitting of the space-regularity,* (6.10′)
that is at the basis of compensated compactness (as in the div-curl example).

Scale-Integration and Scale-Disintegration. We shall be concerned with the integration of (6.10) w.r.t. y for a cyclically maximal monotone mapping $\alpha(\cdot, x, y)$ – namely with $\alpha(\cdot, x, y) = \partial\varphi(\cdot, x, y)$ for a normal integrand φ such that $\varphi(\xi, \cdot, \cdot) \in \mathcal{F}$ (cf. (1.4)) for any $\xi \in \mathbf{R}^3$. (The assumption of cyclical monotonicity may however be dropped, by using the recently developed theory of *representative functionals.)* We claim that there exists an integrand $\varphi_0(\cdot, x)$ such that, whenever $u, w \in L^2(\Omega \times \mathcal{Y})^3$ fulfill (6.7),

$$\begin{aligned} &w(x,y) \in \partial\varphi(u(x,y), x, y) \quad \text{for a.e. } (x,y) \in \Omega \times \mathcal{Y} \\ &\Rightarrow \quad \hat{w}(x) \in \partial\varphi_0(\hat{u}(x), x) \quad \text{for a.e. } x \in \Omega. \end{aligned} \tag{6.11}$$

This means that, if (6.7) holds and u minimizes the functional

$$\begin{aligned} &\Phi_w(v) : \big\{v \in L^2(\Omega \times \mathcal{Y})^3,\ \nabla_y \cdot v = 0 \ \text{ in } \mathcal{D}'(\Omega \times \mathcal{Y})\big\} \to \mathbf{R} : \\ &v \mapsto \iint_{\Omega \times \mathcal{Y}} [\varphi(v(x,y), x, y) - w(x,y) \cdot v(x,y)]\, dxdy, \end{aligned} \tag{6.12}$$

then $\hat{u}$ minimizes the homogenized functional

$$\Phi_{0\hat{w}}(\bar{v}) = \int_\Omega [\varphi_0(\bar{v}(x), x) - \hat{w}(x) \cdot \bar{v}(x)]\, dx \quad \big(\bar{v} \in L^2(\Omega)^3\big). \tag{6.13}$$

We shall refer to the transformation $(u, w) \rightarrow (\hat{u}, \hat{w})$ as a *scale-integration.*

Under the assumption that the functional Φ_w is weakly lower semicontinuous, a converse result of *scale-disintegration* holds: for any pair of functions $\bar{u}, \bar{w} \in L^2(\Omega)^3$,

$$\begin{aligned} &\bar{w}(x) \in \partial\varphi_0(\bar{u}(x), x) \quad \text{for a.e. } x \in \Omega \ \Rightarrow \\ &\exists u, w \in L^2(\Omega \times \mathcal{Y})^3 \text{ such that } \hat{u} = \bar{u}, \hat{w} = \bar{w}, \\ &\nabla_y \cdot u = 0, \quad \nabla_y \times w = 0 \quad \text{in } \mathcal{D}'(\Omega \times \mathcal{Y}), \\ &w(x,y) \in \partial\varphi(u(x,y), x, y) \quad \text{for a.e. } (x,y) \in \Omega \times \mathcal{Y}. \end{aligned} \tag{6.14}$$

Equivalently stated, whenever $\bar{u}$ minimizes the one-scale functional $\Phi_{0\bar{w}}$, there exists a pair (u, w) such that: $\hat{u} = \bar{u}, \hat{w} = \bar{w}$, (6.7) is fulfilled, and u minimizes the two-scale functional Φ_w.

One may thus reconstruct the dependence on the fine-scale variable. We conclude that in the *div-curl case*

$$\text{scale-integration entails no loss of information.} \tag{6.15}$$

This may also be extended to more general settings, characterized by a splitting of the space regularity, cf. (6.10').

Although here we just discussed the div-curl set-up, analogous results hold in more general settings characterized by the splitting of the space regularity that is typical of compensated compactness, cf. (6.10'). These developments rest

(i) on the variational structure of the problem, that is implicit in the cyclical monotonicity of the operator, and

(ii) on the orthogonality property (6.9).

This approach may schematically be illustrated through the following implications:

$$\text{splitting of the space-regularity} \to \text{fine-scale structure}, \tag{6.16}$$

namely, the occurrence of a fine-scale structure in a composite material may be accounted for by the splitting of the space-regularity, see e.g. (6.6); (2)

$$\text{P.D.E.s with oscillating coefficients} \to \text{corresponding } \textit{homogeneous} \text{ P.D.E.s in } \mathcal{Y}, \qquad (6.16)$$

e.g. (6.7) is a consequence of the results of two-scale convergence of the curl and div operators of Sect. 4;

$$\text{homogeneity of the P.D.E.s in } \mathcal{Y} \to L^2(\mathcal{Y})\text{-orthogonality}, \tag{6.17}$$

$$\text{cyclical monotonicity and orthogonality} \to \text{scale-integration and scale-disintegration}. \tag{6.18}$$

On the other hand, whenever one of the two dependent variables has full (i.e., H^1) regularity, then that variable has no fine-scale structure (that is, it does not depend on y).

Scale-transformations may also be extended to non-cyclically monotone operators.

Homogenization by Two-Scale Convergence and Scale-Integration. The above results of scale-transformation allow us to develop a two-scale method for the homogenization of nonlinear P.D.E.s. This consists in the following steps:

(i) derivation of a two-scale problem via two-scale convergence from a sequence of ε-problems,

(ii) derivation of a coarse-scale model by scale-integration of the two-scale problem,

(iii) retrieval of the two-scale problem from the coarse-scale formulation via scale-disintegration.

The two-scale model provides full information on both the coarse- and fine-scale structure of the system. The one-scale problem however is the real goal, because it provides a more synthetical representation of the system by using a smaller number of independent variables: this one is the homogenized problem. The third step provides the equivalence between the two problems. We emphasize that only after this final step we are genuinely entitled to regard the coarse-scale problem as the homogenized version of the first one, for it is guaranteed that no spurious pair solves this one-scale formulation. This especially applies to the cases where the coarse-scale solution need not be unique.

A relation may be established between this procedure of homogenization by two-scale convergence with scale-integration and the classical one based on De Giorgi's notion of Γ-convergence. More specifically, it is shown for the magnetostatic example that the customary (one-scale) Γ-convergence is equivalent to the two-scale Γ-convergence (as defined in Sect. 2) combined with a scale-transformation.

(2) here *homogeneous* means *with vanishing second member.*

Homogenization of a Model of Magnetostatics. We illustrate the above developments by means of a simple model of magnetostatics: we couple a constitutive relation with the magnetostatic equations

$$H(x) \in \partial\varphi(B(x), x, x/\varepsilon), \qquad \text{for } x \in \mathbf{R}^3 \tag{6.19}$$

$$\nabla \cdot B = 0, \quad \nabla \times H = J_\varepsilon \quad \text{in } \mathbf{R}^3, \tag{6.20}$$

for a prescribed electric current field $J_\varepsilon(x) = J(x, x/\varepsilon)$. Passing to the two-scale limit as ε vanishes, we derive a two-scale relation coupled with the fine- and coarse-scale magnetostatic equations

$$H(x,y) \in \partial\varphi(B(x,y), x, y) \qquad \text{for } (x,y) \in \mathbf{R}^3 \times \mathcal{Y}, \tag{6.21}$$

$$\nabla_y \cdot B = 0, \quad \nabla_y \times H = 0 \qquad \text{in } \mathbf{R}^3 \times \mathcal{Y}, \tag{6.22}$$

$$\nabla \cdot \hat{B} = 0, \quad \nabla \times \hat{H} = \hat{J} \qquad \text{in } \mathbf{R}^3. \tag{6.23}$$

We then introduce a homogenized constitutive function φ_0, and formulate a coarse-scale problem by coupling the integrated constitutive relation

$$\hat{H}(x) \in \partial\varphi_0(\hat{B}(x), x) \qquad \text{for } (x,y) \in \mathbf{R}^3 \times \mathcal{Y} \tag{6.24}$$

with the coarse-scale magnetostatic equations (6.23). We show that conversely from any coarse-scale solution we may reconstruct a solution of the two-scale problem (6.21)—(6.23). As we already remarked, this equivalence shows that there is no loss of information by scale-integration.

7. Scale-Transformations of Cyclically Monotone Operators

In this section we state the result that is at the basis of the discussion of the previous section.

Let B be a separable real Banach space B with norm $|\cdot|$, and denote by $\mathcal{B}(B)$ the Borel family associated to the strong topology.

Let us assume that we are given a measurable integrand φ with prescribed growth at infinity:

$$\varphi : B \times \mathcal{Y} \to B \;\text{ is measurable w.r.t. } \mathcal{B}(B) \otimes \mathcal{L}(\mathcal{Y}), \tag{7.1}$$

$$\begin{aligned} &\exists p \in]1, +\infty[, \exists c_1, c_2 > 0, \exists f_1, f_2 \in L^1(\mathcal{Y}) : \\ &c_1|\xi|^p - f_1(y) \le \varphi(\xi, y) \le c_2|\xi|^p + f_2(y) \quad \forall \xi \in B, \text{ for a.e. } y. \end{aligned} \tag{7.2}$$

The function $v \mapsto \varphi(v(y), y)$ is thus an element of $L^p(\mathcal{Y}; B)$ for any v of this space.

We shall label spaces of integrable functions having vanishing mean by appending the index $*$, set $p' = p/(p-1)$, and assume that

$$V \subset L^p_*(\mathcal{Y}; B), \quad Z \subset L^{p'}_*(\mathcal{Y}; B'), \tag{7.3}$$

$$V, Z \text{ are closed, convex, and nonempty,} \tag{7.4}$$

$$\int_{\mathcal{Y}} \langle v(y), z(y) \rangle \, dy = 0 \qquad \forall v \in V, \forall z \in Z. \tag{7.5}$$

We shall denote by $\varphi^*(\cdot, y)$ the convex conjugate function of $\varphi(\cdot, y)$, cf. (A.15) (in the Appendix). By (7.2) and by the definition of conjugate function, it is easily checked that φ^* has growth at infinity of degree p':

$$\begin{aligned} &\exists \tilde{c}_1, \tilde{c}_2 > 0, \exists f_1, f_2 \in L^1(\mathcal{Y}) : \\ &\tilde{c}_1|\eta|^{p'} - f_2(y) \le \varphi^*(\eta, y) \le \tilde{c}_2|\eta|^{p'} + f_1(y) \quad \forall \eta \in B', \text{ for a.e. } y. \end{aligned} \tag{7.6}$$

Let us now set

$$\begin{aligned} &\Phi(\xi,v) := \int_{\mathcal{Y}} \varphi(\xi+v(y),y)\,dy \qquad \forall \xi\in B, \forall v\in L^p_*(\mathcal{Y};B),\\ &\varphi_0(\xi) := \inf_{v\in V} \Phi(\xi,v), \end{aligned} \tag{7.7}$$

$$\begin{aligned} &\Psi(\eta,v) := \int_{\mathcal{Y}} \varphi^*(\eta+v(y),y)\,dy \qquad \forall \eta\in B', \forall v\in L^{p'}_*(\mathcal{Y};B'),\\ &\psi_0(\eta) := \inf_{v\in Z} \Psi(\eta,v). \end{aligned} \tag{7.8}$$

The functional $\Psi(\cdot,v)$ is the convex conjugate of $\Phi(\cdot,v)$ for any v; whenever φ_0 is convex and lower semicontinuous, ψ_0 is in turn its convex conjugate. Let us now see some properties of these two integral functions.

Theorem 7.2 *Let $V,Z,\varphi,\varphi_0,\psi_0$ fulfill (7.1)—(7.7).*
(i) If $u\in L^p(\mathcal{Y};B)$ and $w\in L^{p'}(\mathcal{Y};B')$ are such that

$$\tilde u\in V,\quad \tilde w\in Z, \tag{7.9}$$
$$w(y)\in\partial\varphi(u(y),y) \qquad \textit{for a.e. } y, \tag{7.10}$$

then

$$\hat w\in\partial\varphi_0(\hat u),\qquad \hat u\in\partial\psi_0(\hat w). \tag{7.11}$$

(ii) Under the hypothesis (7.1), whenever

$$r\in B,\quad s\in B',\quad s\in\partial\varphi_0(r), \tag{7.12}$$

there exist $\tilde u\in L^p_*(\mathcal{Y};B)$ and $\tilde w\in L^{p'}_*(\mathcal{Y};B')$ that fulfill (7.9) and are such that $u:=r+\tilde u$ and $w:=s+\tilde w$ satisfy (7.10).

The inclusion (7.10) and its integrated counterpart (7.11) may thus be regarded as essentially equivalent.

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