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Équations aux dérivées partielles/*Partial Differential Equations*

A new approach to evolution

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Abstract. Let B be a Banach space, and $\varphi_0, \varphi_1 : B \to]-\infty, +\infty]$ be two given functions. A generalized solution of the equation $\nabla\varphi_1\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right) + \nabla\varphi_0(u) = 0$ in B' is introduced via the search for a minimal element of a certain partial order relation on the trajectories. This is based on the use of the *evolution potential* $[\Phi(u)](t) := \int_0^t \varphi_1\left(\frac{\mathrm{d}u}{\mathrm{d}\tau}(\tau)\right)\mathrm{d}\tau + \varphi_0(u(t))$. This approach is extended to higher order ODEs and PDEs. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Une nouvelle approche á l'évolution

Résumé. *Soit B un espace de Banach réel, et $\varphi_0, \varphi_1 : B \to]-\infty, +\infty]$ deux fonctions données. Une solution généralisée de l'équation $\nabla\varphi_1\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right) + \nabla\varphi_0(u) = 0$ dans B' est formulée au moyen de la recherche d'un élément minimal pour une certaine relation d'ordre partiel sur les trajectoires. Ceci est basé sur l'utilisation du potentiel évolutif $[\Phi(u)](t) := \int_0^t \varphi_1\left(\frac{\mathrm{d}u}{\mathrm{d}\tau}(\tau)\right)\mathrm{d}\tau + \varphi_0(u(t))$. Cette approche est généralisée à des équations non linéaires d'ordre supérieur, et est aussi appliquée à des équations différentielles aux dérivées partielles.* © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Soit B un espace de Banach. On suppose que :

$$\varphi_1 : B \longrightarrow]-\infty, +\infty] \text{ est propre, convexe et semi-continue inférieurement;} \tag{1}$$

$$\varphi_0 : B \longrightarrow]-\infty, +\infty] \text{ est propre et faiblement semi-continue inférieurement;} \tag{2}$$

$$\exists p \in]1, +\infty[,\ \exists C_1 > 0,\ \exists C_2 \in \mathbb{R} : \forall v \in B, \quad \varphi_1(v) \geqslant C_1\|v\|^p + C_2; \tag{3}$$

$$\exists q \in [1, p[,\ \exists \widehat{C}_1, \widehat{C}_2 \in \mathbb{R} \ : \ \forall v \in B, \quad \varphi_0(v) \geqslant \widehat{C}_1\|v\|^q + \widehat{C}_2. \tag{4}$$

Soit $u^0 \in B$ tel que $\varphi_0(u^0) < +\infty$. On définit le *potentiel évolutif* :

$$\big[\Phi(u)\big](t) := \int_0^t \varphi_1\big(\mathrm{D}u(\tau)\big)\,\mathrm{d}\tau + \varphi_0\big(u(t)\big), \quad \forall u \in \mathrm{W}^{1,p}_{\mathrm{loc}}\big(\mathbb{R}^+; B\big) \ \left(\mathrm{D} := \frac{\mathrm{d}}{\mathrm{d}\tau}\right),$$

Note présentée par Jacques-Louis LIONS.

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et on suppose que $X := \{v \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^+; B) : v(0) = u^0,\ [\Phi(v)](t) < +\infty$ pour tout $t \geqslant 0\}$ est non vide. Par exemple, $[\Phi(u^0)](t) = t\varphi_1(0) + \varphi_0(u^0) < +\infty$ pour tout $t \geqslant 0$, si $\varphi_1(0) < +\infty$. Pour tout $\delta > 0$ on pose :

$$\text{pour tout } u, v \in X,\ u \preceq_\delta v \text{ si et seulement si ou bien } u = v \text{ dans } \mathbb{R}^+,$$

$$\text{ou bien, en posant } \hat{t} := \inf\{t > 0 : u(t) \neq v(t)\},\ \Phi(u) < \Phi(v) \text{ dans }]\hat{t}, \hat{t} + \delta[,$$

et on pose aussi :

$$\forall u, v \in X,\ u \preceq v \text{ si et seulement si } \exists \delta > 0 \ :\ u \preceq_\delta v.$$

Ce sont des relations d'ordre. Tout élément minimal u pour $\preceq$ est appelé une *courbe Φ-minimale*, et est caractérisé par la propriété suivante :

$$\forall w \in X \setminus \{u\},\ \text{en posant } \hat{t} := \inf\{t > 0 : w(t) \neq u(t)\},$$

$$\forall \tilde{t} > \hat{t},\ \exists t \in]\hat{t}, \tilde{t}[:\ [\Phi(u)](t) \leqslant [\Phi(w)](t).$$

THÉORÈME 1. – *Supposons que les hypothèses* (1)–(4) *sont vérifiées, et que $\varphi_0 : B \to \mathbb{R}$ est lipschitzienne au voisinage de u^0. Si u est un point intérieur de X et est Φ-minimale, alors, en notant par $\partial_C\varphi_0$ la dérivée généralisée au sens de Clarke, on a* :

$$\partial\varphi_1(\mathrm{D}u) + \partial_C\varphi_0(u) \ni 0 \quad \textit{dans } B', \textit{ p.p. dans } \mathbb{R}^+.$$

On peut donner des exemples où il n'existe aucune courbe Φ-minimale. On dit alors que $u \in X$ est une *courbe Φ-minimale généralisée* si :

$$\exists \{u_n\} \subset X : u_n \text{ est minimal par rapport à } \preceq_{1/n},\ \forall n \in \mathbb{N},$$

$$u_n \to u \text{ faiblement dans } \mathrm{W}^{1,p}(0, T; B),\ \forall T > 0, \text{ lorsque } n \to \infty.$$

THÉORÈME 2. – *Supposons que les hypothèses* (1)–(4) *sont vérifiées, et que $\varphi_1(0) < +\infty$, $\varphi_0(u^0) < +\infty$. Alors il existe une courbe Φ-minimale généralisée.*

On peut aussi appliquer cette approche à des équations d'ordre supérieur. Par exemple, pour l'équation $\sum_{j=0}^m \nabla\varphi_j(\mathrm{D}^j u) = 0$ le potentiel évolutif est de la forme :

$$[\Phi(v)](t) := \sum_{j=1}^m \int_0^t \frac{(t-\tau)^{j-1}}{(j-1)!} \varphi_j(\mathrm{D}^j v(\tau))\, \mathrm{d}\tau + \varphi_0(v(t)).$$

On peut aussi traiter de cette façon certaines équations aux dérivées partielles, par exemple (4.1), (4.2), (4.3).

1. Introduction

Let B be a Banach space with norm $\|\cdot\|$ and topological dual B', $\varphi_0, \varphi_1 : B \to \mathbb{R}$ be two given functions, and $u^0 \in B$. We consider the problem of finding an a.e. differentiable function $u : \mathbb{R}^+ \to B$ such that,

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denoting by ∇ a suitable generalization either of the Fréchet differential or of the subdifferential:

$$\begin{cases} \nabla\varphi_1\left(\dfrac{\mathrm{d}u}{\mathrm{d}t}\right) + \nabla\varphi_0(u) = 0 & \text{in } B', \text{ a.e. in } \mathbb{R}^+, \\ u(0) = u^0. \end{cases} \tag{1.1}$$

We propose a new approach for this problem, and for several higher order nonlinear ODEs and PDEs (*see* [6] for details).

The equation $\frac{\mathrm{d}u}{\mathrm{d}t} + \nabla\varphi_0(u) = 0$ has been extensively studied in Hilbert spaces (*see*, e.g., [1,3–5]). In [2], De Giorgi introduced the rather general concept of *minimizing movements*; this idea inspired the present author in the search for a new formulation of evolution problems.

2. Evolution potential and minimal curves

Let B be a reflexive Banach space, and assume that:

$$\varphi_1 : B \longrightarrow]-\infty, +\infty] \text{ is proper, convex and lower semicontinuous;} \tag{2.1}$$

$$\varphi_0 : B \longrightarrow]-\infty, +\infty] \text{ is proper and weakly lower semicontinuous;} \tag{2.2}$$

$$\exists p \in]1, +\infty[,\ \exists C_1 > 0,\ \exists C_2 \in \mathbb{R} \,:\, \forall v \in B, \quad \varphi_1(v) \geqslant C_1\|v\|^p + C_2; \tag{2.3}$$

$$\exists q \in [1, p[,\ \exists \widehat{C}_1, \widehat{C}_2 \in \mathbb{R} \,:\, \forall v \in B, \quad \varphi_0(v) \geqslant \widehat{C}_1\|v\|^q + \widehat{C}_2. \tag{2.4}$$

Let us fix any $u^0 \in B$ such that $\varphi_0(u^0) < +\infty$, and define the *evolution potential*:

$$\big[\Phi(v)\big](t) := \int_0^t \varphi_1\big(\mathrm{D}v(\tau)\big)\,\mathrm{d}\tau + \varphi_0\big(v(t)\big), \quad \forall t \geqslant 0,\ \forall v \in \mathrm{W}^{1,p}_{\mathrm{loc}}\big(\mathbb{R}^+; B\big)\ \left(D := \frac{\mathrm{d}}{\mathrm{d}\tau}\right). \tag{2.5}$$

We assume that:

$$X := \big\{v \in \mathrm{W}^{1,p}_{\mathrm{loc}}\big(\mathbb{R}^+; B\big) : v(0) = u^0,\ \big[\Phi(v)\big](t) < +\infty,\ \forall t \geqslant 0\big\} \neq \varnothing; \tag{2.6}$$

for instance, $[\Phi(u^0)](t) = t\varphi_1(0) + \varphi_0(u^0) < +\infty$ for any $t \geqslant 0$, whenever $\varphi_1(0) < +\infty$. For any $\delta > 0$ we set:

$$\text{for all } u, v \in X,\ u \preceq_\delta v \text{ if and only if, either } u = v \text{ in } \mathbb{R}^+,$$
$$\text{or, setting } \hat{t} := \inf\big\{t > 0 : u(t) \neq v(t)\big\},\ \Phi(u) < \Phi(v) \text{ in }]\hat{t}, \hat{t} + \delta[.$$

This is an order relation (it would not be so, if '$<$' were replaced by '$\leqslant$'). A minimal element is easily constructed step by step in time. For any u, $v \in X$ and any δ', δ'' such that $0 < \delta' < \delta''$, if $u \preceq_{\delta''} v$, then $u \preceq_{\delta'} v$. We then set $u \preceq v$ whenever $u \preceq_\delta v$ for some $\delta > 0$; this also is an order relation. Any minimal element u of $\preceq$ will be called a Φ-*minimal curve*. This holds for u if and only if:

$$\forall w \in X \setminus \{u\},\ \text{setting } \hat{t} := \inf\big\{t > 0 : w(t) \neq u(t)\big\},$$
$$\forall \tilde{t} > \hat{t},\ \exists t \in]\hat{t}, \tilde{t}[\,:\ \big[\Phi(u)\big](t) \leqslant \big[\Phi(w)\big](t),$$

and entails that:

$$\begin{aligned} &\text{for a.a. } \hat{t} > 0,\ \forall v \in B \setminus \{0\},\ \text{setting } z_{\hat{t},v}(t) := (t - \hat{t})^+ v,\ \forall t \geqslant 0, \\ &\limsup_{t \to \hat{t}^+} \frac{[\Phi(u + z_{\hat{t},v})](t) - [\Phi(u)](t)}{\|z_{\hat{t},v}(t)\|} \geqslant 0. \end{aligned} \tag{2.7}$$

THEOREM 1. – *Assume that* (2.1)–(2.4) *hold, let* Φ *and* X *be defined as in* (2.5) *and* (2.6), *and let* u *be an interior point of* X. *Then*:

(i) *if* φ_0 *is Lipschitz-continuous in a neighbourhood of* $u(t)$ *for a.a.* $t > 0$ *and* (2.7) *is fulfilled, then* (*denoting by* $\partial_C \varphi_0$ *the Clarke generalized gradient of* φ_0):

$$\partial\varphi_1\big(\mathrm{D}u(t)\big) + \partial_C\varphi_0\big(u(t)\big) \ni 0 \quad \textit{in } B', \textit{ for a.a. } t > 0;$$

(ii) *conversely,* (2.7) *holds if* φ_0 *is continuously Gâteaux-differentiable at* $u(t)$ *for a.a.* $t > 0$ *and*:

$$\partial\varphi_1\big(\mathrm{D}u(t)\big) + \nabla\varphi_0\big(u(t)\big) \ni 0 \quad \textit{in } B', \textit{ for a.a. } t > 0.$$

Φ-minimal curves may fail to exist even for $B = \mathbb{R}^2$. We then introduce a weaker notion via a closure property: $u \in X$ is called a *generalized* Φ*-minimal curve* if:

$$\exists\{u_n\} \subset X \ : \ u_n \text{ is minimal with respect to } \preceq_{1/n} \text{ for all } n \in \mathbb{N}, \text{ and}$$
$$u_n \to u \text{ weakly in } \mathrm{W}^{1,p}(0,T;B) \text{ for all } T > 0, \text{ as } n \to \infty.$$

THEOREM 2. – *Assume that* (2.1)–(2.4) *are fulfilled, that* $\varphi_1(0) < +\infty$, *and that* Φ *and* X *are defined as in* (2.5) *and* (2.6). *Then there exists a generalized* Φ*-minimal curve in* X.

Remarks. – (i) Under the hypotheses of Theorem 2.2, assume that φ_0 is convex. For any $h > 0$, let $\{u_h^n\}_{n\in\mathbb{N}}$ fulfill the *implicit Euler scheme*:

$$\partial\varphi_1\left(\frac{u_h^n - u_h^{n-1}}{h}\right) + \partial\varphi_0\big(u_h^n\big) \ni 0 \quad \text{in } B', \text{ for } n = 1, 2, \ldots \ \big(u_h^0 := u^0\big).$$

The time interpolate, u_h, of the nodal values $\{u_h^n\}_{n\in\mathbb{N}}$ is then minimal with respect to $\preceq_\delta$, for any $\delta > h$.

(ii) If φ_1 is a *duality mapping*, equation $(1.1)_1$ represents steepest descent along the graph of the function φ_0. One can then define an order relation in terms of φ_0, and introduce corresponding (generalized) minimal curves (*see* [6]).

(iii) The above results can be extended to nonhomogeneous equations of the form $\partial\varphi_1(\mathrm{D}u) + \partial\varphi_0(u) \ni f(t)$.

3. Higher order equations

Let $m \in \mathbb{N}$, and assume that:

$$\varphi_m : B \longrightarrow]-\infty, +\infty] \text{ is proper, convex and lower semicontinuous;} \tag{3.1}$$
$$\varphi_0, \ldots, \varphi_{m-1} : B \longrightarrow]-\infty, +\infty] \text{ are proper and weakly lower semicontinuous;} \tag{3.2}$$
$$\exists p \in]1, +\infty[, \ \exists C_1 > 0, \ \exists C_2 \in \mathbb{R} \ : \ \forall v \in B, \quad \varphi_m(v) \geqslant C_1\|v\|^p + C_2; \tag{3.3}$$
$$\exists\, q \in [1, p[, \ \exists \widehat{C}_1, \widehat{C}_2 \in \mathbb{R} \ : \ \forall j \in \{0, \ldots, m-1\}, \ \forall v \in B, \quad \varphi_j(v) \geqslant \widehat{C}_1\|v\|^q + \widehat{C}_2. \tag{3.4}$$

Let us fix any $u^j \in B$ such that $\varphi_j(u^j) < +\infty$ for $j = 0, \ldots, m-1$, define the evolution potential:

$$\big[\Phi_m(v)\big](t) := \sum_{j=1}^{m} \int_0^t \frac{(t-\tau)^{j-1}}{(j-1)!} \varphi_j\big(\mathrm{D}^j v(\tau)\big)\,\mathrm{d}\tau + \varphi_0\big(v(t)\big), \quad \forall t \geqslant 0, \ \forall v \in \mathrm{W}^{m,p}_{\mathrm{loc}}\big(\mathbb{R}^+; B\big), \tag{3.5}$$

and assume that:

$$X_m := \left\{ v \in \mathrm{W}^{m,p}_{\mathrm{loc}}\left(\mathbb{R}^+; B\right) : \mathrm{D}^j v(0) = u^j \ (j = 0, \ldots, m-1),\ \left[\Phi_m(v)\right](t) < +\infty,\ \forall t \geqslant 0 \right\} \neq \varnothing.$$

Denoting by D^{-1} the time integral in $]0,t[$, (3.5) also reads $\Phi_m(v) := \sum_{j=0}^m \mathrm{D}^{-j}\varphi_j(\mathrm{D}^j v)$.

If $u \in X_m$ is a Φ_m-minimal curve, then:

$$\text{for a.a. } \hat{t} > 0, \forall v \in B \setminus \{0\},\ \text{setting } z(t) := \left[(t-\hat{t})^+\right]^m v \text{ for any } t \geqslant 0,$$
$$\limsup_{t \to \hat{t}^+} \frac{[\Phi_m(u+z)](t) - [\Phi_m(u)](t)}{\|z(t)\|} \geqslant 0. \tag{3.6}$$

THEOREM 3. – *Assume that* (3.1)–(3.4) *hold, let* Φ_m *and* X_m *be defined as above, and let* u *be an interior point of* X_m. *Then*:

(i) *if* φ_j *is Lipschitz-continuous in a neighbourhood of* $\mathrm{D}^j u(t)$, *for a.a.* $t > 0$ *for* $j = 0, \ldots, m-1$, *and if* (3.6) *holds, then* $\partial\varphi_m(\mathrm{D}^m u(t)) + \sum_{j=0}^{m-1} \partial_C \varphi_j(\mathrm{D}^j u(t)) \ni 0$ *in* B', *for a.a.* $t > 0$;

(ii) *conversely,* (3.6) *holds if* φ_j *is continuously Gâteaux-differentiable at* $\mathrm{D}^j u(t)$ *for a.a.* $t > 0$, *for* $j = 0, \ldots, m-1$, *and if* $\partial\varphi_m(\mathrm{D}^m u(t)) + \sum_{j=0}^{m-1} \nabla\varphi_j(\mathrm{D}^j u(t)) \ni 0$ *in* B', *for a.a.* $t > 0$.

One can define generalized Φ_m-minimal curves as above, and provide an existence result.

4. PDEs

Let Ω be a Euclidean domain, and set $\nabla := \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N}\right)$. For $j = 0, \ldots, m$, let $n_j \in \mathbb{N}$, $p_j \in]1, +\infty[$, $\psi_j : \mathbb{R} \to]-\infty, +\infty]$ be a convex and lower semicontinuous function, and $P_j(\nabla)$ be a linear differential operator such that:

$$\exists C_{j1} > 0,\ \exists C_{j2} \in \mathbb{R}\ :\ \forall v \in B_j := \mathrm{W}^{n_j,p_j}_0(\Omega),$$
$$\varphi_j(v) := \int_\Omega \psi_j\left(P_j(\nabla)v\right) \mathrm{d}x \geqslant C_{j1}\|v\|^{p_j}_{B_j} + C_{j2}.$$

We assume that $n_0 \geqslant \cdots \geqslant n_m$, $p_0 \geqslant \cdots \geqslant p_m$, and prescribe $u^j \in B_j$ for $j = 0, \ldots, m-1$. We then define the evolution potential $\Phi_m(v)$ as in (3.5), for any $v \in \mathrm{L}^{p_0}_{\mathrm{loc}}(\mathbb{R}^+; B_0)$ such that $\mathrm{D}^j v \in \mathrm{L}^{p_j}_{\mathrm{loc}}(\mathbb{R}^+; B_j)$ for $j = 1, \ldots, m$. Thus $\Phi_m(v) := \sum_{j=0}^m \int_\Omega \mathrm{D}^{-j}\psi_j(\mathrm{D}^j P_j(\nabla)v(x,\cdot))\,\mathrm{d}x$.

Any Φ_m-minimal curve is a generalized solution of the Cauchy problem associated with the following nonlinear PDE of order m in time:

$$\sum_{j=0}^m P_j(\nabla)^* \partial\psi_j\left(\mathrm{D}^j P_j(\nabla)u\right) \ni 0 \quad \text{in } \mathcal{D}'(\Omega), \text{ a.e. in } \mathbb{R}^+,$$

coupled with homogeneous boundary conditions. Here $P_j(\nabla)^*$ represents the transposed operator of $P_j(\nabla)$. The extension to vector functions $u : \Omega \to \mathbb{R}^N$ is straightforward.

Examples include a large class of nonlinear parabolic and hyperbolic inclusions, e.g.:

$$\partial\psi_1(\mathrm{D}u) - \nabla\cdot\partial\psi_0(\nabla u) \ni 0 \qquad \left(u : \Omega \times \mathbb{R}^+ \to \mathbb{R}\right); \tag{4.1}$$
$$\partial\psi_2\left(\mathrm{D}^2 u\right) + \partial\psi_1(\mathrm{D}u) - \nabla\cdot\partial\psi_0(\nabla u) \ni 0 \qquad \left(u : \Omega \times \mathbb{R}^+ \to \mathbb{R}\right); \tag{4.2}$$
$$\partial\psi_2\left(\mathrm{D}^2 u\right) + \partial\psi_1(\mathrm{D}u) + \mathrm{curl}\,\partial\psi_0(\mathrm{curl}\,u) \ni 0 \qquad \left(\Omega \subset \mathbb{R}^3,\ u : \Omega \times \mathbb{R}^+ \to \mathbb{R}^3\right). \tag{4.3}$$

5. Other equations

The above procedure can be extended to several other Cauchy problems. Here we provide some examples; for each of them we present the corresponding evolution potential, Φ, without specifying the function spaces. Our first generalization reads:

$$\begin{cases} \partial\varphi_1(\mathrm{D}u) + \partial\varphi_0(u) + \mathrm{D}\partial\psi_1(u) \ni 0 \quad \text{in } B', \text{ for a.a. } t > 0, \\ u(0) = u^0, \quad \partial\psi_1(u(0)) \ni \xi; \end{cases}$$
$$\Phi(v) := \mathrm{D}^{-1}\big[\mathrm{D}^{-2}\varphi_1(\mathrm{D}v) + \mathrm{D}^{-1}\varphi_0(v) + \psi_1(v) - \langle \xi, v\rangle\big].$$

More generally, we fix any $h, k, m \in \mathbb{N}$ and consider the following integro-differential inclusion:

$$\begin{cases} \displaystyle\sum_{j=0}^{m} \partial\varphi_j\big(\mathrm{D}^j u\big) + \sum_{i=-h}^{k} \mathrm{D}^i \partial\psi_i(u) \ni 0 \quad \text{in } B', \text{ for a.a. } t > 0, \\ \mathrm{D}^j u(0) = u^j \ (j = 0, \ldots, m-1), \quad \displaystyle\sum_{i=s}^{k} \big[\mathrm{D}^{i-s}\partial\psi_i(u)\big](0) \ni \xi^s \ (s = 1, \ldots, k); \end{cases}$$
$$\Phi(v) := \mathrm{D}^{-1}\left(\sum_{j=0}^{m} \mathrm{D}^{-k-j}\varphi_j\big(\mathrm{D}^j v\big) + \sum_{i=-h}^{k} \mathrm{D}^{i-k}\psi_i(v) - \sum_{s=1}^{k} \frac{t^{k-s}}{(k-s)!}\big\langle \xi^s, v\big\rangle\right).$$

The following integro-differential inclusion is even more general:

$$\begin{cases} \displaystyle\sum_{i=-h}^{k}\sum_{j=0}^{m} \mathrm{D}^i \partial\varphi_{ij}\big(\mathrm{D}^j u\big) \ni 0 \quad \text{in } B', \text{ for a.a. } t > 0, \\ \mathrm{D}^j u(0) = u^j \ (j = 0, \ldots, m-1), \quad \displaystyle\sum_{i=s}^{k}\sum_{j=0}^{m} \big[\mathrm{D}^{i-s}\partial\varphi_{ij}\big(\mathrm{D}^j u\big)\big](0) \ni \xi^s \ (s = 1, \ldots, k); \end{cases}$$
$$\Phi(v) := \mathrm{D}^{-1}\left(\sum_{i=-h}^{k}\sum_{j=0}^{m} \mathrm{D}^{i-k-j}\varphi_{ij}\big(\mathrm{D}^j v\big) - \sum_{s=1}^{k} \frac{t^{k-s}}{(k-s)!}\big\langle \xi^s, v\big\rangle\right).$$

All of these examples fit in the general theory.

References

[1] Brézis H., Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, North-Holland, Amsterdam, 1973.
[2] De Giorgi E., New problems on minimizing movements, in: Lions J.-L., Baiocchi C. (Eds.), Boundary Value Problems for P.D.E.s and Applications, Masson, Paris, 1993, pp. 81–98.
[3] De Giorgi E., Marino A., Tosques M., Problemi di evoluzione in spazi metrici e curve di massima pendenza, Atti Accad. Naz. Lincei, Cl. Sci. Mat. Fis. Natur. 68 (1980) 180–187.
[4] Degiovanni M., Marino A., Tosques M., Evolution equations with lack of convexity, Nonlin. Anal. T.M.A. 9 (1985) 1401–1443.
[5] Marino A., Saccon C., Tosques M., Curves of maximal slope and parabolic inequalities on nonconvex constraints, Ann. Scuola Norm. Sup. Pisa 16 (1989) 281–330.
[6] Visintin A., A minimization principle for evolution, (to appear).