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Two-scale convergence of some integral functionals

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Abstract Nguetseng's notion of *two-scale convergence* is reviewed, and some related properties of integral functionals are derived. The coupling of two-scale convergence with convexity and monotonicity is then investigated, and a two-scale version is provided for *compactness by strict convexity.* The *div-curl lemma* of Murat and Tartar is also extended to two-scale convergence, and applications are outlined.

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0 Introduction

In the pioneering paper [44] Nguetseng introduced the following notion, that Allaire then developed in [1], naming it *two-scale convergence;* see also the seminal work [4]. Let Ω be a domain of $\mathbf{R}^N$ ($N \geq 1$), and set $Y := [0,1[^N$. A bounded sequence $\{u_\varepsilon\}$ of $L^2(\Omega)$ is said (weakly) *two-scale convergent* to $u \in L^2(\Omega \times Y)$ if and only if

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)\ \psi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \iint_{\Omega\times Y} u(x,y)\ \psi(x,y)\,\mathrm{d}x\mathrm{d}y, \tag{1}$$

for any smooth function $\psi : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ that is Y-periodic w.r.t. the second argument. (The reader will notice that u_ε depends just on x, whereas u is a function of x and y.)

This notion was introduced by Nguetseng as a tool for *homogenization,* namely the search for effective models representing the macroscopic behaviour of mesoscopically inhomogeneous materials—a theory that has a well-established tradition, see e.g. the monographs [6,11,15,27,29,35,38,50] and their bibliographies. That approach

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provided a rigorous justification of the classical method of asymptotic expansions, cf. e.g. [6,11,50], thus offering an alternative to the classical Tartar *method of oscillating test functions,* cf. [43,51,53], for periodic homogenization. The seminal works [1,44] were followed by a large number of papers on two-scale convergence, mostly devoted to applications. Two-scale convergence was also extended to the nonperiodic setting e.g. in [20,21,45,46].

Beyond applications to homogenization, two-scale convergence and its natural extension to several scales [2] offer a new point of view for *multiscaling*—a classical topic of applied mathematics that plays a fundamental role in physics and engineering and nowadays is attracting a renewed interest among mathematicians. The theory of two-scale convergence may be regarded as a mathematical object in itself, and may also be equipped with a calculus of its own, see [59].

In order to give the reader a vague idea of the richness of this setting, we point out that, aside the familiar notions of weak and strong ordinary *(single-scale)* convergence, one defines a weak and a strong two-scale convergence in L^p-spaces. One has thus to face no less than four notions of L^p-type convergence. Several other concepts of functional analysis have a natural two-scale extension, see [59]. Nguetseng's theory thus looks worth of investigation also independently of applications, that however are numerous and relevant. It is also in this spirit that the present work has been written; more is left to be done: for instance the notion of Γ-convergence also has a two-scale extension, that will be studied apart.

Here, we deal with some properties of two-scale convergence, and extend results that are known to hold for single-scale convergence, along the lines of [57]–[59]. The properties that we study include convexity, monotonicity, compactness by strict convexity, and compensated compactness in the form of the classical *div-curl lemma.* Our results may be used for the homogenization of variational problems and of either linear or nonlinear partial differential equations, and indeed have recently been applied in [60–62].

Let us now briefly illustrate the structure of the paper. Section 1 is preparatory: there we review the notions of weak and strong two-scale convergence (that we denote by $\underset{2}{\rightharpoonup}$ and $\underset{2}{\to}$, resp.) in L^p-spaces, and state some simple properties of integral functionals. Our approach is based on a scale-trasformation formula, see Lemma 1.1 (cf. [57,58]), by which we are able to reduce two-scale to single-scale convergence. This technique was inspired by [4,14,22,36] and by the notion of *periodic unfolding* of [26].

In Sect. 2 we use the above properties to extend to two-scale convergence some results concerning passage to the limit in subdifferentials. Here and elsewhere in this paper we mimic typical procedures of analogous results of single-scale convergence. Most of our developments rest on convex analysis, see e.g. the monographs [30,33,34,49], and results are expressed using the notions of subdifferential, convex conjugate function, and so on.

In Sect. 3 we address analogous questions for maximal monotone functions, see e.g. the monographs [5,10,16,19,37]. We confine ourselves to single-valued functions, although after [25,31,39] one may expect that these results take over to multi-valued functions (that are the most natural framework for maximal monotonicity).

In Sect. 4 we extend the method of *compactness by strict convexity* of [56], see also [3,7,8,17,47,54,55]. More specifically we show that $u_\varepsilon \underset{2}{\rightharpoonup} u$ entails $u_\varepsilon \underset{2}{\to} u$ in $L^1(\Omega \times Y)^M$ whenever

(i) $u_\varepsilon(x) \in K(x, x/\varepsilon)$ (a closed convex subset of $\mathbf{R}^M$) for a.e. x and any ε, and
(ii) $u(x,y)$ is an extremal point of $K(x,y)$ for a.e. $(x,y) \in \Omega \times Y$

(the mapping $K(x,\cdot)$ is here assumed to be Y-periodic for a.e. x).

Loosely speaking, the rational behind this statement is that sequences converging to extremal points of their range cannot exhibit large oscillations, and that this applies to single- as well as to two-scale convergence. We also attain a similar conclusion whenever the hypotheses (i) and (ii) are replaced by

$$\int_\Omega \varphi(u_\varepsilon(x), x, x/\varepsilon)\,\mathrm{d}x \to \iint_{\Omega\times Y} \varphi(u(x,y), x, y)\,\mathrm{d}x\mathrm{d}y, \tag{2}$$

φ being a strictly convex integrand that is Y-periodic w.r.t. its third argument. In particular, the u_εs might be gradients, that is, $u_\varepsilon = \nabla v_\varepsilon$ for some sequence of functions $\{v_\varepsilon\}$. This improvement from weak to strong two-scale convergence may find applications in the homogenization of problems of the calculus of variations and of nonlinear partial differential equations.

In Sect. 5 we prove some simple two-scale extensions of the *div-curl lemma* [40,41, 51]—one of the basic results of Murat and Tartar's theory of *compensated compactness* [52,53]. First, we show that if

$$u_\varepsilon \underset{2}{\rightharpoonup} u, \quad w_\varepsilon \underset{2}{\rightharpoonup} w \quad \text{in } L^2(\mathbf{R}^N \times Y)^N, \tag{3}$$

$$\{\nabla \times u_\varepsilon\} \text{ is bounded in } L^2(\Omega)^{N^2}, \quad \{\nabla \cdot w_\varepsilon\} \text{ is bounded in } L^2(\Omega), \tag{4}$$

then

$$\int_{\mathbf{R}^N} u_\varepsilon(x)\cdot w_\varepsilon(x)\,\theta(x)\,\mathrm{d}x \to \iint_{\mathbf{R}^N\times Y} u(x,y)\cdot w(x,y)\,\theta(x)\,\mathrm{d}x\mathrm{d}y \quad \forall\theta \in \mathcal{D}(\mathbf{R}^N). \tag{5}$$

This result was already stated in [57], and may be used in the homogenization of nonlinear problems issued from electromagnetism and fluid dynamics. In view of applications to continuum mechanics, we then prove a similar theorem for the symmetrized gradient and the divergence of tensor-valued functions. These statements partly rest on results of two-scale convergence of the curl and divergence operators, that extend the fundamental Theorem 3 of [44] about the two-scale behaviour of the ordinary gradient, see [59]. In [63] a rather different two-scale extension of the div-curl lemma is addressed.

In [18] Briane and Casado-Díaz recently showed that

$$(3),(4) \quad \not\Rightarrow \quad u_\varepsilon\cdot w_\varepsilon \underset{2}{\to} u\cdot w \quad \text{in } \mathcal{D}'_\sharp(\mathbf{R}^N\times Y); \tag{6}$$

that is, in (5) the test function θ cannot be assumed to depend (periodically) on y. These authors pointed out that this failure may be due to the onset of oscillations slower than $1/\varepsilon$, and provided a corresponding counterexample. Here, we construct a slightly different counterexample, that is based on the lack of control on the x-derivatives of the two-scale limit functions u and w.

Finally, we derive a statement that combines some of the results of this paper, in view of the application to the homogenization of nonlinear electromagnetic processes and of phase transitions, cf. [60–62].

1 Two-scale convergence and integral functionals

This section is preparatory. Here, we reformulate the notion of two-scale convergence and review some of its basic properties, using a scale-decomposition (or *periodic unfolding)* along the lines of [26,57,58]. We then deal with the behaviour of some integral functionals w.r.t. two-scale convergence.

1.1 Two-scale decomposition

Throughout this paper, we denote by $\mathcal{Y}$ the set $Y = [0,1[^N$ equipped with the topology of the N-dimensional torus, identify $\mathcal{Y}$ with Y in an obvious way, and then identify any function on Y with its Y-periodic extension to $\mathbf{R}^N$. For any $\varepsilon > 0$ we set

$$\begin{aligned} &\hat{n}(x) := \max\{n \in \mathbf{Z} : n \le x\}, \quad \hat{r}(x) := x - \hat{n}(x) \; (\in [0,1[) \qquad \forall x \in \mathbf{R}, \\ &\mathcal{N}(x) := (\hat{n}(x_1), ..., \hat{n}(x_N)) \in \mathbf{Z}^N, \quad \mathcal{R}(x) := x - \mathcal{N}(x) \in \mathcal{Y} \qquad \forall x \in \mathbf{R}^N. \end{aligned} \tag{1.1}$$

Thus, $x = \varepsilon[\mathcal{N}(x/\varepsilon) + \mathcal{R}(x/\varepsilon)]$ for any $x \in \mathbf{R}^N$. With reference to the scale-factor ε, $\varepsilon\mathcal{N}(x/\varepsilon)$ and $\mathcal{R}(x/\varepsilon)$ may respectively, be regarded as coarse-scale and fine-scale variables. We also set

$$S_\varepsilon(x,y) := \varepsilon\mathcal{N}(x/\varepsilon) + \varepsilon y \qquad \forall (x,y) \in \mathbf{R}^N \times \mathcal{Y}, \forall \varepsilon > 0. \tag{1.2}$$

Note that, as $S_\varepsilon(x,y) = x + \varepsilon[y - \mathcal{R}(x/\varepsilon)]$,

$$S_\varepsilon(x,y) \to x \quad \text{uniformly in } \mathbf{R}^N \times \mathcal{Y}, \text{ as } \varepsilon \to 0. \tag{1.3}$$

We assume that Ω is a (possibly unbounded) domain of $\mathbf{R}^N$, and extend any function defined on Ω to $\mathbf{R}^N \setminus \Omega$ with vanishing value. We also denote by $\mathcal{L}(\Omega)$ ($\mathcal{B}(\Omega)$, resp.) the σ-algebra of Lebesgue- (Borel-, resp.) measurable subsets of Ω, define $\mathcal{L}(\mathcal{Y})$ and $\mathcal{B}(\mathcal{Y})$ similarly, and denote by $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots$ the σ-algebra generated by any finite family $\mathcal{A}_1, \mathcal{A}_2, \ldots$ of σ-algebras.

The next result will allow us to express in terms of single-scale convergence a number of properties concerning two-scale convergence.

Lemma 1.1 [58] *Let $f = f_1 + f_2$ where*

$$\begin{aligned} &f_1 : \Omega \times \mathcal{Y} \to \mathbf{R} \text{ is measurable either w.r.t. } \mathcal{B}(\Omega) \otimes \mathcal{L}(\mathcal{Y}) \text{ or w.r.t. } \mathcal{L}(\Omega) \otimes \mathcal{B}(\mathcal{Y}), \\ &\text{and moreover} \\ &\text{either } \|f_1\|_{L^\infty(\Omega)} \in L^1(\mathcal{Y}) \text{ and } f_1 \text{ has compact support, or } \|f_1\|_{L^\infty(\mathcal{Y})} \in L^1(\Omega), \end{aligned} \tag{1.4}$$

$$\begin{aligned} &f_2(x,y) = \sum_{i=1}^{m} u_i(x) v_i(y) \quad \text{for a.e. } x \in \Omega \text{ and a.e. } y \in \mathcal{Y}, \\ &\text{with } u_i \in L^{p_i}(\Omega), v_i \in L^{p_i'}(\mathcal{Y}),\ p_i \in [1, +\infty],\ \text{for } i = 1, \ldots, m, \end{aligned} \tag{1.5}$$

for some $m \in \mathbf{N}$ (by p_i' we denote the conjugate index of p_i).

Let us extend $f(\cdot, y)$ to $\mathbf{R}^N \setminus \Omega$ with vanishing value for any y. For any $\varepsilon > 0$ the functions

$$\mathbf{R}^N \to \mathbf{R} : x \mapsto f(x, x/\varepsilon), \quad \mathbf{R}^N \times \mathcal{Y} \to \mathbf{R} : (x,y) \mapsto f(S_\varepsilon(x,y), y)$$

are then integrable, and

$$\int_{\mathbf{R}^N} f(x, x/\varepsilon)\, \mathrm{d}x = \iint_{\mathbf{R}^N \times Y} f(S_\varepsilon(x,y), y)\, \mathrm{d}x\mathrm{d}y \qquad \forall \varepsilon > 0. \tag{1.6}$$

For any $p \in [1,+\infty]$ *and any* $\varepsilon > 0$, *the operator* $A_\varepsilon : g \mapsto g \circ S_\varepsilon$ *is then a* (*nonsurjective*) *linear isometry* $L^p(\mathbf{R}^N) \to L^p(\mathbf{R}^N \times \mathcal{Y})$.

Proof If f_1 is measurable w.r.t. the σ-algebra generated by $\mathcal{B}(\mathbf{R}^N) \times \mathcal{L}(\mathcal{Y})$, then there exists a Borel function $\tilde{f}_1 : \mathbf{R}^N \times \mathcal{Y} \to \mathbf{R}$ such that $\tilde{f}_1(\cdot, y) = f_1(\cdot, y)$ in $\mathbf{R}^N$ for a.e. $y \in \mathcal{Y}$. Thus, $f_1(x, x/\varepsilon) = \tilde{f}_1(x, x/\varepsilon)$ for a.e. $x \in \mathbf{R}^N$, and this function is measurable. An analogous argument with exchanged roles of x and y applies if f_1 is measurable w.r.t. the σ-algebra generated by $\mathcal{L}(\mathbf{R}^N) \times \mathcal{B}(\mathcal{Y})$. On the other hand, the measurability of $x \mapsto f_2(x, x/\varepsilon)$ is obvious.

The function $(x,y) \mapsto f(S_\varepsilon(x,y), y)$ is also measurable, for the mapping $(x,y) \mapsto (S_\varepsilon(x,y), y)$ is piecewise constant w.r.t. x and affine w.r.t. y. As $\mathbf{R}^N = \bigcup_{m \in \mathbf{Z}^N} (\varepsilon m + \varepsilon \mathcal{Y})$ and $\mathcal{N}(x/\varepsilon) = m$ for any $x \in \varepsilon m + \varepsilon \mathcal{Y}$, we have

$$\begin{aligned}\int_{\mathbf{R}^N} f(x, x/\varepsilon)\,\mathrm{d}x &= \sum_{m \in \mathbf{Z}^N} \int_{\varepsilon m + \varepsilon Y} f(x, x/\varepsilon)\,\mathrm{d}x = \varepsilon^N \sum_{m \in \mathbf{Z}^N} \int_Y f(\varepsilon[m+y], y)\,\mathrm{d}y \\ &= \sum_{m \in \mathbf{Z}^N} \int_{\varepsilon m + \varepsilon Y} \mathrm{d}x \int_Y f(\varepsilon[\mathcal{N}(x/\varepsilon) + y], y)\,\mathrm{d}y = \int_{\mathbf{R}^N} \mathrm{d}x \int_Y f(S_\varepsilon(x,y), y)\,\mathrm{d}y.\end{aligned}$$

By writing (1.6) for $f(x,y) = |g(x)|$ for a.e. (x,y), we get the final statement for any $p \in [1,+\infty[$, and then by passage to the limit also for $p = \infty$. The operator A_ε is not onto, for $g \circ S_\varepsilon$ is piecewise constant w.r.t. x for any $g \in L^p(\mathbf{R}^N)$. □

1.2 Two-scale convergence in $L^p(\Omega \times \mathcal{Y})$

By ε we shall denote the generic element of an arbitrary but fixed, vanishing sequence of positive real numbers; e.g., $\varepsilon = \{1, 1/2, ..., 1/n, ...\}$. Let $p \in [1,+\infty]$; for any sequence of measurable functions, $u_\varepsilon : \Omega \to \mathbf{R}$, and any measurable function, $u : \Omega \times \mathcal{Y} \to \mathbf{R}$, we say that u_ε strongly *two-scale converges* to u in $L^p(\Omega \times \mathcal{Y})$, whenever $u_\varepsilon \circ S_\varepsilon \to u$ strongly in the latter space. We similarly define weak (and for $p = \infty$ weak star) two-scale convergence. Thus, for any $p \in [1,+\infty]$

$$\begin{aligned} u_\varepsilon \underset{2}{\to} u \ \text{ in } L^p(\Omega \times \mathcal{Y}) \quad &\Leftrightarrow \quad u_\varepsilon \circ S_\varepsilon \to u \ \text{ in } L^p(\Omega \times \mathcal{Y}), \\ u_\varepsilon \underset{2}{\rightharpoonup} u \ \text{ in } L^p(\Omega \times \mathcal{Y}) \quad &\Leftrightarrow \quad u_\varepsilon \circ S_\varepsilon \rightharpoonup u \ \text{ in } L^p(\Omega \times \mathcal{Y}), \\ u_\varepsilon \underset{2}{\overset{*}{\rightharpoonup}} u \ \text{ in } L^\infty(\Omega \times \mathcal{Y}) \quad &\Leftrightarrow \quad u_\varepsilon \circ S_\varepsilon \overset{*}{\rightharpoonup} u \ \text{ in } L^\infty(\Omega \times \mathcal{Y}). \end{aligned} \tag{1.7}$$

In [58] it is shown that this generalizes the original definitions of [1,44]. The extension of these definitions to vector-valued functions is obvious. Let us now set

$$\begin{aligned}\mathcal{F} := \big\{ \varphi : \mathbf{R}^M \times \Omega \times \mathcal{Y} \to \mathbf{R} \cup \{+\infty\} \ \text{ measurable} \\ \text{either w.r.t. } \mathcal{B}(\mathbf{R}^M) \otimes \mathcal{B}(\Omega) \otimes \mathcal{L}(\mathcal{Y}) \ \text{ or w.r.t. } \mathcal{B}(\mathbf{R}^M) \otimes \mathcal{L}(\Omega) \otimes \mathcal{B}(\mathcal{Y}) \big\}.\end{aligned} \tag{1.8}$$

The elements of $\mathcal{F}$ are Caratheodory functions, cf. e.g. [15, p. 30]. (This class might be enlarged by including terms of the form (1.5), but the possibility of attaining the value $+\infty$ makes the formulation of an all-inclusive class rather cumbersome.) In the next statement, we gather some known properties of integral functionals (cf. e.g. [29, Chap. 1]) that will be used in the sequel.

Lemma 1.2 *Let $\varphi \in \mathcal{F}$ and $p \in [1,+\infty[$. Then:*

(i)

$$\text{if } v : \Omega \times \mathcal{Y} \to \mathbf{R}^M \text{ is measurable, then } (x,y) \mapsto \varphi(v(x,y),x,y) \text{ is measurable,}$$
$$\text{if } v : \Omega \to \mathbf{R}^M \text{ is measurable, then } x \mapsto \varphi(v(x),x,x/\varepsilon) \text{ is measurable.} \tag{1.9}$$

(ii) *If*

$$\exists c_1 \in \mathbf{R}, \exists f \text{ as in (1.4) or (1.5) such that}$$
$$\varphi(v,x,y) + c_1|v|^p \geq f(x,y) \quad \forall v, \text{ for a.e. } (x,y), \tag{1.10}$$

then the functionals

$$\Phi_\varepsilon : L^p(\Omega)^M \to \mathbf{R} \cup \{+\infty\} : v \mapsto \int_\Omega \varphi(v(x),x,x/\varepsilon)\, \mathrm{d}x \quad \forall \varepsilon > 0,$$
$$\bar{\Phi} : L^p(\Omega \times \mathcal{Y})^M \to \mathbf{R} \cup \{+\infty\} : v \mapsto \iint_{\Omega \times \mathcal{Y}} \varphi(v(x,y),x,y)\, \mathrm{d}x\mathrm{d}y \tag{1.11}$$

are well-defined.

(iii) *If (1.10) is fulfilled and*

$$v \mapsto \varphi(v,x,y) \text{ is lower semicontinuous for a.e. } (x,y), \tag{1.12}$$

then Φ_ε and $\bar{\Phi}$ are lower semicontinuous.

(iv) *If*

$$\exists c_2 > 0, \exists g \text{ as in (1.4) or (1.5) such that}$$
$$|\varphi(v,x,y)| \leq c_2|v|^p + g(x,y) \quad \forall v, \text{ for a.e. } (x,y), \tag{1.13}$$
$$\text{the function } v \mapsto \varphi(v,x,y) \text{ is continuous for a.e. } (x,y), \tag{1.14}$$

then Φ_ε and $\bar{\Phi}$ are finite-valued and strongly continuous.

(v) *If (1.10) is fulfilled and*

$$\text{the function } v \mapsto \varphi(v,x,y) \text{ is convex and lower semicontinuous for a.e. } (x,y), \tag{1.15}$$

then Φ_ε and $\bar{\Phi}$ are convex and weakly lower semicontinuous.

Proof Both statements of (1.9) may be checked via the procedure that we used for the first part of Lemma A1.1. Part (ii) directly follows from (1.9). By Fatou's lemma the functional

$$L^p(\Omega)^M \to \mathbf{R}^+ \cup \{+\infty\} : v \mapsto \int_\Omega [\varphi(v(x),x,x/\varepsilon) + c_1|v(x)|^p - f(x,x/\varepsilon)]\, \mathrm{d}x$$

is lower semicontinuous; the same then holds for Φ_ε. A similar argument applies to $\bar{\Phi}$. Part (iii) is thus established. Part (iv) trivially follows by applying part (iii) to φ and $-\varphi$. Part (v) is a consequence of the closedness and convexity of the epigraph of φ. □

Next we draw some simple consequences concerning two-scale convergence.

Proposition 1.3 *Let $p \in [1, +\infty[$, assume that $\varphi \in \mathcal{F}$ fulfils (1.10), and define $\Phi_\varepsilon, \bar{\Phi}$ as in (1.11). Then:*

(i) *If*

$$\text{the function } (v,x) \mapsto \varphi(v,x,y) \text{ is lower semicontinuous for a.e. } y, \tag{1.16}$$

then

$$u_\varepsilon \underset{2}{\to} u \quad \text{in } L^p(\Omega \times \mathcal{Y})^M \quad \Rightarrow \quad \liminf_{\varepsilon\to 0} \Phi_\varepsilon(u_\varepsilon) \geq \bar{\Phi}(u). \tag{1.17}$$

(ii) *If (1.13) is fulfilled and the function $(v,x) \mapsto \varphi(v,x,y)$ is continuous for a.e. y, then*

$$u_\varepsilon \underset{2}{\to} u \quad \text{in } L^p(\Omega \times \mathcal{Y})^M \quad \Rightarrow \quad \Phi_\varepsilon(u_\varepsilon) \to \bar{\Phi}(u). \tag{1.18}$$

(iii) *If (1.15) and (1.16) are fulfilled, then*

$$u_\varepsilon \underset{2}{\rightharpoonup} u \quad \text{in } L^p(\Omega \times \mathcal{Y})^M \quad \Rightarrow \quad \liminf_{\varepsilon\to 0} \Phi_\varepsilon(u_\varepsilon) \geq \bar{\Phi}(u). \tag{1.19}$$

Proof In view of proving (1.17), first notice that by Lemma 1.1

$$\Phi_\varepsilon(u_\varepsilon) = \int_\Omega \varphi(u_\varepsilon(x), x, x/\varepsilon)\,\mathrm{d}x = \iint_{\Omega\times\mathcal{Y}} \varphi(u_\varepsilon(S_\varepsilon(x,y)), S_\varepsilon(x,y), y)\,\mathrm{d}x\mathrm{d}y.$$

If $u_\varepsilon \underset{2}{\to} u$ in $L^p(\Omega\times\mathcal{Y})^M$ then by definition $u_\varepsilon \circ S_\varepsilon \to u$ in $L^p(\Omega\times\mathcal{Y})^M$, hence also a.e. in $\Omega\times\mathcal{Y}$ for a suitable subsequence that we label by ε'. Therefore, by (1.3) and (1.16)

$$\liminf_{\varepsilon'\to 0} \varphi(u_{\varepsilon'}(S_{\varepsilon'}(x,y)), S_{\varepsilon'}(x,y), y) \geq \varphi(u(x,y), x, y) \qquad \text{for a.e. } (x,y) \in \Omega\times\mathcal{Y}.$$

By (1.10) and Fatou's lemma we then have

$$\begin{aligned}
&\liminf_{\varepsilon'\to 0} \iint_{\Omega\times\mathcal{Y}} \big[\varphi(u_{\varepsilon'}(S_{\varepsilon'}(x,y)), S_{\varepsilon'}(x,y), y) + c_1|u_{\varepsilon'}(S_{\varepsilon'}(x,y))|^p - f(S_{\varepsilon'}(x,y), y)\big]\,\mathrm{d}x\mathrm{d}y \\
&\quad\geq \iint_{\Omega\times\mathcal{Y}} \liminf_{\varepsilon'\to 0} \big[\varphi(u_{\varepsilon'}(S_{\varepsilon'}(x,y)), x, y) + c_1|u_{\varepsilon'}(S_{\varepsilon'}(x,y))|^p - f(S_{\varepsilon'}(x,y), y)\big]\,\mathrm{d}x\mathrm{d}y \\
&\quad\geq \iint_{\Omega\times\mathcal{Y}} \big[\varphi(u(x,y), x, y) + c_1|u(x,y)|^p - f(x,y)\big]\,\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

As

$$\iint_{\Omega\times\mathcal{Y}} \big[c_1|u_{\varepsilon'}(S_{\varepsilon'}(x,y))|^p - f(S_{\varepsilon'}(x,y), y)\big]\,\mathrm{d}x\mathrm{d}y \to \iint_{\Omega\times\mathcal{Y}} \big[c_1|u(x,y)|^p - f(x,y)\big]\,\mathrm{d}x\mathrm{d}y,$$

we then infer that $\liminf_{\varepsilon'\to 0} \Phi_{\varepsilon'}(u_{\varepsilon'}) \geq \bar{\Phi}(u)$. As this argument also applies to any extracted subsequence of $\{u_\varepsilon\}$, (1.17) is established.

By applying part (i) to φ and $-\varphi$ we get part (ii). Part (iii) follows from the weak lower semicontinuity of convex functionals. □

Let us now denote by ∂ the *subdifferential* operator, by φ^* the *conjugate convex function* of φ w.r.t. the first variable, and define Φ_ε^* and $\bar{\Phi}^*$ similarly. For these notions see e.g. [30,33,34,49]. Henceforth, we set $p' = p/(p-1)$ if $p \in]1, +\infty[$, $1' = \infty$ and $\infty' = 1$.

Proposition 1.4 *Let $p \in]1,+\infty[$, let $\varphi \in \mathcal{F}$ fulfil (1.10), and assume that*

$$\exists f \text{ as in (1.4) or (1.5) such that } \varphi(0,x,y) \le f(x,y) \text{ for a.e. } (x,y). \tag{1.20}$$

Then φ^* fulfils a condition like (1.10) with $c_1 = 0$, and, defining Φ_ε and $\bar{\Phi}$ as in (1.11),

$$\Phi_\varepsilon^* : L^{p'}(\Omega)^M \to \mathbf{R} \cup \{+\infty\} : v \mapsto \int_\Omega \varphi^*(v(x),x,x/\varepsilon)\,\mathrm{d}x \quad \forall \varepsilon > 0, \tag{1.21}$$

$$\bar{\Phi}^* : L^{p'}(\Omega \times \mathcal{Y})^M \to \mathbf{R} \cup \{+\infty\} : v \mapsto \iint_{\Omega\times\mathcal{Y}} \varphi^*(v(x,y),x,y)\,\mathrm{d}x\mathrm{d}y, \tag{1.22}$$

$$w_\varepsilon \underset{2}{\rightharpoonup} w \quad \text{in } L^{p'}(\Omega\times\mathcal{Y})^M \quad \Rightarrow \quad \liminf_{\varepsilon\to 0} \Phi_\varepsilon^*(w_\varepsilon) \ge \bar{\Phi}^*(w). \tag{1.23}$$

Proof The definition of convex conjugate function yields $\varphi^* \in \mathcal{F}$. As

$$\varphi^*(w,x,y) = \sup_{v\in\mathbf{R}^M} \{v\cdot w - \varphi(v,x,y)\} \ge -\varphi(0,x,y) \ge -f(x,y)$$
$$\forall w \in \mathbf{R}^M, \text{ for a.e. } (x,y) \in \Omega\times\mathcal{Y},$$

φ^* fulfils a condition like (1.10) with $c_1 = 0$. The representation formulas (1.21) and (1.22) of the convex conjugate functionals Φ_ε^* and $\bar{\Phi}^*$ follow from [48]; see also e.g. [30, Sect. IX.2], [34, Sect. 8.3]. Part (iii) of Proposition 1.3 yields (1.23). □

2 Two-scale convergence and subdifferentials

In this section, we deal with the two-scale convergence of subdifferentials. In particular Theorem 2.1 extends to two-scale convergence a result of single-scale convergence, and is potentially useful for the homogenization of (nonnecessarily stationary) variational problems.

Theorem 2.1 *Let $p \in]1,+\infty[$, $\{\varphi_\varepsilon\}$ be a sequence of $\mathcal{F}$, $\varphi \in \mathcal{F}$, and assume that these functions fulfil (1.10) and (1.20) with f independent of ε. Let us define $\bar{\Phi}$ as in (1.11), and set*

$$\Phi^\varepsilon(v) := \int_\Omega \varphi_\varepsilon(v(x),x,x/\varepsilon)\,\mathrm{d}x \qquad \forall v \in L^p(\Omega)^M, \forall \varepsilon. \tag{2.1}$$

Let $\{u_\varepsilon\}$ and $\{w_\varepsilon\}$ be sequences of $L^p(\Omega)^M$ and $L^{p'}(\Omega)^M$, resp., and

$$w_\varepsilon(x) \in \partial\varphi_\varepsilon(u_\varepsilon(x),x,x/\varepsilon) \qquad \text{for a.e. } x \in \Omega, \forall \varepsilon, \tag{2.2}$$

$$u_\varepsilon \underset{2}{\rightharpoonup} u \quad \text{in } L^p(\Omega\times\mathcal{Y})^M, \quad w_\varepsilon \underset{2}{\rightharpoonup} w \quad \text{in } L^{p'}(\Omega\times\mathcal{Y})^M, \tag{2.3}$$

$$\liminf_{\varepsilon\to 0} \int_\Omega u_\varepsilon\cdot w_\varepsilon\,\mathrm{d}x \le \iint_{\Omega\times\mathcal{Y}} u\cdot w\,\mathrm{d}x\mathrm{d}y, \tag{2.4}$$

$$\liminf_{\varepsilon\to 0} \Phi^\varepsilon(u_\varepsilon) \ge \bar{\Phi}(u), \qquad \liminf_{\varepsilon\to 0} \Phi^{\varepsilon*}(w_\varepsilon) \ge \bar{\Phi}^*(w). \tag{2.5}$$

Then

$$w(x,y) \in \partial\varphi(u(x,y),x,y) \qquad \textit{for a.e. } (x,y) \in \Omega\times\mathcal{Y}, \tag{2.6}$$

$$\int_\Omega u_\varepsilon(x)\cdot w_\varepsilon(x)\,\mathrm{d}x \to \iint_{\Omega\times\mathcal{Y}} u(x,y)\cdot w(x,y)\,\mathrm{d}x\mathrm{d}y, \tag{2.7}$$

$$\Phi^\varepsilon(u_\varepsilon) \to \bar{\Phi}(u), \qquad \Phi^{\varepsilon *}(w_\varepsilon) \to \bar{\Phi}^*(w). \tag{2.8}$$

Proof We shall use the following classical Fenchel's properties, that stem from the definition of convex conjugate function, cf. e.g. [30,33,34,49]. If B is a Banach space, $\langle\cdot,\cdot\rangle$ is the associated duality pairing, and $f: B \to]-\infty,+\infty]$, then

$$\langle u^*,u\rangle \le f(u)+f^*(u^*) \quad \forall u \in \mathrm{Dom}(f), \forall u^* \in \mathrm{Dom}(f^*), \tag{2.9}$$

$$\langle u^*,u\rangle \ge f(u)+f^*(u^*) \quad \Leftrightarrow \quad u^* \in \partial f(u) \quad \forall u \in \mathrm{Dom}(f), \forall u^* \in \mathrm{Dom}(f^*). \tag{2.10}$$

Notice that by (2.9) the latter statement also reads

$$\langle u^*,u\rangle = f(u)+f^*(u^*) \quad \Leftrightarrow \quad u^* \in \partial f(u) \quad \forall u \in \mathrm{Dom}(f), \forall u^* \in \mathrm{Dom}(f^*). \tag{2.11}$$

By Proposition 1.4 the convex conjugates $\{\varphi_\varepsilon^*\}$ and φ^* also fulfil (1.10), with p' instead of p. By (2.2), $w_\varepsilon \in \partial\Phi^\varepsilon(u_\varepsilon)$ in $L^{p'}(\Omega)$. By (2.4), (2.5), (2.9) and (2.10) we then have

$$\iint_{\Omega\times\mathcal{Y}} u\cdot w\,\mathrm{d}x\mathrm{d}y \le \bar{\Phi}(u) + \bar{\Phi}^*(w) \le \liminf_{\varepsilon\to 0}\ \Phi^\varepsilon(u_\varepsilon) + \liminf_{\varepsilon\to 0}\ \Phi^{\varepsilon *}(w_\varepsilon)$$

$$\le \liminf_{\varepsilon\to 0}\ [\Phi^\varepsilon(u_\varepsilon) + \Phi^{\varepsilon *}(w_\varepsilon)] = \liminf_{\varepsilon\to 0} \int_\Omega u_\varepsilon\cdot w_\varepsilon\,\mathrm{d}x \le \iint_{\Omega\times\mathcal{Y}} u\cdot w\,\mathrm{d}x\mathrm{d}y; \tag{2.12}$$

hence equality holds everywhere in the latter formula. (2.7) is thus established at least for a subsequence (however, we shall see that this applies to the whole sequence). In particular, we have

$$\left(\bar{\Phi}(u) + \bar{\Phi}^*(w) =\right) \iint_{\Omega\times\mathcal{Y}} [\varphi(u(x,y),x,y) + \varphi^*(w(x,y),x,y)]\,\mathrm{d}x\mathrm{d}y = \iint_{\Omega\times\mathcal{Y}} u\cdot w\,\mathrm{d}x\mathrm{d}y,$$

whence by (2.9)

$$\varphi(u(x,y),x,y) + \varphi^*(w(x,y),x,y) = u(x,y)\cdot w(x,y) \qquad \text{for a.e. } (x,y) \in \Omega\times\mathcal{Y};$$

by (2.11) we then infer (2.6). Moreover, by (2.5) and (2.12),

$$\liminf_{\varepsilon\to 0}\ \Phi^\varepsilon(u_\varepsilon) = \bar{\Phi}(u), \quad \liminf_{\varepsilon\to 0}\ \Phi^{\varepsilon *}(w_\varepsilon^*) = \bar{\Phi}^*(w^*),$$

possibly up to extracting a subsequence. Since the whole argument applies as ε vanishes along any subsequence, (2.7) and (2.8) hold as ε vanishes along the original sequence. □

Remarks (i) In Theorem 2.1 the functions φ and φ_ε need not be either convex or lower semicontinuous w.r.t. the first argument. However the condition (2.2) entails that the function $\varphi_\varepsilon(\cdot,x,x/\varepsilon)$ coincides with its Γ-regularized (in the sense e.g. of [30, Chap. 1]) at $u_\varepsilon(x)$.

(ii) The hypothesis (2.4) is trivially fulfilled whenever

$$\text{either}\quad u_\varepsilon \underset{2}{\to} u \quad \text{in } L^p(\Omega\times\mathcal{Y})^M \quad \text{or} \quad w_\varepsilon \underset{2}{\to} w \quad \text{in } L^{p'}(\Omega\times\mathcal{Y})^M.$$

On the other hand, in Sect. 4 (cf. Theorem 4.3) we shall see that, if φ (φ^*, resp.) is strictly convex and suitable further conditions hold, then (2.8) entails $u_\varepsilon \underset{2}{\to} u$ in $L^p(\Omega\times\mathcal{Y})^M$ ($w_\varepsilon \underset{2}{\to} w$ in $L^{p'}(\Omega\times\mathcal{Y})^M$, resp.).

(iii) Now we point out a simple variant of Theorem 2.1, in view of an application that we shall illustrate in Proposition 5.12. It is easy to see that this theorem also holds if, for some fixed nonnegative function $\theta \in \mathcal{D}(\Omega)$, (2.4) is replaced by

$$\liminf_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\cdot w_\varepsilon(x)\,\theta(x)\,\mathrm{d}x \le \iint_{\Omega\times\mathcal{Y}} u(x,y)\cdot w(x,y)\,\theta(x)\,\mathrm{d}x\mathrm{d}y,$$

and Φ_ε and Φ are, respectively, replaced by the functionals

$$\Phi_{\theta,\varepsilon} : v \mapsto \int_\Omega \varphi_\varepsilon(v(x),x)\,\theta(x)\,\mathrm{d}x, \quad \Phi_\theta : v \mapsto \iint_{\Omega\times\mathcal{Y}} \varphi(v(x,y),x,y)\,\theta(x)\,\mathrm{d}x\mathrm{d}y. \tag{2.13}$$

In this case, however, (2.6) just holds for a.e. $(x,y) \in D\times\mathcal{Y}$, D being the support of θ, and (2.7) must be replaced by

$$\int_\Omega u_\varepsilon(x)\cdot w_\varepsilon(x)\,\theta(x)\,\mathrm{d}x \to \iint_{\Omega\times\mathcal{Y}} u(x,y)\cdot w(x,y)\,\theta(x)\,\mathrm{d}x\mathrm{d}y. \tag{2.14}$$

The next statement applies if φ_ε is independent of ε.

Corollary 2.2 *Let $p \in [1,+\infty[$, $\varphi \in \mathcal{F}$ fulfil (1.10) and (1.20), and define Φ_ε and $\bar{\Phi}$ as in (1.11). Let $\{u_\varepsilon\}$ and $\{w_\varepsilon\}$ be sequences of $L^p(\Omega)^M$ and $L^{p'}(\Omega)^M$, resp., and*

$$w_\varepsilon(x) \in \partial\varphi(u_\varepsilon(x),x,x/\varepsilon) \qquad \text{for a.e. } x \in \Omega, \forall\,\varepsilon. \tag{2.15}$$

If (2.3) and (2.4) are fulfilled, then (2.6), (2.7) hold and

$$\Phi_\varepsilon(u_\varepsilon) \to \bar{\Phi}(u), \qquad \Phi^*_\varepsilon(w_\varepsilon) \to \bar{\Phi}^*(w). \tag{2.16}$$

Proof This trivially follows from Theorem 2.1, for part (iii) of Propositions 1.3 and 1.4 yield (2.5). □

3 Maximal monotonicity and two-scale convergence

In this section, we prove a result concerning two-scale convergence and maximal monotone functions. Here, we just deal with *single-valued* functions, but also allow for dependence on a further sequence of functions (that we denote by z_ε), in view of the applications of [61,62].

Let us fix two positive integers M and J, a function f such that

$$
\begin{aligned}
& f : \mathbf{R}^M \times \mathbf{R}^J \times \Omega \times \mathcal{Y} \to \mathbf{R}^M, \\
& v \mapsto f(v,t,x,y) \text{ is monotone } \forall\, t \text{ and for a.e. } (x,y), \\
& (v,t) \mapsto f(v,t,x,y) \text{ is continuous for a.e. } (x,y), \text{ and} \\
& (x,y) \mapsto f(v,t,x,y) \ \text{ is measurable either w.r.t. } \mathcal{B}(\Omega)\otimes\mathcal{L}(\mathcal{Y}) \\
& \qquad\qquad\qquad\qquad \text{or w.r.t. } \mathcal{L}(\Omega)\otimes\mathcal{B}(\mathcal{Y}), \ \forall\,(v,t),
\end{aligned}
\tag{3.1}
$$

and set

$$
f_\varepsilon(v,t,x) := f(v,t,x,x/\varepsilon) \qquad \forall\,(v,t), \text{ for a.e. } x, \forall\, \varepsilon > 0. \tag{3.2}
$$

This function is measurable w.r.t. x for any (v,t), and thus is of Caratheodory class.

The first part of the next theorem extends to two-scale convergence a well-known result of the theory of maximal monotone graphs, cf. e.g. [16, p. 27]. The second part was inspired by [19, p. 27]; see also [12], Lemma 5 of [13], [37, p. 183], and Theorem 2.10 of [62].

Theorem 3.1 *Let $p,q \in\,]1,+\infty[$, let f fulfil (3.1), define f_ε as in (3.2), and assume that*

$$
\begin{aligned}
& \exists L > 0, \exists g_1 \in L^1(\Omega) : \forall\,(v,t), \ \textit{for a.e. } (x,y), \\
& |f(v,t,x,y)|^{p'} \le L(|v|^p + |t|^q) + g_1(x).
\end{aligned}
\tag{3.3}
$$

Then: (i) *If two sequences $\{u_\varepsilon\} \subset L^p(\Omega)^M$ and $\{z_\varepsilon\} \subset L^q(\Omega)^J$ are such that*

$$
u_\varepsilon \underset{2}{\rightharpoonup} u \qquad \textit{in } L^p(\Omega\times\mathcal{Y})^M, \tag{3.4}
$$

$$
z_\varepsilon \underset{2}{\to} z \qquad \textit{in } L^q(\Omega\times\mathcal{Y})^J, \tag{3.5}
$$

$$
w_\varepsilon := f_\varepsilon(u_\varepsilon, z_\varepsilon, x) \underset{2}{\rightharpoonup} w \qquad \textit{in } L^{p'}(\Omega\times\mathcal{Y})^M, \tag{3.6}
$$

$$
\liminf_{\varepsilon\to 0} \int_\Omega u_\varepsilon \cdot w_\varepsilon \,\mathrm{d}x \le \iint_{\Omega\times\mathcal{Y}} u\cdot w \,\mathrm{d}x\mathrm{d}y, \tag{3.7}
$$

then

$$
w = f(u(x,y), z(x,y), x, y) \qquad \textit{for a.e. } (x,y) \in \Omega\times\mathcal{Y}. \tag{3.8}
$$

(ii) *If moreover*

$$
\begin{aligned}
& \exists d_1, d_2 > 0, \exists g_2 \in L^1(\Omega) : \forall\,(v,t), \ \textit{for a.e. } (x,y), \\
& f(v,t,x,y)\cdot v + d_1|t|^q \ge d_2|v|^p + g_2(x),
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
& \textit{for a.e. } (x,y) \in \Omega\times\mathcal{Y}, \forall\, v \in \mathbf{R}^M, \forall\,\{v_n\} \subset \mathbf{R}^M, \\
& \forall\ \textit{bounded sequence } \{t_n\} \subset \mathbf{R}^J, \\
& [f(v,t_n,x,y) - f(v_n,t_n,x,y)]\cdot(v - v_n) \to 0 \ \Rightarrow \ v_n \to v,
\end{aligned}
\tag{3.10}
$$

then

$$
\begin{aligned}
& u_\varepsilon \underset{2}{\to} u && \textit{in } L^p(\Omega\times\mathcal{Y})^M, \\
& f_\varepsilon(u_\varepsilon, z_\varepsilon, x) \underset{2}{\to} f(u,z,x,y) && \textit{in } L^{p'}(\Omega\times\mathcal{Y})^M.
\end{aligned}
\tag{3.11}
$$

Loosely speaking, (3.10) may be interpreted as a hypothesis of strict monotonicity of $f(\cdot,t,x,y)$ uniform w.r.t. v, and may also be compared with the property of strict convexity of the next section.

Proof (i) For any $v \in L^p(\Omega \times \mathcal{Y})^M$ it is known (see e.g. Proposition 5 of [26] and Proposition 2.10 of [58]) that there exists a sequence $\{v_\varepsilon\}$ of $L^p(\Omega)^M$ such that

$$v_\varepsilon \underset{2}{\to} v \qquad \text{in } L^p(\Omega \times \mathcal{Y})^M. \tag{3.12}$$

By the monotonicity of $f_\varepsilon(\cdot, z_\varepsilon, x)$,

$$\int_\Omega [w_\varepsilon(x) - f_\varepsilon(v_\varepsilon, z_\varepsilon, x)]\cdot(u_\varepsilon - v_\varepsilon)\,\mathrm{d}x \ge 0 \qquad \forall\,\varepsilon. \tag{3.13}$$

By the definition of two-scale convergence, (3.5) and (3.12) mean that

$$z_\varepsilon \circ S_\varepsilon \underset{2}{\to} z \quad \text{in } L^q(\Omega \times \mathcal{Y})^J, \qquad v_\varepsilon \circ S_\varepsilon \underset{2}{\to} v \quad \text{in } L^p(\Omega \times \mathcal{Y})^M. \tag{3.14}$$

Hence, as ε vanishes along a suitable subsequence that we still label by ε,

$$\begin{aligned} &v_\varepsilon(S_\varepsilon(x,y)) \to v(x,y), \quad z_\varepsilon(S_\varepsilon(x,y)) \to z(x,y),\\ &f_\varepsilon(v_\varepsilon(S_\varepsilon(x,y)), z_\varepsilon(S_\varepsilon(x,y)), S_\varepsilon(x,y)) \to f(v(x,y), z(x,y), x, y)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for a.e. } (x,y) \in \Omega \times \mathcal{Y}. \end{aligned} \tag{3.15}$$

For any sequence of measurable subsets $\{A_n\}$ of $\Omega \times \mathcal{Y}$ such that their $2N$-dimensional Lebesgue measure vanishes as $n \to \infty$, by (3.3) and (3.14)

$$\begin{aligned} &\sup_\varepsilon \iint_{A_n} \left|f_\varepsilon(v_\varepsilon, z_\varepsilon, \cdot) \circ S_\varepsilon\right|^{p'} \mathrm{d}x\mathrm{d}y\\ &\quad\le \sup_\varepsilon \iint_{A_n} \left(L|v_\varepsilon \circ S_\varepsilon|^p + L|z_\varepsilon \circ S_\varepsilon|^q + g_1(x)\right) \mathrm{d}x\mathrm{d}y \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

By the classical Vitali convergence theorem, the two latter statements yield

$$f_\varepsilon(v_\varepsilon(S_\varepsilon(x,y)), z_\varepsilon(S_\varepsilon(x,y)), S_\varepsilon(x,y)) \to f(v(x,y), z(x,y), x, y) \qquad \text{in } L^{p'}(\Omega \times \mathcal{Y})^M,$$

namely $f_\varepsilon(v_\varepsilon, z_\varepsilon, x) \underset{2}{\to} f(v, z, x, y)$ in $L^{p'}(\Omega \times \mathcal{Y})^M$.

Passing to the inferior limit as $\varepsilon \to 0$ in (3.13) and using (3.7) we then get

$$\iint_{\Omega\times\mathcal{Y}} [w(x,y) - f(v(x,y), z(x,y), x, y)]\cdot[u(x,y) - v(x,y)]\,\mathrm{d}x\mathrm{d}y \ge 0 \qquad \forall\, v \in L^p(\Omega \times \mathcal{Y})^M,$$

whence (3.6) follows by the maximal monotonicity of f w.r.t. its first argument.

(ii) Now we come to the proof of (3.11), and use an argument that is partially analogous to that of Lemma 5 of [13]. First let us set

$$P_\varepsilon(x) := [f_\varepsilon(u_\varepsilon, z_\varepsilon, x) - f_\varepsilon(u, z_\varepsilon, x)]\cdot[u_\varepsilon(x) - u(x)] \qquad \text{for a.e. } x \in \Omega, \forall\, \varepsilon > 0. \tag{3.16}$$

Notice that by (3.1), (3.3) and (3.5).

$$f_\varepsilon(u, z_\varepsilon, x) \underset{2}{\to} f(u, z, x, y) \qquad \text{in } L^{p'}(\Omega \times \mathcal{Y})^M.$$

By (3.4) we then infer that $\liminf_{\varepsilon\to 0}\int_\Omega P_\varepsilon(x)\,\mathrm{d}x \le 0$. As $P_\varepsilon \ge 0$, for a suitable sequence that we label by ε' we then have

$$P_{\varepsilon'} \to 0 \qquad \text{in } L^1(\Omega). \tag{3.17}$$

Hence, there exists a subsequence that we label by ε'' such that $P_{\varepsilon''} \to 0$ a.e. in Ω. By (3.5) we may also assume that $z_{\varepsilon''} \underset{2}{\to} z$ a.e. in $\Omega\times\mathcal{Y}$ without loss of generality. By (3.10) we then get

$$u_{\varepsilon''} \underset{2}{\to} u \qquad \text{a.e. in } \Omega\times\mathcal{Y}, \tag{3.18}$$

whence, recalling (3.6),

$$w_{\varepsilon''} := f_{\varepsilon''}(u_{\varepsilon''}, z_{\varepsilon''}, x) \underset{2}{\to} f(u,z,x,y) = w \qquad \text{a.e. in } \Omega\times\mathcal{Y}. \tag{3.19}$$

By (3.3)—(3.5) we have

$$\int_\Omega f_{\varepsilon''}(u, z_{\varepsilon''}, x)\cdot(u_{\varepsilon''} - u)\,\mathrm{d}x \to 0. \tag{3.20}$$

We then get

$$\begin{aligned}
&\int_\Omega w_{\varepsilon''}\cdot u_{\varepsilon''}\,\mathrm{d}x = \text{ (by (3.16))}\\
&\int_\Omega P_{\varepsilon''}(x)\,\mathrm{d}x + \int_\Omega f_{\varepsilon''}(u, z_{\varepsilon''}, x)\cdot(u_{\varepsilon''} - u)\,\mathrm{d}x + \int_\Omega f_{\varepsilon''}(u_{\varepsilon''}, z_{\varepsilon''}, x)\cdot u\,\mathrm{d}x\\
&\text{(by (3.6), (3.17), (3.20))} \ \to \iint_{\Omega\times\mathcal{Y}} w\cdot u\,\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

We claim that this entails that

$$\xi_{\varepsilon''} := (w_{\varepsilon''}\cdot u_{\varepsilon''})\circ S_{\varepsilon''} \to w\cdot u =: \xi \qquad \text{in } L^1(\Omega\times\mathcal{Y})^3, \tag{3.21}$$

namley $w_{\varepsilon''}\cdot u_{\varepsilon''} \underset{2}{\to} w\cdot u$ in $L^1(\Omega\times\mathcal{Y})^3$. In view of proving this statement, first notice that we already know that

$$\xi_{\varepsilon''} \ge 0, \quad \xi_{\varepsilon''} \to \xi \quad \text{a.e. in } \Omega\times\mathcal{Y}, \quad \iint_{\Omega\times\mathcal{Y}} \xi_{\varepsilon''}\,\mathrm{d}x\mathrm{d}y \to \iint_{\Omega\times\mathcal{Y}} \xi\,\mathrm{d}x\mathrm{d}y.$$

Setting $B_{\varepsilon''} := \{(x,y)\in\Omega\times\mathcal{Y} : \xi_{\varepsilon''}(x,y) \le \xi(x,y)\}$, we also have

$$\iint_{\Omega\times\mathcal{Y}} |\xi_{\varepsilon''} - \xi|\,\mathrm{d}x\mathrm{d}y = 2\iint_{B_{\varepsilon''}} (\xi - \xi_{\varepsilon''})\,\mathrm{d}x\mathrm{d}y + \iint_{\Omega\times\mathcal{Y}} (\xi_{\varepsilon''} - \xi)\,\mathrm{d}x\mathrm{d}y.$$

The Lebesgue dominated convergence theorem then yields (3.21).

By (3.9) we have

$$d_2|u_{\varepsilon''}\circ S_{\varepsilon''}|^p \le (u_{\varepsilon''}\circ S_{\varepsilon''})\cdot(w_{\varepsilon''}\circ S_{\varepsilon''}) - d_1|z_{\varepsilon''}\circ S_{\varepsilon''}|^q - g_2 \qquad \text{a.e. in } \Omega\times\mathcal{Y};$$

(3.5) and (3.21) then entail that the sequence $\{|u_{\varepsilon''} \circ S_{\varepsilon''}|^p\}$ is equiintegrable in $\Omega \times \mathcal{Y}$. By (3.3) and (3.5) the same then applies to $\{|w_{\varepsilon''} \circ S_{\varepsilon''}|^{p'}\}$. By (3.18) and (3.19) the Vitali convergence theorem then yields

$$\begin{aligned} u_{\varepsilon''} &\underset{2}{\to} u && \text{in } L^p(\Omega \times \mathcal{Y})^M, \\ w_{\varepsilon''} &\underset{2}{\to} w = f(u,z,x,y) && \text{in } L^{p'}(\Omega \times \mathcal{Y})^M. \end{aligned} \tag{3.22}$$

As both limits are independent of the extracted subsequence, the whole sequences converge. □

4 Two-scale compactness by strict convexity

In this section, we extend another result of single-scale convergence: we show that weak two-scale convergence and a suitable extremality condition entail strong two-scale convergence. We then apply this statement to strictly convex functionals of the form (1.11), and derive a result of two-scale compactness.

First, we review some standard notions, still assuming that Ω is a possibly unbounded Euclidean domain. Let us consider a multi-valued mapping

$$K : \Omega \to \mathcal{P}(\mathbf{R}^M) \quad (M \geq 1) \qquad \text{such that } K(x) \text{ is closed for a.e. } x. \tag{4.1}$$

We say that K is *measurable* whenever there exists a countable family $\{k_m\}$ of measurable functions $\Omega \to \mathbf{R}^M$ such that (denoting by $\bar{A}$ the closure of any set A)

$$\overline{\bigcup_{m \in \mathbf{N}} \{k_m(x)\}} = K(x) \qquad \text{for a.e. } x \in \Omega; \tag{4.2}$$

see e.g. [23, Sect. 3.2], [34, p. 326]. We say that K is *upper semicontinuous* if an only if, for any $x \in \Omega$ and any open set $V \supset K(x)$, the set $\{y \in \Omega : K(y) \subset V\}$ is a neighbourhood of x. We say that a point ξ of a closed set $A \subset \mathbf{R}^M$ is an *extremal point* of A whenever

$$\forall \xi', \xi'' \in A, \forall \lambda \in]0,1[, \quad \xi = \lambda \xi' + (1-\lambda)\xi'' \quad \Rightarrow \quad \xi = \xi' = \xi''.$$

We shall denote by $d(v, H)$ the distance of a point $v \in \mathbf{R}^M$ from a nonempty set $H \subset \mathbf{R}^M$. Let us review the notion of (single-scale) *compactness by extremality.*

Proposition 4.1 *Let $H : \Omega \to \mathcal{P}(\mathbf{R}^M)$ be a measurable multi-valued mapping and $\{w_\varepsilon\}$ be a sequence of $L^1(\Omega)^M$. If*

$$H(x) \text{ is closed and convex, for a.e. } x \in \Omega, \tag{4.3}$$

$$w_\varepsilon \rightharpoonup w \qquad \text{in } L^1(\Omega)^M, \tag{4.4}$$

$$d(w_\varepsilon, H) \to 0 \qquad \text{a.e. in } \Omega, \text{ as } \varepsilon \to 0, \tag{4.5}$$

$$w(x) \text{ is an extremal point of } H(x), \text{ for a.e. } x \in \Omega, \tag{4.6}$$

then

$$w_\varepsilon \to w \qquad \text{in } L^1(\Omega)^M. \tag{4.7}$$

(After [7], $H(x)$ may be replaced by the closed convex hull of the set of limit points of the sequence $\{w_\varepsilon(x)\}$, for a.e. $x \in \Omega$; in this case the hypotheses (4.3) and (4.5) are not needed.)

Proof This statement extends Theorem 1 of [56], where it is assumed that

$$w_\varepsilon(x) \in H(x) \qquad \text{for a.e. } x \in \Omega, \forall\, \varepsilon,$$

in place of the weaker condition (4.5). Here, we derive Proposition 4.1 from this slightly less general result. For any ε and a.e. $x \in \Omega$, there exists $w'_\varepsilon(x) \in H(x)$ such that $|w_\varepsilon(x) - w'_\varepsilon(x)| = d(w_\varepsilon(x), H(x))$. Setting $w''_\varepsilon := w_\varepsilon - w'_\varepsilon$, we have $|w''_\varepsilon| = d(w_\varepsilon, H) \leq |w_\varepsilon - w|$ a.e. in Ω; by (4.4), the sequence $\{w''_\varepsilon\}$ is then equi-integrable. By (4.5) we then have $w''_\varepsilon \to 0$ in $L^1(\Omega)^M$, whence $w'_\varepsilon = w_\varepsilon - w''_\varepsilon(x) \rightharpoonup w$ in $L^1(\Omega)^M$. Applying Theorem 1 of [56] to the sequence $\{w'_\varepsilon\}$ we then get $w'_\varepsilon \to w$ in $L^1(\Omega)^M$, whence (4.7). □

From this result now we derive an analogous statement for two-scale convergence. We shall assume that the distance of a sequence of functions $\{u_\varepsilon\} = \{u_\varepsilon(x)\}$ from a multi-valued mapping $K = K(x,y)$ vanishes a.e. in the sense of two-scale convergence. This notion of convergence to a multi-valued mapping is not encompassed by the definitions of Sect. 1; we express it as follows:

$$d(u_\varepsilon(S_\varepsilon(x,y)), K(x,y)) \to 0 \qquad \text{for a.e. } (x,y) \in \Omega \times \mathcal{Y}.$$

Theorem 4.2 *Let K be a measurable multi-valued function $\Omega \times \mathcal{Y} \to \mathcal{P}(\mathbf{R}^M)$ such that*

$$K(x,y) \text{ is closed and convex for a.e. } (x,y) \in \Omega \times \mathcal{Y}. \tag{4.8}$$

For a.e. x, let us then identify $K(x,\cdot)$ with its Y-periodic extension to $\mathbf{R}^N$. Let $p \in [1, +\infty[$, $\{u_\varepsilon\}$ be a sequence in $L^p(\Omega)^M$ such that (defining S_ε as in (1.2))

$$u_\varepsilon \underset{2}{\rightharpoonup} u \qquad \text{in } L^p(\Omega \times \mathcal{Y})^M, \tag{4.9}$$

$$d(u_\varepsilon(S_\varepsilon(x,y)), K(x,y)) \to 0 \quad \text{for a.e. } (x,y) \in \Omega \times \mathcal{Y}, \tag{4.10}$$

$$u(x,y) \text{ is an extremal point of } K(x,y) \quad \text{for a.e. } (x,y) \in \Omega \times \mathcal{Y}. \tag{4.11}$$

Then

$$p = 1 \quad \Rightarrow \quad u_\varepsilon \underset{2}{\to} u \qquad \text{in } L^1(\Omega \times \mathcal{Y})^M, \tag{4.12}$$

$$1 < q < p \quad \Rightarrow \quad u_\varepsilon \underset{2}{\to} u \qquad \text{in } L^q_{\text{loc}}(\Omega \times \mathcal{Y})^M. \tag{4.13}$$

Proof If $p = 1$ we apply Proposition 4.1 with $(x,y) \in \Omega \times \mathcal{Y}$ in place of $x \in \Omega$, and with $w_\varepsilon = u_\varepsilon \circ S_\varepsilon$, $w = u$ and $H = K$. This yields $u_\varepsilon \circ S_\varepsilon \to u$ in $L^1(\Omega \times \mathcal{Y})^M$, that is (4.12).

For $1 < q < p$ we use a standard procedure. In this case, $u_\varepsilon \underset{2}{\to} u$ in $L^1_{\text{loc}}(\Omega \times \mathcal{Y})^M$ by (4.12). Thus, for any fixed bounded measurable subset $\tilde{\Omega}$ of Ω, $u_\varepsilon \circ S_\varepsilon \to u$ in measure in $\tilde{\Omega} \times \mathcal{Y}$, and this sequence is bounded in $L^p(\tilde{\Omega} \times \mathcal{Y})^M$. By the Egoroff theorem, for any $\delta > 0$ there exist $A_\delta \subset \tilde{\Omega} \times \mathcal{Y}$ and a subsequence $\{\bar{u}_{\varepsilon'}\} := \{u_{\varepsilon'} \circ S_{\varepsilon'}\}$ such that (setting $A'_\delta := (\tilde{\Omega} \times \mathcal{Y}) \setminus A_\delta$ and denoting by $|\cdot|_N$ the N-dimensional Lebesgue measure)

$$|A'_\delta|_N \leq \delta, \qquad \bar{u}_{\varepsilon'} \to u \quad \text{uniformly in } A_\delta.$$

By the Schwarz–Hölder inequality we then have

$$\iint_{\tilde{\Omega} \times \mathcal{Y}} |u - \bar{u}_{\varepsilon'}|^q \mathrm{d}x\mathrm{d}y \leq \iint_{A'_\delta} |u - \bar{u}_{\varepsilon'}|^q \mathrm{d}x\mathrm{d}y + |A_\delta|_N \sup_{A_\delta} |u - \bar{u}_{\varepsilon'}|^q$$
$$\leq \|u - \bar{u}_{\varepsilon'}\|^q_{L^p(A'_\delta)^M} |A'_\delta|_N^{(p-q)/p} + |A_\delta|_N \sup_{A_\delta} |u - \bar{u}_{\varepsilon'}|^q,$$

and the right side vanishes as first $\varepsilon' \to 0$ and then $\delta \to 0$. Thus $u_{\varepsilon'} \circ S_{\varepsilon'} \to u$ in $L^q(\tilde{\Omega} \times \mathcal{Y})^M$. Since from any subsequence of $\{u_\varepsilon \circ S_\varepsilon\}$ it is possible to extract a converging sequence like this, we conclude that the whole original sequence converges in $L^q(\tilde{\Omega} \times \mathcal{Y})^M$. This completes the proof of (4.13). □

Part (ii) of the next statement extends part (i) of Theorem 2 of [56] to two-scale convergence.

Theorem 4.3 *Let a function $\varphi \in \mathcal{F}$ (cf. (1.8)) fulfil (1.10) for $p = 1$, and assume that*

$$\text{the function } v \mapsto \varphi(v, x, y) \text{ is convex for a.e. } (x, y), \tag{4.14}$$

$$\text{the function } (v, x) \mapsto \varphi(v, x, y) \text{ is lower semicontinuous for a.e. } y. \tag{4.15}$$

Let $\{u_\varepsilon\}$ be a sequence in $L^1(\Omega)^M$ and $u \in L^1(\Omega \times \mathcal{Y})^M$. Then:

(i) *If*

$$u_\varepsilon \underset{2}{\rightharpoonup} u \qquad \text{in } L^1(\Omega \times \mathcal{Y})^M, \tag{4.16}$$

$$\int_\Omega \varphi(u_\varepsilon(x), x, x/\varepsilon)\, \mathrm{d}x \to \iint_{\Omega\times\mathcal{Y}} \varphi(u(x,y), x, y)\, \mathrm{d}x\mathrm{d}y \neq +\infty, \tag{4.17}$$

then

$$\varphi(u_\varepsilon(x), x, x/\varepsilon) \underset{2}{\rightharpoonup} \varphi(u(x,y), x, y) \qquad \text{in } L^1(\Omega \times \mathcal{Y}). \tag{4.18}$$

(ii) *If (4.16) and (4.17) are fulfilled and*

$$\begin{aligned} &(u(x,y), \varphi(u(x,y), x, y)) \text{ is an extremal point of} \\ &K(x,y) := \{(v,z) \in \mathbf{R}^{M+1} : z \geq \varphi(v,x,y)\} \text{ for a.e. } (x,y) \in \Omega \times \mathcal{Y}, \end{aligned} \tag{4.19}$$

then

$$u_\varepsilon \underset{2}{\to} u \qquad \text{in } L^1(\Omega \times \mathcal{Y})^M, \tag{4.20}$$

$$\varphi(u_\varepsilon(x), x, x/\varepsilon) \underset{2}{\to} \varphi(u(x,y), x, y) \qquad \text{in } L^1(\Omega \times \mathcal{Y}). \tag{4.21}$$

Obviously (4.19) is fulfilled whenever $\varphi(\cdot, x, y)$ is strictly convex for a.e. (x, y). But (4.19) is slightly more general: for instance $(0, 0)$ is an extremal point of $\{(v, z) \in \mathbf{R}^2 : z \geq (v^+)^2\}$, but $v \mapsto (v^+)^2$ is not strictly convex, not even in a neighbourhood of $v = 0$.

Proof We claim that

$$\varphi(u_\varepsilon(x), x, x/\varepsilon) \underset{2}{\rightharpoonup} \varphi(u(x,y), x, y) \qquad \text{in } L^1(\Omega \times \mathcal{Y}). \tag{4.22}$$

This argument mimics that of Lemma 3 of [56]. By Lemma 1.1, (4.17) also reads

$$\iint_{\Omega\times\mathcal{Y}} \varphi(u_\varepsilon(S_\varepsilon(x,y)), S_\varepsilon(x,y), y)\, \mathrm{d}x\mathrm{d}y \to \iint_{\Omega\times\mathcal{Y}} \varphi(u(x,y), x, y)\, \mathrm{d}x\mathrm{d}y. \tag{4.23}$$

The condition (4.16) is tantamount to $u_\varepsilon \circ S_\varepsilon \rightharpoonup u$ in $L^1\big(\Omega \times \mathcal{Y}\big)^M$. For any fixed pair of measurable sets $A \subset \Omega$ and $B \subset \mathcal{Y}$, by part (iii) of Proposition 1.3 we then have

$$\liminf_{\varepsilon\to 0} \iint_{A\times B} \varphi(u_\varepsilon(S_\varepsilon(x,y)), S_\varepsilon(x,y), y)\, \mathrm{d}x\mathrm{d}y \geq \iint_{A\times B} \varphi(u(x,y), x, y)\, \mathrm{d}x\mathrm{d}y;$$

a similar inequality holds if $A \times B$ is replaced by $(\Omega \times \mathcal{Y}) \setminus (A \times B)$. By (4.23) we then get

$$\iint_{A\times B} \varphi(u_\varepsilon(S_\varepsilon(x,y)), S_\varepsilon(x,y), y)\, \mathrm{d}x\mathrm{d}y \to \iint_{A\times B} \varphi(u(x,y), x, y)\, \mathrm{d}x\mathrm{d}y. \tag{4.24}$$

By (1.10) (with $p = 1$) and (4.17) the sequence

$$\{\varphi(u_\varepsilon(S_\varepsilon(x,y)), S_\varepsilon(x,y), y) + c_1|u_\varepsilon(S_\varepsilon(x,y))| - f(x,y)\}$$

is nonnegative and bounded in $L^1(\Omega\times\mathcal{Y})$. By (4.16) the same applies to $\{|u_\varepsilon(S_\varepsilon(x,y))|\}$, hence

$$\{\varphi(u_\varepsilon(S_\varepsilon(x,y)), S_\varepsilon(x,y), y)\} \text{ is bounded in } L^1(\Omega\times\mathcal{Y}).$$

As the linear space spanned by characteristic functions of subsets of the form $A\times B$ is dense in $L^\infty(\Omega\times\mathcal{Y})$, by (4.24) we then infer that

$$\varphi(u_\varepsilon(S_\varepsilon(x,y)), S_\varepsilon(x,y), y) \rightharpoonup \varphi(u(x,y), x, y) \qquad \text{in } L^1(\Omega\times\mathcal{Y}),$$

that is (4.22).

(ii) Let us now set

$$\begin{aligned} w_\varepsilon(x) &:= (u_\varepsilon(x), \varphi(u_\varepsilon(x), x, x/\varepsilon)) && \text{for a.e. } x \in \Omega, \\ w(x,y) &:= (u(x,y), \varphi(u(x,y), x, y)) && \text{for a.e. } (x,y) \in \Omega\times\mathcal{Y}; \end{aligned}$$

thus

$$w_\varepsilon(S_\varepsilon(x,y)) = (u_\varepsilon(S_\varepsilon(x,y)), \varphi(u_\varepsilon(S_\varepsilon(x,y)), S_\varepsilon(x,y), y)) \qquad \text{for a.e. } (x,y) \in \Omega\times\mathcal{Y}.$$

By (4.16) and (4.22), $w_\varepsilon \underset{2}{\rightharpoonup} w$ in $L^1(\Omega\times\mathcal{Y})^{M+1}$. Notice that

$$\begin{aligned} &d(w_\varepsilon(S_\varepsilon(x,y)), K(x,y)) \\ &\quad \le [\varphi(u_\varepsilon(S_\varepsilon(x,y)), x, y)) - \varphi(u_\varepsilon(S_\varepsilon(x,y)), S_\varepsilon(x,y), y)]^+ \quad \text{for a.e. } (x,y) \in \Omega\times\mathcal{Y}; \end{aligned}$$

by (1.3) and (4.15) as $\varepsilon \to 0$ the latter quantity vanishes a.e.. The condition (4.10) is thus fulfilled with w_ε and w in place of u_ε and u, resp.. Theorem 4.2 then yields $w_\varepsilon \underset{2}{\to} w$ in $L^1(\Omega\times\mathcal{Y})^{M+1}$, that is, (4.20) and (4.21). □

Corollary 4.4 *If (4.17) is replaced by*

$$\int_A \varphi(u_\varepsilon(x), x, x/\varepsilon)\, \mathrm{d}x \to \iint_{A\times\mathcal{Y}} \varphi(u(x,y), x, y)\, \mathrm{d}x\mathrm{d}y \ne +\infty \qquad \forall \text{ bounded measurable } A \subset \Omega, \tag{4.25}$$

then Theorem 4.3 holds with $L^1_{\mathrm{loc}}(\Omega)^M$ and $L^1_{\mathrm{loc}}(\Omega\times\mathcal{Y})^M$ in place of $L^1(\Omega)^M$ and $L^1(\Omega\times\mathcal{Y})^M$, resp..

Corollary 4.5 *Let $p \in\,]1, +\infty[$ and a function $\varphi \in \mathcal{F}$ (cf. (1.8)) fulfil (4.14), (4.15) and be such that*

$$\begin{aligned} &\exists c_3 > 0, \exists f \text{ as in (1.4) or (1.5) such that} \\ &\varphi(v, x, y) \ge c_3|v|^p + f(x,y) \quad \forall v, \text{ for a.e. } (x,y);. \end{aligned} \tag{4.26}$$

Let a sequence $\{u_\varepsilon\}$ *of* $L^p(\Omega)^M$*, and* $u \in L^p(\Omega\times\mathcal{Y})^M$ *fulfil* (4.17), (4.19)*, and be such that*

$$u_\varepsilon \underset{2}{\rightharpoonup} u \qquad \text{in } L^p(\Omega\times\mathcal{Y})^M. \tag{4.27}$$

Then

$$u_\varepsilon \underset{2}{\to} u \qquad \text{in } L^p(\Omega\times\mathcal{Y})^M. \tag{4.28}$$

Proof By (4.27), $u_\varepsilon \underset{2}{\rightharpoonup} u$ in $L^1_{\rm loc}(\Omega\times\mathcal{Y})^M$. Moreover (4.17) and the procedure of part (i) of Theorem 4.3 yields (4.25). By Corollary 4.4 then $u_\varepsilon \underset{2}{\to} u$ in $L^1_{\rm loc}(\Omega\times\mathcal{Y})^M$, whence a.e. in $\Omega\times\mathcal{Y}$ as ε vanishes along a suitable subsequence $\{\varepsilon'\}$. By (1.3), (4.14) and (4.15) then

$$\begin{aligned}&\liminf_{\varepsilon'\to 0} \varphi(u_{\varepsilon'}(S_{\varepsilon'}(x,y)), S_{\varepsilon'}(x,y), y) - f(S_{\varepsilon'}(x,y), y)\\ &\quad \ge \varphi(u(x,y), x, y) - f(x,y) \qquad \text{for a.e. } (x,y)\in\Omega\times\mathcal{Y},\end{aligned}$$

and the latter function is integrable in $\Omega\times\mathcal{Y}$. Moreover by (4.26),

$$\begin{aligned}&|u(x,y) - u_{\varepsilon'}(S_{\varepsilon'}(x,y))|^p \le 2^p|u(x,y)|^p + 2^p|u_{\varepsilon'}(S_{\varepsilon'}(x,y))|^p\\ &\quad\le \frac{2^p}{c_3}\big[\varphi(u(x,y),x,y)) - f(x,y)\big]\\ &\qquad + \frac{2^p}{c_3}\big[\varphi(u_{\varepsilon'}(S_{\varepsilon'}(x,y)), S_{\varepsilon'}(x,y), y) - f(S_{\varepsilon'}(x,y), y)\big] \quad \text{for a.e. } (x,y)\in\Omega\times\mathcal{Y}, \forall\,\varepsilon'.\end{aligned}$$

By (4.23), Lemma 4.6 below then yields $u_{\varepsilon'}\circ S_{\varepsilon'} \to u$ in $L^p(\Omega\times\mathcal{Y})^M$, namely (4.28) for the subsequence $\{u_{\varepsilon'}\}$. As the limit is independent of the subsequence, we infer that the whole sequence $\{u_\varepsilon\}$ two-scale converges. □

Lemma 4.6 [56] *Let* A *be any measurable Euclidean set, and two sequences of measurable functions* $v_n : A \to \mathbf{R}^M$ *and* $w_n : A\to\mathbf{R}$ *be such that*

$$\begin{aligned}&|v_n| \le z_n \quad \forall n \text{ a.e. in } A, \quad v_n \to v,\ z_n\to z \quad \text{a.e. in } A,\\ &z\in L^1(A), \qquad \int_A z_n\,\mathrm{d}x \to \int_A z\,\mathrm{d}x.\end{aligned} \tag{4.29}$$

Then $v\in L^1(A)^M$ *and* $\int_A v_n \mathrm{d}x \to \int_A v\mathrm{d}x$.

We now draw a simple consequence from Theorem 2.1.

Corollary 4.7 *Let the assumptions of Theorem 2.1 be fulfilled, as well as* (4.14), (4.15) *and* (4.19)*. Then* (4.28) *holds.*

Proof It suffices to apply Corollary 4.5, for Corollary 2.2 provides (4.17). □

Remarks This section rests on the method of *compactness by strict convexity* as developed in [56]. The results of that paper were then extended in several other works, e.g. [3,7,8,17,47,54,55], that might also suggest further two-scale analogs.

5 Two-scale div-curl lemma and related results

In this section, we derive two simple two-scale extensions of the classical *div-curl lemma* of [40] and of one of its variants. We also briefly illustrate how the two-scale div-curl lemma and some of the results of the previous sections can be applied to the homogenization of nonlinear partial differential equations.

5.1 First-order operators of electromagnetism

Let us first set

$$(\nabla\times v)_{ij} := \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \quad \forall i,j \in \{1,...,N\}, \quad \nabla\cdot v := \sum_{i=1}^{N} \frac{\partial v_i}{\partial x_i},$$

$$L^2_{\rm rot}(\mathbf{R}^N)^N := \left\{v \in L^2(\mathbf{R}^N)^N : \nabla\times v \in L^2(\mathbf{R}^N)^{N^2}\right\},$$

$$L^2_{\rm div}(\mathbf{R}^N)^N := \left\{v \in L^2(\mathbf{R}^N)^N : \nabla\cdot v \in L^2(\mathbf{R}^N)\right\}.$$

It is known that these are Hilbert spaces equipped with the respective graph norms.

Let us now review a simplified form of the classical (*single-scale*) div-curl lemma.

Theorem 5.1 [40] (*Single-Scale Div-Curl Lemma – I*) *Let* $\{u_\varepsilon\}$ *and* $\{w_\varepsilon\}$ *be two bounded sequences of* $L^2_{\rm rot}(\mathbf{R}^N)^N$ *and* $L^2_{\rm div}(\mathbf{R}^N)^N$*, resp.. If*

$$u_\varepsilon \rightharpoonup u, \quad w_\varepsilon \rightharpoonup w \qquad in\ L^2(\mathbf{R}^N)^N, \tag{5.1}$$

then $u_\varepsilon\cdot w_\varepsilon \to u\cdot w$ *in* $\mathcal{D}'(\mathbf{R}^N)$*, that is,*

$$\int_{\mathbf{R}^N} u_\varepsilon(x)\cdot w_\varepsilon(x)\, \theta(x)\, {\rm d}x \to \int_{\mathbf{R}^N} u(x)\cdot w(x)\, \theta(x)\, {\rm d}x \qquad \forall \theta \in \mathcal{D}(\mathbf{R}^N). \tag{5.2}$$

We extend this result to two-scale convergence via the next two lemmata. First, we set

$$\hat{v} := \int_{\mathcal{Y}} v(y)\, {\rm d}y, \quad \tilde{v} := v - \hat{v} \qquad \forall v \in L^1(\mathcal{Y})^N. \tag{5.3}$$

Henceforth, by appending the index $*$ to a space of functions over $\mathcal{Y}$ we shall denote the subspace of functions having vanishing mean: e.g.

$$L^2_*(\mathcal{Y}) := \left\{v \in L^2(\mathcal{Y}) : \hat{v} = 0\right\}, \quad H^1_*(\mathcal{Y}) := \left\{v \in H^1(\mathcal{Y}) : \hat{v} = 0\right\}.$$

Lemma 5.2 *For any* $u \in L^2_*(\mathcal{Y})^N$*, if* $\nabla\times u = 0$ *in* $\mathcal{D}'(\mathcal{Y})^{N^2}$ *then there exists a scalar potential* $\eta \in H^1(\mathcal{Y})$ *such that* $u = \nabla\eta$ *a.e. in* $\mathcal{Y}$*, and conversely.*

Proof Setting $n\cdot y := \sum_{i=1}^N n_i y_i$ for any $n = (n_1,...,n_N) \in \mathbf{Z}^N$ and any $y = (y_1,...,y_N) \in \mathcal{Y}$, we may represent u via Fourier series as follows

$$u(y) = \textstyle\sum_{n\in\mathbf{Z}^N} a_n \exp(2\pi i n\cdot y) \qquad \text{for a.e. } y \in \mathcal{Y},$$

$$\text{where } a_n := \int_{\mathcal{Y}} u(y)\, \exp(-2\pi i n\cdot y)\, {\rm d}y\ (\in \mathbf{C}^N) \qquad \forall n \in \mathbf{Z}^N;$$

moreover $\sum_{n\in\mathbf{Z}^N}|a_n|^2 = \|u\|^2_{L^2(\mathcal{Y})^N} < +\infty$. Denoting by a_n^j the jth component of a_n for $j = 1, ..., N$ and setting $\bar{0} := (0, ..., 0)$, as $\nabla\times u = 0$ we have

$$a_{\bar{0}}^j = 0 \quad \text{for } j = 1, ..., N, \qquad n_i a_n^j = n_j a_n^i \quad \forall n \in \mathbf{Z}^N.$$

For any $n \in \mathbf{Z}^N$, a_n is thus parallel to n; that is there exists $k_n \in \mathbf{C}$ such that $a_n = k_n n$ if $n \neq \bar{0}$, whereas $k_{\bar{0}}$ is undetermined. Setting

$$\eta(y) := (2\pi)^{-1}\sum_{n\in\mathbf{Z}^N} k_n \exp(2\pi i n\cdot y) \qquad \text{for a.e. } y \in \mathcal{Y},$$

we then have $\eta \in H^1(\mathcal{Y})$ and $u = \nabla\eta$ a.e. in $\mathcal{Y}$. □

Lemma 5.3 *If* $u, w \in L^2(\mathbf{R}^N\times\mathcal{Y})^N$ *are such that*

$$\nabla_y\times u = 0 \quad \text{in } \mathcal{D}'(\mathbf{R}^N\times\mathcal{Y})^{N^2}, \quad \nabla_y\cdot w = 0 \quad \text{in } \mathcal{D}'(\mathbf{R}^N\times\mathcal{Y}), \tag{5.4}$$

then

$$\iint_{\mathbf{R}^N\times\mathcal{Y}} u(x,y)\cdot w(x,y)\theta(x)\,\mathrm{d}x\mathrm{d}y = \int_{\mathbf{R}^N} \hat{u}(x)\cdot\hat{w}(x)\theta(x)\,\mathrm{d}x \qquad \forall\theta \in L^\infty(\mathbf{R}^N). \tag{5.5}$$

Proof Setting $z(x,y) := u(x,y) - \hat{u}(x)$ for a.e. $(x,y) \in \mathbf{R}^N\times\mathcal{Y}$, we have

$$z \in L^2\big(\mathbf{R}^N; L^2_*(\mathcal{Y})^N\big), \qquad \nabla_y\times z(x,\cdot) = 0 \quad \text{in } \mathcal{D}'(\mathbf{R}^N\times\mathcal{Y})^{N^2}, \text{ for a.e. } x \in \mathbf{R}^N.$$

After Lemma 5.2 then there exists a potential $\eta \in L^2\big(\mathbf{R}^N; H^1_*(\mathcal{Y})\big)$ such that $z = \nabla_y\eta$ a.e. in $\mathbf{R}^N\times\mathcal{Y}$. Therefore,

$$\begin{aligned}&\iint_{\mathbf{R}^N\times\mathcal{Y}} u(x,y)\cdot w(x,y)\theta(x)\,\mathrm{d}x\mathrm{d}y - \int_{\mathbf{R}^N} \hat{u}(x)\cdot\hat{w}(x)\theta(x)\,\mathrm{d}x\\ &\quad= \iint_{\mathbf{R}^N\times\mathcal{Y}} z(x,y)\cdot w(x,y)\theta(x)\,\mathrm{d}x\mathrm{d}y = \int_{\mathbf{R}^N}\mathrm{d}x\,\theta(x)\int_{\mathcal{Y}} \nabla_y\eta(x,y)\cdot w(x,y)\,\mathrm{d}y = 0.\end{aligned}$$

□

We are now able to state a two-scale version of Theorem 5.1.

Theorem 5.4 (*Two-Scale Div-Curl Lemma – I*) *Let* $\{u_\varepsilon\}$ *and* $\{w_\varepsilon\}$ *be two bounded sequences of* $L^2_{\mathrm{rot}}(\mathbf{R}^N)^N$ *and* $L^2_{\mathrm{div}}(\mathbf{R}^N)^N$, *resp.. If*

$$u_\varepsilon \underset{2}{\rightharpoonup} u, \quad w_\varepsilon \underset{2}{\rightharpoonup} w \qquad \text{in } L^2(\mathbf{R}^N\times\mathcal{Y})^N, \tag{5.6}$$

then

$$\begin{aligned}&\int_{\mathbf{R}^N} u_\varepsilon(x)\cdot w_\varepsilon(x)\,\theta(x)\,\mathrm{d}x \to \int_{\mathbf{R}^N} \hat{u}(x)\cdot\hat{w}(x)\,\theta(x)\,\mathrm{d}x\\ &\quad= \iint_{\mathbf{R}^N\times\mathcal{Y}} u(x,y)\cdot w(x,y)\,\theta(x)\,\mathrm{d}x\mathrm{d}y \qquad \forall\theta \in \mathcal{D}(\mathbf{R}^N).\end{aligned} \tag{5.7}$$

Proof By (5.6), $u_\varepsilon \rightharpoonup \hat{u}$ and $w_\varepsilon \rightharpoonup \hat{w}$ in $L^2(\mathbf{R}^N)^N$. Hence, by Theorem 5.1

$$\int_{\mathbf{R}^N} u_\varepsilon(x)\cdot w_\varepsilon(x)\,\theta(x)\,\mathrm{d}x \to \int_{\mathbf{R}^N} \hat{u}(x)\cdot\hat{w}(x)\,\theta(x)\,\mathrm{d}x \quad \forall\theta\in\mathcal{D}(\mathbf{R}^N).$$

By the boundedness hypotheses and by known results of two-scale convergence, cf. [1,44], (5.4) is fulfilled. This yields (5.5), whence (5.7) follows. □

5.2 A Variant of the Div-Curl Lemma

First, we state a simple variant of Theorem 5.1 that is especially relevant because of its connection with J. Ball's notion of *polyconvexity,* cf. e.g. [9,15,28]. For the sake of simplicity here we select $N = 3$.

Proposition 5.5 [42] (*Single-Scale Div-Curl Lemma – II*) *Let* $\{u_\varepsilon\}$ *and* $\{w_\varepsilon\}$ *be two bounded sequences of* $L^2_{\mathrm{rot}}(\mathbf{R}^3)^3$*. If*

$$u_\varepsilon \rightharpoonup u, \quad w_\varepsilon \rightharpoonup w \qquad \text{in } L^2(\mathbf{R}^3)^3, \tag{5.8}$$

then $u_\varepsilon \times w_\varepsilon \to u\times w$ *in* $\mathcal{D}'(\mathbf{R}^3)^3$*, that is,*

$$\int_{\mathbf{R}^3} u_\varepsilon(x)\times w_\varepsilon(x)\cdot\theta(x)\,\mathrm{d}x \to \int_{\mathbf{R}^3} u(x)\times w(x)\cdot\theta(x)\,\mathrm{d}x \qquad \forall\theta\in\mathcal{D}(\mathbf{R}^3)^3. \tag{5.9}$$

Next we provide a two-scale version.

Proposition 5.6 (*Two-Scale Div-Curl Lemma – II*) *Let* $\{u_\varepsilon\}$ *and* $\{w_\varepsilon\}$ *be two bounded sequences of* $L^2_{\mathrm{rot}}(\mathbf{R}^3)^3$*. If*

$$u_\varepsilon \underset{2}{\rightharpoonup} u, \quad w_\varepsilon \underset{2}{\rightharpoonup} w \qquad \text{in } L^2(\mathbf{R}^3\times\mathcal{Y})^3, \tag{5.10}$$

then

$$\begin{aligned}\int_{\mathbf{R}^3} u_\varepsilon(x)\times w_\varepsilon(x)\cdot\theta(x)\,\mathrm{d}x &\to \int_{\mathbf{R}^3} \hat{u}(x)\times\hat{w}(x)\cdot\theta(x)\,\mathrm{d}x \\ &= \iint_{\mathbf{R}^3\times\mathcal{Y}} u(x,y)\times w(x,y)\cdot\theta(x)\,\mathrm{d}x\mathrm{d}y \qquad \forall\theta\in\mathcal{D}(\mathbf{R}^3)^3.\end{aligned} \tag{5.11}$$

Proof The argument mimics that of Theorem 5.4 and is based on the next lemma. □

Lemma 5.7 *If* $u, w \in L^2(\mathbf{R}^3\times\mathcal{Y})^3$ *are such that*

$$\nabla_y\times u = \nabla_y\times w = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^3\times\mathcal{Y})^3, \tag{5.12}$$

then

$$\iint_{\mathbf{R}^3\times\mathcal{Y}} u(x,y)\times w(x,y)\cdot\theta(x)\,\mathrm{d}x\mathrm{d}y = \int_{\mathbf{R}^3} \hat{u}(x)\times\hat{w}(x)\cdot\theta(x)\,\mathrm{d}x \quad \forall\theta\in L^\infty(\mathbf{R}^3)^3. \tag{5.13}$$

Proof The argument concides with that of Lemma 5.3, with the only variant that in the final formula the scalar product is replaced by the vector product. □

5.3 First-order operators of continuum mechanics

Other compensated compactness results also have a two-scale extension. A statement analogous to the div-curl lemma concerns the space of $N\times N$-tensors and involves the symmetrized gradient and the divergence:

$$(\nabla^s u)_{ij} := \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) \qquad \forall i,j, \forall u \in H^1(\mathbf{R}^N)^N,$$

$$(\nabla\cdot v)_i := \sum_{j=1}^{N} \frac{\partial v_{ij}}{\partial x_j} \qquad \forall i, \forall v \in H^1(\mathbf{R}^N)^{N^2};$$

here N is any positive integer, and of course $N = 2$ or 3 in mechanical applications. Let us set

$$U := \nabla^s H^1(\mathbf{R}^N)^N = \{u \in L^2(\mathbf{R}^N)^{N^2} : u = \nabla^s v \text{ for some } v \in H^1(\mathbf{R}^N)^N\},$$

$$W := \{w \in L^2(\mathbf{R}^N)^{N^2} : \nabla\cdot w \in L^2(\mathbf{R}^N)^N\}; \tag{5.14}$$

both are Hilbert spaces: U is equipped with the topology induced by $H^1(\mathbf{R}^N)^N$, and W with the graph norm. A variant of Theorem 5.1 reads as follows. (Here we set $u:v := \sum_{i,j=1}^N u_{ij}v_{ij}$ for any $u, v \in \mathbf{R}^{N^2}$.)

Theorem 5.8 [40] (*Single-Scale Div-Curl Lemma – III*) *Let* $\{u_\varepsilon\}$ *and* $\{w_\varepsilon\}$ *be two bounded sequences of U and W, resp.. If*

$$u_\varepsilon \rightharpoonup u, \quad w_\varepsilon \rightharpoonup w \qquad \text{in } L^2(\mathbf{R}^N)^{N^2}, \tag{5.15}$$

then $u_\varepsilon : w_\varepsilon \to u:w$ *in* $\mathcal{D}'(\mathbf{R}^N)$, *that is,*

$$\int_{\mathbf{R}^N} u_\varepsilon(x):w_\varepsilon(x)\,\theta(x)\,\mathrm{d}x \to \int_{\mathbf{R}^N} u(x):w(x)\,\theta(x)\,\mathrm{d}x \qquad \forall\theta \in \mathcal{D}(\mathbf{R}^N). \tag{5.16}$$

Next we show a two-scale version of this result.

Theorem 5.9 (*Two-Scale Div-Curl Lemma – III*) *Let* $\{u_\varepsilon\}$ *and* $\{w_\varepsilon\}$ *be two bounded sequences of U and W, resp.. If*

$$u_\varepsilon \underset{2}{\rightharpoonup} u, \quad w_\varepsilon \underset{2}{\rightharpoonup} w \qquad \text{in } L^2(\mathbf{R}^N\times\mathcal{Y})^{N^2}, \tag{5.17}$$

then

$$\int_{\mathbf{R}^N} u_\varepsilon(x):w_\varepsilon(x)\,\theta(x)\,\mathrm{d}x \to \int_{\mathbf{R}^N} \hat{u}(x):\hat{w}(x)\,\theta(x)\,\mathrm{d}x$$
$$= \iint_{\mathbf{R}^N\times\mathcal{Y}} u(x,y):w(x,y)\,\theta(x)\,\mathrm{d}x\mathrm{d}y \qquad \forall\theta \in \mathcal{D}(\mathbf{R}^N). \tag{5.18}$$

Proof By (5.17), $u_\varepsilon \rightharpoonup \hat{u}$ and $w_\varepsilon \rightharpoonup \hat{w}$ in $L^2(\mathbf{R}^N)^{N^2}$. Hence by Theorem 5.8

$$\int_{\mathbf{R}^N} u_\varepsilon(x):w_\varepsilon(x)\,\theta(x)\,\mathrm{d}x \to \int_{\mathbf{R}^N} \hat{u}(x):\hat{w}(x)\,\theta(x)\,\mathrm{d}x \qquad \forall\theta \in \mathcal{D}(\mathbf{R}^N). \tag{5.19}$$

By the boundeness of u_ε in U there exists a bounded sequence $\{v_\varepsilon\}$ of $H^1(\mathbf{R}^N)^N$ such that $u_\varepsilon = \nabla^s v_\varepsilon$ for any ε. By Lemma 5.10 below then there exist $v \in H^1(\mathbf{R}^N)^N$ and $v_1 \in L^2\big(\mathbf{R}^N; H^1(\mathcal{Y})^N\big)$ such that

$$u_\varepsilon = \nabla^s v_\varepsilon \underset{2}{\rightharpoonup} u = \nabla^s v + \nabla^s_y v_1 \quad \text{in } L^2(\mathbf{R}^N \times \mathcal{Y})^{N^2},$$

whence $\hat{u} = \nabla^s v$ a.e. in $\mathbf{R}^N$. Moreover, by the boundedness of w_ε in W, $\nabla^s_y \cdot w = 0$ in $\mathcal{D}'(\mathbf{R}^N \times \mathcal{Y})^N$, whence

$$\iint\limits_{\mathbf{R}^N\times\mathcal{Y}} \nabla^s_y v_1(x,y) : w(x,y)\, \theta(x)\, \mathrm{d}x\mathrm{d}y = 0 \qquad \forall \theta \in \mathcal{D}(\mathbf{R}^N).$$

Therefore,

$$\iint\limits_{\mathbf{R}^N\times\mathcal{Y}} u(x,y) : w(x,y)\, \theta(x)\, \mathrm{d}x\mathrm{d}y = \iint\limits_{\mathbf{R}^N\times\mathcal{Y}} [\hat{u}(x) + \nabla^s_y v_1(x,y)] : w(x,y)\, \theta(x)\, \mathrm{d}x\mathrm{d}y$$
$$= \int\limits_{\mathbf{R}^N} \hat{u}(x) \cdot \hat{w}(x)\, \theta(x)\, \mathrm{d}x \quad \forall \theta \in \mathcal{D}(\mathbf{R}^N).$$

(5.18) then follows from (5.19). □

Lemma 5.10 [59] *Let $\{v_\varepsilon\}$ be a bounded sequence of $H^1(\mathbf{R}^N)^N$. Then there exist $v \in H^1(\mathbf{R}^N)^N$ and $v_1 \in L^2\big(\mathbf{R}^N; H^1_*(\mathcal{Y})^N\big)$ such that, as $\varepsilon \to 0$ along the extracted subsequence,*

$$\nabla^s v_\varepsilon \underset{2}{\rightharpoonup} \nabla^s v + \nabla^s_y v_1 \qquad \text{in } L^2(\mathbf{R}^N \times \mathcal{Y})^{N^2}. \tag{5.20}$$

Remarks (i) Theorems 5.4 and 5.9 differ from each other, not only for the different frameworks but also in the following respect: in the former statement u_ε is assumed to be bounded in the domain of a first-order differential operator, whereas in the latter one u_ε is required to be bounded in the range of a first-order differential operator. Actually it is the latter condition that is used in both arguments.
Lemma 5.2 reduces the first hypothesis to the latter condition. On the other hand, there is no strict analog of that lemma in the framework of Theorems 5.9, for it is known that U is just the kernel of a second-order (rather than first-order) linear differential operator, see e.g. [32, p. 105]: for any $w \in L^2_*(\mathcal{Y})^9_s$, $w \in V$ if and only if

$$\frac{\partial^2 w_{i\ell}}{\partial y_j \partial y_m} + \frac{\partial^2 w_{jm}}{\partial y_i \partial y_\ell} - \frac{\partial^2 w_{j\ell}}{\partial y_i \partial y_m} - \frac{\partial^2 w_{im}}{\partial y_j \partial y_\ell} = 0 \qquad \text{for } i,j,\ell,m = 1,2,3.$$

(ii) Theorems 5.1, 5.6 and 5.8 are just basic examples of (single-scale) compensated compactness, and more general statements may be found e.g. in [40,41]. Further two-scale extensions look feasible, but are not addressed here.

5.4 A negative result

The next statement seems to limit the possibilities offered by two-scale compensated compactness.

Proposition 5.11 *Assume that the hypotheses of Theorem 5.4 are fulfilled and that*

$$u_\varepsilon \underset{2}{\rightharpoonup} u, \quad w_\varepsilon \underset{2}{\rightharpoonup} w \quad \text{in } L^2(\mathbf{R}^3 \times \mathcal{Y})^3. \tag{5.21}$$

This does not entail $u_\varepsilon \cdot w_\varepsilon \underset{2}{\rightharpoonup} u \cdot w$ in $\mathcal{D}'(\mathbf{R}^3 \times \mathcal{Y})$; that is, this does not guarantee that

$$\int_{\mathbf{R}^3} u_\varepsilon(x) \cdot w_\varepsilon(x)\, \theta(x, x/\varepsilon)\, \mathrm{d}x \to \iint_{\mathbf{R}^3 \times \mathcal{Y}} u(x,y) \cdot w(x,y)\, \theta(x,y)\, \mathrm{d}x\mathrm{d}y \quad \forall \theta \in \mathcal{D}(\mathbf{R}^3 \times \mathcal{Y}). \tag{5.22}$$

This result was recently proved in [18] by Briane and Casado-Díaz, who provided a counterexample based on the lack of control on oscillations of period $k\varepsilon$ for $k \in \mathbf{N}^N$ with $|k| > 1$. Here, we construct a slightly different counterexample, which involves oscillations of period $\sqrt{\varepsilon}$. Loosely speaking this failure of compactness might be ascribed to the lack of control on the x-derivatives of the two-scale limit functions u and w.

Proof Let us fix any $v \in \mathcal{D}(\mathbf{R}^3)$ ($v \not\equiv 0$), any $c \in \mathbf{R}^3 \setminus \{0\}$, and set

$$f_\varepsilon(x) := \sin(|x|/\sqrt{\varepsilon})v(x), \quad g_\varepsilon := c \sin(|x|/\sqrt{\varepsilon})v(x) \quad \forall x \in \mathbf{R}^3, \forall \varepsilon > 0.$$

Thus,

$$\begin{aligned} &f_\varepsilon \underset{2}{\rightharpoonup} 0 \quad \text{in } L^2(\mathbf{R}^3 \times \mathcal{Y}), \qquad g_\varepsilon \underset{2}{\rightharpoonup} 0 \quad \text{in } L^2(\mathbf{R}^3 \times \mathcal{Y})^3, \\ &f_\varepsilon(x) g_\varepsilon(x) = c\,[\sin(|x|/\sqrt{\varepsilon})]^2 v(x)^2 \rightharpoonup \tfrac{1}{2} c v(x)^2 \not\equiv 0 \quad \text{in } L^1(\mathbf{R}^3)^3. \end{aligned} \tag{5.23}$$

Let us also fix any $\varphi, \psi \in H^1(\mathcal{Y})$ ($\varphi, \psi \not\equiv 0$) and set

$$u_\varepsilon(x) := \varepsilon \nabla[f_\varepsilon(x)\varphi(x/\varepsilon)], \quad w_\varepsilon(x) := \varepsilon \nabla \times [g_\varepsilon(x)\psi(x/\varepsilon)] \quad \text{for a.e. } x \in \mathbf{R}^3.$$

Notice that

$$\begin{aligned} u_\varepsilon(x) &= \varepsilon[\nabla f_\varepsilon(x)]\varphi(x/\varepsilon) + f_\varepsilon(x)\nabla\varphi(x/\varepsilon) \underset{2}{\rightharpoonup} 0 \quad \text{in } L^2(\mathcal{Y})^3, \\ w_\varepsilon(x) &= \varepsilon[\nabla \times g_\varepsilon(x)]\psi(x/\varepsilon) + g_\varepsilon(x) \times \nabla\psi(x/\varepsilon) \underset{2}{\rightharpoonup} 0 \quad \text{in } L^2(\mathcal{Y})^3. \end{aligned} \tag{5.24}$$

The hypotheses of Theorem 5.4 are thus fulfilled. But by (5.23) and (5.24)

$$u_\varepsilon(x) \cdot w_\varepsilon(x) \underset{2}{\rightharpoonup} \tfrac{1}{2} v(x)^2 \nabla\varphi(y) \cdot c \times \nabla\psi(y) \qquad \text{in } L^1(\mathbf{R}^3 \times \mathcal{Y})^3,$$

and in general the latter function does not vanish identically. □

Remarks (i) Although we proved this negative result just in the setting of Theorems 5.4, an analogous counterexample could be constructed in the framework of Proposition 5.6 and of Theorem 5.9.

(ii) Due to the relevance of the div-curl lemma and of other results of compensated compactness, it might be of some interest to establish (5.2) by amending the hypotheses of Theorem 5.4. This is achieved in [63] by replacing the curl and the divergence by corresponding *approximate two-scale operators* (in the sense of [58]). This also provides some control on the x-derivatives of the two-scale limit functions u and w.

5.5 An application

Finally we briefly illustrate an application of Theorem 5.4 and of results of Sects. 2,4. Details may be found in [60–62], where these properties are used to study the homogenization of nonlinear electromagnetic processes and of phase transitions.

Proposition 5.12 (i) *Let $\varphi \in \mathcal{F}$ (cf. (1.8)) be such that*

$$\text{the function } v \mapsto \varphi(v,x,y) \text{ is strictly convex for a.e. } (x,y), \tag{5.25}$$

$$\text{the function } (v,x) \mapsto \varphi(v,x,y) \text{ is lower semicontinuous for a.e. } y, \tag{5.26}$$

$$\exists c_4 \in \mathbf{R}, \exists f \text{ as in (1.4) or (1.5) such that}$$
$$\varphi(v,x,y) + c_4|v|^2 \geq f(x,y) \quad \forall v, \text{ for a.e. } (x,y). \tag{5.27}$$

Let $\{u_\varepsilon\}$ and $\{w_\varepsilon\}$ be two bounded sequences of $L^2_{\text{rot}}(\mathbf{R}^3)^3$ and $L^2_{\text{div}}(\mathbf{R}^3)^3$, resp., such that

$$w_\varepsilon(x) \in \partial\varphi(u_\varepsilon(x), x, x/\varepsilon) \qquad \text{for a.e. } x \in \mathbf{R}^3, \tag{5.28}$$

$$u_\varepsilon \underset{2}{\rightharpoonup} u, \quad w_\varepsilon \underset{2}{\rightharpoonup} w \quad \text{in } L^2(\mathbf{R}^3 \times \mathcal{Y})^3. \tag{5.29}$$

Then

$$w(x,y) \in \partial\varphi(u(x,y),x,y) \quad \text{for a.e. } (x,y) \in \mathbf{R}^3 \times \mathcal{Y}. \tag{5.30}$$

(ii) *If moreover $\varphi(\cdot,x,y)$ is strictly convex for a.e. (x,y), then*

$$u_\varepsilon \underset{2}{\to} u \qquad \text{in } L^2_{\text{loc}}(\mathbf{R}^3 \times \mathcal{Y})^3. \tag{5.31}$$

(iii) *If φ^* also fulfils properties like (5.25) and (5.27) and $\varphi^*(\cdot,x,y)$ is strictly convex for a.e. (x,y), then*

$$w_\varepsilon \underset{2}{\to} w \quad \text{in } L^2_{\text{loc}}(\mathbf{R}^3 \times \mathcal{Y})^3. \tag{5.32}$$

Proof By Theorem 5.4

$$\int_{\mathbf{R}^3} u_\varepsilon(x)\cdot w_\varepsilon(x)\,\theta(x)\,\mathrm{d}x \to \iint_{\mathbf{R}^3\times\mathcal{Y}} u(x,y)\cdot w(x,y)\,\theta(x)\,\mathrm{d}x\mathrm{d}y \quad \forall\theta \in \mathcal{D}(\mathbf{R}^3).$$

By the Remark (iii) after Theorem 2.1, (5.30) then follows and, defining Φ_ε and $\bar{\Phi}$ as in (1.11) for any bounded domain $\Omega \subset \mathbf{R}^3$,

$$\Phi_\varepsilon(u_\varepsilon) \to \bar{\Phi}(u), \qquad \Phi^*_\varepsilon(u_\varepsilon) \to \bar{\Phi}^*(u).$$

Corollary 4.5 then yields $u_\varepsilon \underset{2}{\to} u$ in $L^2(\Omega \times \mathcal{Y})^3$, that is (5.31). The same argument applies to φ^* and provides (5.32). □

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