CLASSICAL SPACES — A. Visintin — 2017

Contents: 1. Hölder spaces. 2. Regularity of Euclidean domains.

Note. The bullet • and the asterisk $*$ are respectively used to indicate the most relevant results and complements. The symbol [] follows statements the proof of which has been omitted, whereas [Ex] is used to propose the reader to fill in the argument as an exercise.

Here are some abbreviations that are used throughout:

a.a. = almost any; resp. = respectively; w.r.t. = with respect to.

p': conjugate exponent of p, that is, $p' := p/(p-1)$ if $1 < p < +\infty$, $1' := \infty$, $\infty' := 1$.

$\mathbf{N}_0 := \mathbf{N} \setminus \{0\}$; $\mathbf{R}^N_+ := \mathbf{R}^{N-1} \times]0, +\infty[$. $|A|$:= measure of the measurable set A.

1. Hölder spaces

First we state a result, which provides a procedure to construct normed spaces, and is easily extended from the product of two spaces to that of a finite family. This technique is very convenient, and we shall repeatedly use it.

Proposition 1.1 *Let A and B be two normed spaces and $p \in [1, +\infty]$. Then:*

(i) The vector space $A \times B$ is a normed space equipped with the p-norm *of the product:*

$$\begin{aligned} &\|(v,w)\|_p := \left(\|v\|_A^p + \|w\|_B^p\right)^{1/p} \qquad \text{if } 1 \le p < +\infty, \\ &\|(v,w)\|_\infty := \max\{\|v\|_A, \|w\|_B\}. \end{aligned} \tag{1.1}$$

Let us denote this space by $(A \times B)_p$. These norms are mutually equivalent.

(ii) If A and B are Banach spaces, then $(A \times B)_p$ is a Banach space.

(iii) If A and B are separable (reflexive, resp.), then $(A \times B)_p$ is also separable (reflexive, resp.).

(iv) If A and B are uniformly convex and $1 < p < +\infty$, then $(A \times B)_p$ is uniformly convex.

(v) If A and B are inner-product spaces (Hilbert spaces, resp.), equipped with the scalar product $(\cdot,\cdot)_A$ and $(\cdot,\cdot)_B$, resp., then $(A \times B)_2$ is an inner-product space (a Hilbert space, resp.) equipped with the scalar product

$$\big((u_1, v_1), (u_2, v_2)\big)_2 := (u_1, u_2)_A + (v_1, v_2)_B \qquad \forall (u_1, v_1), (u_2, v_2) \in (A \times B)_2.$$

$\|(\cdot,\cdot)\|_2$ is then the corresponding Hilbert norm.

(vi) $F \in (A \times B)'_p$ (the dual space of $(A \times B)_p$) iff there exists a (unique) pair $(g, h) \in A' \times B'$ such that

$$\langle F, (u, v)\rangle = {}_{A'}\langle g, u\rangle_A + {}_{B'}\langle h, v\rangle_B \qquad \forall (u, v) \in (A \times B)_p. \tag{1.2}$$

In this case

$$\|F\|_{(A\times B)'_p} = \|(g, h)\|_{(A'\times B')_{p'}}. \tag{1.3}$$

The mapping $(A \times B)'_p \to (A' \times B')_{p'} : F \mapsto (g, h)$ is indeed an isometric surjective isomorphism.

(We omit the simple argument, that rests upon classical properties of Banach spaces.)

A variant of the above result consists in equipping Banach spaces with the graph norm, associated to a linear operator.

Spaces of Continuous Functions. Throughout this section, by K we shall denote a compact subset of $\mathbf{R}^N$, and by Ω a (possibly unbounded) domain of $\mathbf{R}^N$.

The linear space of continuous functions $K \to \mathbf{C}$, denoted by $C^0(K)$, is a Banach space equipped with the sup-norm $p_K(v) := \sup_{x\in K} |v(x)|$ (this is even a maximum). The corresponding topology induces the uniform convergence.

The linear space of continuous functions $\Omega \to \mathbf{C}$, denoted by $C^0(\Omega)$, is a locally convex Fréchet space equipped with a family of seminorms: $\{p_{K_n}\}$. Here $\{K_n \subset\subset \Omega : n \in \mathbf{N}\}$ is a nondecreasing sequence of compact sets that invades Ω, namely $\bigcup_{n\in\mathbf{N}} K_n = \Omega$. (1) For instance, one may take

$$K_n = \{x \in \Omega : |x| \le n, \operatorname{dist}(x, \partial\Omega) \ge 1/n\} \qquad \forall n \in \mathbf{N}.$$

This topology induces the **locally uniform convergence.**

The linear space of *bounded continuous* functions $\Omega \to \mathbf{C}$, denoted by $C^0_b(\Omega)$, is also a Banach space equipped with the sup-norm $p_\Omega(v) := \sup_{x\in\Omega} |v(x)|$, and is thus a subspace of $C^0(\Omega)$.

As Ω is a metric space, we may also deal with uniformly continuous functions. In the literature, the linear space of *bounded and uniformly continuous functions* $\Omega \to \mathbf{C}$ is often denoted by $BUC(\Omega)$ or $C^0(\bar{\Omega})$, as these functions have a unique continuous extension to $\bar{\Omega}$. The latter notation is customary but slightly misleading: for instance,

$$C^0\big(\overline{\mathbf{R}^N}\big) \ne C^0\big(\mathbf{R}^N\big) \tag{1.4}$$

although obviously $\overline{\mathbf{R}^N} = \mathbf{R}^N$. If Ω is bounded then $K := \bar{\Omega}$ is compact, and $C^0(\bar{\Omega})$ may be identified with the space $C^0(K)$ that we defined above. Notice that $C^0(\bar{\Omega})$ $(= BUC(\Omega))$ is a closed subspace of $C^0_b(\Omega)$ for any domain Ω of $\mathbf{R}^N$, and the inclusion is strict; for instance,

$$\{x \mapsto \sin(1/x)\} \in C^0_b(]0,1[) \setminus C^0(\overline{]0,1[}), \qquad \{x \mapsto \sin(x^2)\} \in C^0_b\big(\mathbf{R}\big) \setminus C^0\big(\bar{\mathbf{R}}\big). \tag{1.5}$$

In this section we shall see several other spaces *over* $\bar{\Omega}$ that are included into the corresponding space *over* Ω.

Spaces of Hölder-Continuous Functions. Let us fix any $\lambda \in]0,1]$. The bounded continuous functions $v : \Omega \to \mathbf{C}$ such that

$$p_{\Omega,\lambda}(v) := \sup_{x,y\in\Omega, x\ne y} \frac{|v(x) - v(y)|}{|x-y|^\lambda} < +\infty \tag{1.6}$$

are said **Hölder-continuous** of index (or exponent) λ, and form a linear space that we denote by $C^{0,\lambda}(\bar{\Omega})$ and equip with the graph norm. If $\lambda = 1$ these functions are said to be **Lipschitz continuous.** Obviously Hölder functions are uniformly continuous, so $C^{0,\lambda}(\bar{\Omega}) \subset C^0(\bar{\Omega})$. The functional $p_{\Omega,\lambda}$ is a seminorm on $C^0(\Omega)$. [Ex]

Proposition 2.1 *For any $\lambda \in]0,1]$, $C^{0,\lambda}(\bar{\Omega})$ is a Banach space when equipped with the norm $p_\Omega + p_{\Omega,\lambda}$.*

The functions $v : \Omega \to \mathbf{C}$ that are Hölder-continuous of index λ in any compact set $K \subset \Omega$ are called **locally Hölder-continuous.** They form a Fréchet space, denoted by $C^{0,\lambda}(\Omega)$, when equipped with the family of seminorms $\{p_K + p_{K,\lambda} : K \subset\subset \Omega\}$. Notice that

$$C^{0,\lambda}(\bar{\Omega}) \subset C^{0,\nu}(\bar{\Omega}) \qquad \forall \lambda, \nu \in]0,1], \nu < \lambda, [Ex] \tag{1.7}$$

with continuous injections. (2) For instance for any $\lambda \in]0,1]$, the function $x \mapsto |x|^\lambda$ is an element of $C^{0,\lambda}(\mathbf{R})$, but not of $C^{0,\nu}(\mathbf{R})$ for any $\nu > \lambda$, and not of $C^{0,\lambda}(\bar{\mathbf{R}})$ (here also the traditional notation is not very helpful).

(1) We remind the reader that Fréchet spaces are linear spaces that are also complete metric spaces and such that the linear operations are continuous.

(2) All the injections between function spaces will be continuous; so we shall not point it out any more.

Notice that $\bigcup_{\lambda\in]0,1]} C^{0,\lambda}([0,1]) \neq C^0([0,1])$; e.g., the function

$$u(x) := 1/\log(x/2) \qquad \forall x \in \,]0,1], \qquad u(0) = 0 \tag{1.8}$$

is continuous, but is not Hölder-continuous for any index λ. [Ex]

On the other hand $\bigcap_{\lambda\in]0,1[} C^{0,\lambda}([0,1]) \neq C^{0,1}([0,1])$; e.g., the function

$$u(x) := x\log(x/2) \qquad \forall x \in \,]0,1], \qquad u(0) = 0 \tag{1.8$'$}$$

is Hölder-continuous for any index λ, but is not Lipschitz-continuous. [Ex]

Spaces of Differentiable Functions. Let us assume that Ω and λ are as above and that $m \in \mathbf{N}$. Let us recall the multi-index notation, and set $D_i := \partial/\partial x_i$ for $i = 1, ..., N$.

We claim that the functions $\Omega \to \mathbf{C}$ that are m-times differentiable and are bounded and continuous jointly with their derivatives up to order m form a Banach space, denoted by $C^m_b(\Omega)$, when equipped with the norm

$$p_{\Omega,m}(v) := \sum_{|\alpha|\leq m} \sup_{x\in\Omega} |D^\alpha v(x)| \qquad \forall m \in \mathbf{N}. \tag{1.9}$$

This is easily seen because, setting

$$k(m) := \frac{(N+m)!}{N!m!} = \text{ number of the multi-indices } \alpha \in \mathbf{N}^N \text{ such that } |\alpha| \leq m, \tag{1.10}$$

the mapping $C^m_b(\Omega) \to C^0_b(\Omega)^{k(m)} : v \mapsto \{D^\alpha v : |\alpha| \leq m\}$ is an isomorphism between $C^m_b(\Omega)$ and its range. Indeed, if $D^\alpha u_n \to u_\alpha$ uniformly in Ω for any $\alpha \in \mathbf{N}^N$ such that $|\alpha| \leq m$, then $u_\alpha = D^\alpha u_0$; thus $u_n \to u_0$ in $C^m_b(\Omega)$. For instance, $C^1_b(\mathbf{R}^2)$ is isomorphic to $\{(w, w_1, w_2) \in C^0_b(\mathbf{R}^2)^3 : w_i = \partial w/\partial x_i$ in $\mathbf{R}^2$, for $i = 1, 2\}$. Here one may define a norm via Proposition 1.1.

The functions $\Omega \to \mathbf{C}$ that are continuous with their derivatives up to order m form a locally convex Fréchet space equipped with the family of seminorms $\{p_{K,m} : K \subset\subset \Omega\}$. This space is denoted by $C^m(\Omega)$ (or by $\mathcal{E}^m(\Omega)$).

The linear space of the functions $\Omega \to \mathbf{C}$ that are bounded with their derivatives up to order m, and whose derivatives of order m are Hölder-continuous of index λ, may be equipped with the norm

$$p_{\Omega,m,\lambda}(v) := \sum_{|\alpha|\leq m} \sup_{x\in\Omega} |D^\alpha v(x)| + \sum_{|\alpha|=m} p_{\Omega,\lambda}(D^\alpha v), \tag{1.11}$$

with $p_{\Omega,\lambda}$ as above. By Proposition 1.1, this is a Banach space, that we denote by $C^{m,\lambda}(\bar{\Omega})$.

The linear space of the functions $\Omega \to \mathbf{C}$ whose derivatives up to order m are Hölder-continuous of index λ in any compact set $K \subset \Omega$ can be equipped with the family of seminorms $\{p_{K,m,\lambda} : K \subset\subset \Omega\}$. This is a locally convex Fréchet space, denoted by $C^{m,\lambda}(\Omega)$.

It is also convenient to set

$$\left.\begin{aligned} &C^{m,0}(\bar{\Omega}) = C^m(\bar{\Omega}) := \{v \in C^m(\Omega) : D^\alpha v \in C^0(\bar{\Omega}), \forall\alpha, |\alpha| \leq m\},\\ &C^{m,0}(\Omega) = C^m(\Omega),\\ &C^\infty(\bar{\Omega}) = \bigcap_{m\in\mathbf{N}} C^m(\bar{\Omega}), \qquad C^\infty(\Omega) = \bigcap_{m\in\mathbf{N}} C^m(\Omega). \end{aligned}\right. \qquad \forall m \in \mathbf{N}. \tag{1.12}$$

In passing notice that $C^\infty(\bar{\Omega}) \cap L^p(\Omega)$ is a dense subset of $L^p(\Omega)$ for any $p \in [1, +\infty[$. This may be proved by convolution with a regularizing kernel.

Some Embeddings. We say that a topological space A is **embedded** into another topological space B whenever $A \subset B$ and the injection operator $A \to B$ (which is then called an *embedding)* is continuous. [3]

For any $m \in \mathbf{N}$, some embeddings are obvious within the class of C^m-spaces,

$$m \geq \ell \quad \Rightarrow \quad C^m(\bar{\Omega}) \subset C^\ell(\bar{\Omega}), \tag{1.13}$$

as well within that of $C^{m,\lambda}$-spaces:

$$\nu \leq \lambda \quad \Rightarrow \quad C^{m,\lambda}(\bar{\Omega}) \subset C^{m,\nu}(\bar{\Omega}) \qquad \forall m. \tag{1.14}$$

Concerning inclusions between spaces of the two classes, apart from obvious ones like $C^{m,\lambda}(\bar{\Omega}) \subset C^m(\bar{\Omega})$, some regularity is needed for the domain. [4]

Proposition 2.2 *Let either* $\Omega = \mathbf{R}^N$, *or* $\Omega \in C^{0,1}$ [5] *and bounded. Then*

$$C^{m+1}(\bar{\Omega}) \subset C^{m,\lambda}(\bar{\Omega}) \qquad \forall m, \forall \lambda \in [0,1].[] \tag{1.15}$$

From this inclusion it easily follows that

$$C^{m_2,\lambda_2}(\bar{\Omega}) \subset C^{m_1,\lambda_1}(\bar{\Omega}) \qquad \text{if } m_1 < m_2, \forall \lambda_1, \lambda_2 \in [0,1]. \tag{1.16}$$

A Counterexample. The next example shows that some regularity is actually needed for (1.15) to hold. Let us set

$$\Omega := \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 < 1,\ y < |x|^{1/2}\}. \tag{1.17}$$

Of course $\Omega \in C^{0,1/2} \setminus C^{0,\nu}$ for any $\nu > 1/2$. [5] For any $a \in \,]1,2[$, the function $v : \Omega \to \mathbf{R} : (x,y) \mapsto (y^+)^a \ \mathrm{sign}(x)$ belongs to $C^1(\bar{\Omega}) \setminus C^{0,\nu}(\bar{\Omega})$ for any $\nu > a/2$. [Ex]

We just considered embeddings for Banach spaces "on $\bar{\Omega}$". It is easy to see that these results yield analogous statements for the corresponding Fréchet spaces "on Ω".

Synthesis. For any domain $\Omega \subset \mathbf{R}^N$, we have introduced the Banach spaces

$$C^0_b(\Omega), \quad C^0(\bar{\Omega}) \ (= BUC(\Omega)), \quad C^{0,\lambda}(\bar{\Omega}) \quad \forall \lambda \in \,]0,1],$$

and the Fréchet spaces

$$C^0(\Omega), \quad C^{0,\lambda}(\Omega) \quad \forall \lambda \in \,]0,1].$$

For any $m \in \mathbf{N}$, assuming that Ω is regular enough (e.g., it coincides with the interior of $\bar{\Omega}$), we have introduced the Banach spaces

$$C^m_b(\Omega), \quad C^m(\bar{\Omega}), \quad C^{m,\lambda}(\bar{\Omega}) \quad \forall \lambda \in \,]0,1],$$

and the Fréchet spaces

$$C^m(\Omega), \quad C^{m,\lambda}(\Omega) \quad \forall \lambda \in \,]0,1], \quad C^\infty(\bar{\Omega}), \quad C^\infty(\Omega).$$

Exercises. 1. Show that $\bigcup_{\lambda \in]-1,1[} C^{0,\lambda}(\Omega) \neq C^0(]-1,1[)$.

2. Show that $\bigcap_{\lambda \in]0,1[} C^{0,\lambda}(]-1,1[) \neq C^{0,1}(]-1,1[)$.

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(3) We shall use some notions of regularity of domains that are defined in the next section ...

(4) The regularity of domains is defined in the next section.

(5) See the definition in the next section ...

(5) According to the definition of the next section ...

2. Regularity of Euclidean Domains

Open subsets of $\mathbf{R}^N$ may be very irregular; e.g., consider $\bigcup_{n\in\mathbf{N}} B(q_n, 2^{-n})$, where $\{q_n\}$ is an enumeration of $\mathbf{Q}^N$. This set is open and has finite measure, but it is obviously dense in $\mathbf{R}^N$.

Several notions may be used to define the regularity of a Euclidean open set Ω, or rather that of its boundary Γ. Here we just introduce two of them.

Open Sets of Class $C^{m,\lambda}$. Let us denote by $B_N(x,R)$ the ball of $\mathbf{R}^N$ of center x and radius R. For any $m \in \mathbf{N}$ and $0 \le \lambda \le 1$, we say that Ω is of class $C^{m,\lambda}$ (here $C^{m,0}$ stays for C^m), and write $\Omega \in C^{m,\lambda}$, iff for any $x \in \Gamma$ there exist:

(i) two positive constants $R = R_x$ and δ_x,
(ii) a mapping $\varphi_x : B_{N-1}(0,R) \to \mathbf{R}$ of class $C^{m,\lambda}$,
(iii) a Cartesian system of coordinates $y_1, ..., y_N$,

such that the point x is characterized by $y_1 = ... = y_N = 0$ in this Cartesian system, and, for any $y' := (y_1, ..., y_{N-1}) \in B_{N-1}(0,R)$,

$$\begin{aligned}
&y_N = \varphi(y') && \Rightarrow && (y', y_N) \in \Gamma, \\
&\varphi(y') < y_N < \varphi(y') + \delta && \Rightarrow && (y', y_N) \in \Omega, \\
&\varphi(y') - \delta < y_N < \varphi(y') && \Rightarrow && (y', y_N) \notin \bar{\Omega}.
\end{aligned} \tag{2.1}$$

This means that Γ is an $(N-1)$-dimensional manifold (without boundary) of class $C^{m,\lambda}$, and that Ω locally stays only on one side of Γ. We say that Ω is a continuous (Lipschitz, Hölder, resp.) open set whenever it is of class C^0 ($C^{0,1}$, $C^{0,\lambda}$ for some $\lambda \in]0,1]$, resp.). [6]

For instance, the domain

$$\Omega_{a,b,\lambda} := \{(x,y) \in \mathbf{R}^2 : x > 0, ax^{1/\lambda} < y < bx^{1/\lambda}\} \qquad \forall\lambda \le 1, \forall a,b \in \mathbf{R}, a < b \tag{2.2}$$

is of class $C^{0,\lambda}$ iff $a < 0 < b$. [Ex]

We say that Ω is **uniformly of class** $C^{m,\lambda}$ iff

$$\Omega \in C^{m,\lambda}, \quad \inf_{x\in\Gamma} R_x > 0, \quad \inf_{x\in\Gamma} \delta_x > 0, \quad \sup_{x\in\Gamma} \|\varphi_x\|_{C^{m,\lambda}(B_{N-1}(x,R))} < +\infty. \tag{2.3}$$

For instance, by compactness, this is fulfilled by any bounded domain Ω of class $C^{m,\lambda}$. [Ex]

Cone Property. The above notion of regularity of open sets is not completely satisfactory, as it excludes sets like e.g. a ball with deleted center. We then introduce a further regularity notion.

We say that Ω has the cone property iff there exist $a, b > 0$ such that, defining the finite open cone

$$C_{a,b} := \left\{x := (x_1, ..., x_N) : x_1^2 + ... + x_{N-1}^2 \le b x_N^2,\ 0 < x_N < a\right\},$$

any point of Ω is the vertex of a cone contained in Ω and congruent to $C_{a,b}$. For instance, any ball with deleted center and the plane sets

$$\begin{aligned}
&\Omega_1 := \{(\rho,\theta) : 1 < \rho < 2, 0 < \theta < 2\pi\} \qquad (\rho,\theta : \text{polar coordinates}), \\
&\Omega_2 := \{(x,y) \in \mathbf{R}^2 : |x|, |y| < 1, x \ne 0\}
\end{aligned} \tag{2.4}$$

have the cone property, but are not of class C^0. [Ex]

Proposition 2.1 *Any bounded Lipschitz domain has the cone property. [Ex]*

For unbounded Lipschitz domains this may fail; $\Omega := \{(x,y) \in \mathbf{R}^2 : x > 1, 0 < y < 1/x\}$ is a counterexample. Note that a domain Ω is bounded whenever it has the cone property and $|\Omega| < +\infty$. []

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[6] This notation refers to the Hölder spaces, that are defined half-a-page ahead ...