Elements of Convex Analysis

1 The Legendre Transform. omissis

2 Convex Lower Semicontinuous Functions.

We deal with a real Banach space B, although most of the results of this chapter also hold for any real separated locally convex space. By $\langle \cdot, \cdot \rangle$ we denote the duality pairing between B and its topological dual B^*, as well as that between B^* and B. If not otherwise specified, we refer to the *strong* topology.

Proposition 2.0 *(i) Any set $K \subset B$ is closed and convex iff I_K is closed and convex.*

(ii) Any function $f : B \to]-\infty, +\infty]$ is lower semicontinuous and convex iff $\operatorname{epi}(f)$ *is closed and convex.*

(iii) If $\{K_i\}_{i \in I}$ is a family of closed convex subsets of B, then $\bigcap_i K_i$ is closed and convex.

(iv) If $\{f_i\}_{i \in I}$ is a family of lower semicontinuous convex functions $B \to]-\infty, +\infty]$, then their upper hull $f(\cdot) := \sup_i f_i(\cdot)$ is lower semicontinuous and convex.

• **Proposition 2.1** *(i) Any convex set $K \subset B$ is strongly closed iff it is weakly closed.*

(ii) Any convex function $F : B \to]-\infty, +\infty]$ is strongly lower semicontinuous iff it is weakly lower semicontinuous.

Proof. Let K be a strongly closed subset of B. If $u \notin K$, then by a corollary of the Hahn-Banach theorem the compact set $\{u\}$ can be strongly separated from K by means of a closed hyperplane. Hence u does not belong to the weak closure of K, and this proves the first statement.

The second statement then follows from part (ii) of Proposition 2.0. $\square$

Remark. Of course the same result also holds for B^*. However, if B is not reflexive, a subset of B^* may be convex and weakly (equivalently, strongly) closed without being weakly star closed. For instance, this occurs for the half-space $\{u^* \in B^* : \langle u^{**}, u^* \rangle \geq 0\}$ for any $u^{**} \in B^{**} \setminus B$. (1)

In order to avoid any reference to B^{**}, we deal with the duality between B^* and B, rather than that between B^* and B^{**}. This can be expressed by saying that B and B^* are regarded as *spaces in duality.* We shall then often use the weak star topology instead of the weak one.

The argument of Proposition 2.1 also yields the following result.

• **Proposition 2.2** *(i) Any set $K \subset B$ ($K \neq B$) is convex and closed iff it is the intersection of a (nonempty) family of closed half-spaces.*

(ii) Any function $F : B \to]-\infty, +\infty]$ is lower semicontinuous and convex iff it is the upper hull of a (nonempty) family of continuous affine functions. (2) *[]*

(Part (i) easily follows from the Hahn-Banach theorem.)

We denote by $\Gamma(B)$ the class of convex lower semicontinuous functions $B \to]-\infty, +\infty]$, and by $\Gamma_0(B)$ the subclass of functions not identically equal to $+\infty$.

For any set $K \subset B$, the smallest closed convex subset of B which contains K is called the *closed convex hull* of K. It coincides with the intersection of all the closed convex subsets of B which contain K. By Proposition 2.2(i), it also coincides with the intersection of the closed half-spaces which contain K. As the closure of any convex set is convex, that hull also coincides with $\overline{\operatorname{co}}(K)$, namely, the closure of the convex hull of K. (But it may not coincide with the convex hull of the

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(1) Here $\langle \cdot, \cdot \rangle$ represents the duality pairing between B^{**} and B^*.

(2) In passing note that the convex set B is the intersection of the empty family, and $F \equiv -\infty$ is the upper hull of the empty family of functions.

closure of K, for this may not be closed. The set $K := \{(0,0)\} \cup \{(x,y) \in (\mathbf{R}^+)^2 : xy \geq 1\}$ is a counterexample.)

Similarly, let us consider any function $F : B \to]-\infty, +\infty]$ which has a convex and lower semicontinuous lower bound. By Proposition 2.2(ii), F has then a continuous affine lower bound, and the upper hull of these lower bounds is the largest lower bound of F in $\Gamma(B)$. It is called the *Γ-regularized function* of F, and its epigraph coincides with the closed convex hull of the epigraph of F. []

Regularity Properties of Convex Functions. Convexity is a source of regularity, as it is shown by the following classical result.

*** Theorem 2.2'.** *(Alexandrov) If $F : \mathbf{R}^N \to \mathbf{R}$ is convex, then it is twice differentiable a.e. in $\mathbf{R}^N$. []*

*** Theorem 2.3** *Let $F : B \to]-\infty, +\infty]$ be convex and upperly bounded in a (nonempty) open set $A \subset B$. Then F is locally Lipschitz continuous in* $\mathrm{int}(\mathrm{Dom}(F))$. [2] *[]*

Whenever F is convex and continuous at a point, the latter result applies.

Corollary 2.4 *If $F : \mathbf{R}^N \to]-\infty, +\infty]$ is convex, then it is locally Lipschitz continuous in* $\mathrm{int}(\mathrm{Dom}(F))$.

Proof. $\mathrm{int}(\mathrm{Dom}(F))$ contains an *N-simplex* S, i.e., a set of convex combinations of $N+1$ affinely independent points of $\mathrm{int}(\mathrm{Dom}(F))$. By the convexity, F is bounded in the interior of S, which is nonempty. It then suffices to apply the latter theorem. □

• **Corollary 2.5** *If $F \in \Gamma_0(B)$, then it is locally Lipschitz continuous in* $\mathrm{int}(\mathrm{Dom}(F))$. *[]*

Exercises. (i) Prove that the interior and the closure of any convex subset of B are convex.

(ii) Show that $\mathrm{co}(\bar{K}) \subset \overline{\mathrm{co}}(K)$. Check that the opposite inclusion fails for $K := (\mathbf{R} \times \{1\}) \cup \{(0,0)\}$.

(iii) Let $F : B \to]-\infty, +\infty]$ be convex. Show that F is continuous at $u_0 \in \mathrm{Dom}(B)$ whenever it is upper semicontinuous at the same point u_0.

Hint: Apply Theorem 2.3.

(iv) Let $F : B \to]-\infty, +\infty]$ be convex. Show that any point of relative minimum for F is also of absolute minimum.

(v) Prove that a convex subset of a Banach space is weakly closed iff it is sequentially weakly closed, and that a convex function is lower semicontinuous iff it is sequentially lower semicontinuous.
□

3 The Fenchel Transform.

Let $F : B \to]-\infty, +\infty]$ be *proper* (i.e., with nonempty domain). We define the *conjugate function* [2]

$$F^*(u^*) := \sup_{u \in B} \{\langle u^*, u\rangle - F(u)\} \ \big(= \sup\{\langle u^*, u\rangle - r : (u,r) \in \mathrm{epi}(F)\}\big) \qquad \forall u^* \in B^*. \tag{3.1}$$

As B^* is a Banach space, for any proper function $G : B^* \to]-\infty, +\infty]$, the conjugate function $G^* : B^{**} \to]-\infty, +\infty]$ might be defined by using the duality pairing between B^* and its dual B^{**}.

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(2) By $\mathrm{int}(A)$ we denote the interior of the set A.

(2) This has nothing to do with the transposed L^* of a linear operator L.

However, it seems more convenient to deal with the duality pairing between B^* and B, as we did above. We then restrict G^* to B: [3]

$$G^*(u) := \sup_{u^* \in B^*} \{\langle u^*, u\rangle - G(u^*)\} \qquad \forall u \in B. \tag{3.2}$$

If F is as above and F^* is also proper, we introduce the *biconjugate function* of F:

$$F^{**}(u) := \sup_{u^* \in B^*} \{\langle u^*, u\rangle - F^*(u^*)\} \ \big(= \sup\{\langle u^*, u\rangle - r : (u^*, r) \in \mathrm{epi}(F^*)\}\big) \qquad \forall u \in B. \tag{3.3}$$

Similarly, if G^* is proper, we define the biconjugate function of a proper function $G : B^* \to]-\infty, +\infty]$:

$$G^{**}(u^*) := \sup_{u \in B}\{\langle u^*, u\rangle - G^*(u)\} \qquad \forall u^* \in B^*. \tag{3.4}$$

Note that

$$F^*(0) = -\inf_{u \in B} F(u), \qquad F^{**}(0) = -\inf_{u^* \in B^*} F^*(u). \tag{3.5}$$

For any proper function $F : B \to]-\infty, +\infty]$, F^* is proper iff F has a continuous affine lower bound.

We may then summarize the above definitions as follows:

(i) If $F : B \to]-\infty, +\infty]$ is proper, then $F^* : B^* \to]-\infty, +\infty]$ is defined as in (3.1).

(ii) If $G : B^* \to]-\infty, +\infty]$ is proper, then $G^* : B^{**} \to]-\infty, +\infty]$ is defined. Most often one restricts G^* to B.

(iii) If $F^* : B^* \to]-\infty, +\infty]$ is proper (i.e., if F has a continuous affine lower bound), then $(F^*)^* : B^{**} \to]-\infty, +\infty]$ is defined. Most often one restricts $(F^*)^*$ to B; we set

$$F^{**} := (F^*)^*\big|_B.$$

(iv) If $G^*\big|_B : B \to]-\infty, +\infty]$ is proper (i.e., if G has a continuous affine lower bound), then $(G^*\big|_B)^* : B^* \to]-\infty, +\infty]$ is defined.

Here is an application of the above setting to economics.

"If we interpret the vector space B as a *commodity space* and accordingly its dual B^* as a *price space,* and if we interpret $F : B \to]-\infty, +\infty]$ as a *cost function* that associates to every commodity $u \in B$ its cost $F(u) \in]-\infty, +\infty]$, then the conjugate function F^* can be interpreted as a *profit function* that associates to every price $u^* \in B^*$ the maximum profit $F^*(u^*) = \sup_{u \in B}\{\langle u^*, u\rangle - F(u)\}$ (since $\langle u^*, u\rangle$ is the value of u when the price system u^* prevails)." [4]

• **Theorem 3.1** *Let $F : B \to]-\infty, +\infty]$ be proper and admit a continuous affine lower bound. Then:*

(i) $F^* \in \mathit{\Gamma}_0(B^*)$ and $F^{**} \in \mathit{\Gamma}_0(B)$. F^* is even weakly star lower semicontinuous.

(ii) (Fenchel-Moreau theorem) *F^{**} is the $\mathit{\Gamma}$-regularized function of F; that is,*

$$F^{**}(u) = \sup\{\langle u^*, u\rangle + \alpha : u^* \in B^*, \alpha \in \mathbf{R}, \langle u^*, u\rangle + \alpha \le F(u)\} \qquad \forall u \in B, \tag{3.5$'$}$$

or also $\mathrm{epi}(F^{**}) = \overline{\mathrm{co}}(\mathrm{epi}(F))$. *[Therefore $F^{**} = F$ whenever $F \in \mathit{\Gamma}_0(B)$.]*

(iii) (Fenchel's inequality and dual Fenchel's inequality)

$$\begin{aligned} &\langle u^*, u\rangle \le F(u) + F^*(u^*) \qquad \forall u \in \mathrm{Dom}(F), \forall u^* \in \mathrm{Dom}(F^*),\\ &\langle u^*, u\rangle \le F^{**}(u) + F^*(u^*) \qquad \forall u \in \mathrm{Dom}(F^{**}), \forall u^* \in \mathrm{Dom}(F^*). \end{aligned} \tag{3.6}$$

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(3) No confusion should arise by denoting this restriction by G^*.

(4) From J.-P. Aubin: Applied Functional Analysis, 1979, p. 211.

Proof. (i) is a consequence of part (ii) of Proposition 2.2.

Let us now come to the proof of (3.5'). Although in this formula u is kept fixed and u^*, α are varied, here we fix any $u^* \in \mathrm{Dom}(F^*)$ and $\alpha \in \mathbf{R}$, and let u vary. By definition of $F^*(u^*)$ we have

$$\langle u^*, u\rangle + \alpha \le F(u) \quad \forall u \in B \quad \Leftrightarrow \quad \alpha \le -F^*(u^*).$$

That is, the function $L_{u^*} : u \mapsto \langle u^*, u\rangle - F^*(u^*)$ is a continuous and affine lower bound of F, and its constant term, $-F^*(u^*)$, is maximal in the family of these lower bounds. This family of functions is parameterized by u^*; its upper hull, F^{**}, is the Γ-regularized function of F, by definition of Γ-regularization. (ii) thus holds.

The inequalities (3.6) directly follows from the definitions of F^* and F^{**}. □

* **Proposition 3.2** *Let X_1, X_2 be vector spaces over* $\mathbf{R}$ *and $f : X_1 \times X_2 \to]-\infty, +\infty]$. Let us define the* infimal value function *$g(u) := \inf_{v\in X_2} f(u, v)$ for any $u \in X_1$.* [5] *If f is convex (quasi-convex, resp.) then g is also convex (quasi-convex, resp.).*

Proof. For $i = 1, 2$, let us fix any $u_i \in \mathrm{Dom}(g)$, any $v_i \in \mathrm{Dom}(f(u_i, \cdot))$, and any $\lambda \in]0, 1[$. If f is convex we have

$$\begin{aligned} g(\lambda u_1 + (1-\lambda)u_2) &\le f(\lambda u_1 + (1-\lambda)u_2, \lambda v_1 + (1-\lambda)v_2) \\ &= f(\lambda(u_1, v_1) + (1-\lambda)(u_2, v_2)) \le \lambda f(u_1, v_1) + (1-\lambda)f(u_2, v_2). \end{aligned}$$

By taking the infimum with respect to (v_1, v_2) we then get $g(\lambda u_1+(1-\lambda)u_2) \le \lambda g(u_1)+(1-\lambda)g(u_2)$.

Similarly, if f is quasi-convex we have

$$\begin{aligned} g(\lambda u_1 + (1-\lambda)u_2) &\le f(\lambda u_1 + (1-\lambda)u_2, \lambda v_1 + (1-\lambda)v_2) \\ &= f(\lambda(u_1, v_1) + (1-\lambda)(u_2, v_2)) \le \max\{f(u_1, v_1), f(u_2, v_2)\}. \end{aligned}$$

By taking the infimum with respect to (v_1, v_2) we get $g(\lambda u_1 + (1-\lambda)u_2) \le \max\{g(u_1), g(u_2)\}$. □

* **The Infimal Convolution.** Let X be a vector space over $\mathbf{R}$. For any $F, G : X \to]-\infty, +\infty]$, we define the *infimal convolution* of F and G by

$$(F\nabla G)(u) := \inf_{v,w\in X: v+w=u} \{F(v) + G(w)\} \; \big(= \inf_{v\in X} \{F(v) + G(u-v)\}\big) \qquad \forall u \in X. \tag{3.7}$$

(The analogy with the usual convolution product is obvious.) This operation is commutative and associative. [Ex] We have [Ex]

$$\mathrm{Dom}(F\nabla G) = \mathrm{Dom}(F) + \mathrm{Dom}(G), \qquad F\nabla I_{\{0\}} = F \qquad \forall F, G : X \to]-\infty, +\infty],$$

$$\mathrm{Dom}(F\nabla G) = \mathrm{Dom}(F) + \mathrm{Dom}(G), \qquad I_A\nabla I_B = I_{A+B} \qquad \forall A, B \subset X,$$

$$\text{if } \; F_1 \le F_2 \; \text{ in } X, \text{ then } \; F_1\nabla G \le F_2\nabla G \; \text{ in } X \qquad \forall F_1, F_2, G : X \to]-\infty, +\infty],$$

$$\mathrm{epi}(F) + \mathrm{epi}(G) \subset \mathrm{epi}(F\nabla G) \qquad \forall F, G : X \to]-\infty, +\infty],$$

$$F\nabla G \quad \text{is convex} \qquad \forall \text{ convex } F, G : X \to]-\infty, +\infty].$$

[5] Notice that g may attain the value $-\infty$.

*** Proposition 3.3** *(i) If $F, G : B \to]-\infty, +\infty]$ are proper, then $(F\nabla G)^* = F^* + G^*$.*

(ii) If $F, G \in \Gamma_0(B)$ and either F or G is continuous at some point $u_0 \in \mathrm{Dom}(F) \cap \mathrm{Dom}(G)$, then $(F+G)^* = F^*\nabla G^*$. []

Part (i) is easily proved. For any $u^* \in B^*$ we have

$$\begin{aligned}(F\nabla G)^*(u^*) &= \sup_{u\in B}\{\langle u^*, u\rangle - \inf_{w\in B}\{F(w) + G(u-w)\}\} \\ &= \sup_{u,w\in B}\{\langle u^*, w\rangle + \langle u^*, u-w\rangle - F(w) - G(u-w)\} \\ &= \sup_{w\in B}\{\langle u^*, w\rangle - F(w)\} + \sup_{v\in B}\{\langle u^*, v\rangle - G(v)\} = F^*(u^*) + G^*(u^*).\end{aligned}$$

If $F, G \in \Gamma_0(B)$, then $(F^*\nabla G^*)^{**} = (F^{**} + G^{**})^* = (F+G)^*$ by part (i). The further assumption of part (ii) allows one to show that $(F^*\nabla G^*)^{**} = F^*\nabla G^*$. []

Exercises. (i) Show that $I_{co(K)} = \mathrm{co}(I_K)$.

(ii) Exhibit a function $f : \mathbf{R} \to]-\infty, +\infty]$ which has no convex lower bound.

(iii) Set $f(u) := u^2$ for any $u \in \mathbf{R} \setminus \{0\}$, $f(0) := 1$. Check that $\mathrm{epi}(\mathrm{co}(f)) \neq \mathrm{co}(\mathrm{epi}(f))$. (Notice that there exists no function $g : \mathbf{R} \to]-\infty, +\infty]$ such that $\mathrm{epi}(g) = \mathrm{co}(\mathrm{epi}(f))$).

(iv) Check that $f : \mathbf{R} \to]-\infty, +\infty]$ is convex iff $\mathrm{Dom}(f)$ is convex and $f\big|_{\mathrm{Dom}(f)}$ is also convex.

(v) Check that the pointwise limit of a sequence of convex functions is convex.

(vi) Let K be a convex subset of a Banach space, equipped with a norm $\|\cdot\|$, and define the distance function, $d_K(u) := \inf\{\|u-v\| : v \in K\}$ for any $u \in B$. Check that $I_K\nabla\|\cdot\| = d_K$.

(vii) Prove the characterization (4.3) of quasi-convex functions.

(viii) The sum of two quasi-convex functions is necessarily quasi-convex?

(ix) Let K be a subset of a real vector space. Does $\mathrm{co}(K)$ coincide with the set of convex combinations of *pairs* of elements of K?

Exercises. (i) Let $p, q \in]1, +\infty[$, $1/p + 1/q = 1$. Check that the function $\mathbf{R} \to \mathbf{R} : v^* \mapsto |v^*|^q/q$ is the conjugate of the function $v \mapsto |v|^p/p$. The Fenchel inequality $(3.6)_1$ then generalizes the classical *Young inequality:* $uu^* \le |u|^p/p + |u^*|^{p'}/p'$ for any $u, u^* \in \mathbf{R}$.

Let Ω be an open subset of $\mathbf{R}^N$. Check that the functional $L^q(\Omega) \to \mathbf{R} : v^* \mapsto (1/q)\int_\Omega |v^*|^q dx$ is the conjugate of the functional $L^p(\Omega) \to \mathbf{R} : v \mapsto (1/p)\int_\Omega |v|^p dx$.

(ii) Check that (3.6) is equivalent to

$$\begin{aligned}&\langle u^*, u\rangle \le r + s \qquad \forall(u, r) \in \mathrm{epi}(F), \forall(u^*, s) \in \mathrm{epi}(F^*), \\ &\langle u^*, u\rangle \le r + s \qquad \forall(u, r) \in \mathrm{epi}(F^{**}), \forall(u^*, s) \in \mathrm{epi}(F^*);\end{aligned}$$

moreover, for instance,

$$\mathrm{epi}(F^*) = \{(u^*, s) \in B^* \times \mathbf{R} : \langle u^*, u\rangle \le r + s,\ \forall(u, r) \in \mathrm{epi}(F)\}.$$

(iii) Under the assumptions of Theorem 3.1, prove that:

(a) $F \le G$ entails $G^* \le F^*$ (whenever G fulfils the conditions which we assumed for F);

(b) $F^{***} := (F^{**})^* = (F^*)^{**} = F^*$;

(iv) Let B be a Banach space equipped with the norm $\|\cdot\|$, and let B^* be equipped with the dual norm $\|\cdot\|_*$. Let $\varphi \in \Gamma_0(\mathbf{R})$ be even, set $F(u) := \varphi(\|u\|)$ for any $u \in B$, and $G(u^*) := \varphi^*(\|u^*\|_*)$ for any $u^* \in B^*$. Show that $F^* = G$.

* (v) Let F be convex. Show that if F is lower semicontinuous at $u \in B$, then $F(u) = F^{**}(u)$.

* (vi) Let B be a Banach space, $u_0 \in B$, set $F(u) := \|u - u_0\|$ for any $u \in B$ and denote the unit ball of B^* by K^*. Check that $F^*(u^*) = I_{K^*}(u^*) + \langle u^*, u_0\rangle$ for any $u^* \in B^*$.

Let us then set $F_c(u) := F(cu)$ for any $u \in B$ and any $c \in \mathbf{R}$. How can F_c^* be represented? □

4 The Subdifferential.

Let $F : B \to]-\infty, +\infty]$ be proper. We define its *subdifferential,* ∂F, as follows:

$$\partial F(u) := \{u^* \in B^* : \langle u^*, u - v\rangle \geq F(u) - F(v),\ \forall v \in B\} \qquad \forall u \in \mathrm{Dom}(F). \tag{4.1}$$

$\partial F(u) = \emptyset$ is not excluded, and we set $\partial F(u) = \emptyset$ for any $u \in B \setminus \mathrm{Dom}(F)$. We also define the *(effective) domain* of ∂F by $\mathrm{Dom}(\partial F) := \{u \in B : \partial F(u) \neq \emptyset\}$, and say that F is subdifferentiable at u iff $\partial F(u) \neq \emptyset$. The elements of $\partial F(u)$ are called *subgradients* of F at u.

The condition (4.1) means that:

(i) the continuous and affine function $L : v \mapsto \langle u^*, v - u\rangle + F(u)$ is a lower bound of F, and

(ii) L is *exact* at u, that is, $L(u) = F(u)$.

If $F^* : B^* \to]-\infty, +\infty]$ is proper, we set

$$\partial F^*(u^*) := \{u \in B : \langle u, u^* - v^*\rangle \geq F^*(u^*) - F^*(v^*),\ \forall v^* \in B^*\} \qquad \forall u^* \in \mathrm{Dom}(F^*), \tag{4.2}$$

and $\partial F^*(u^*) = \emptyset$ for any $u^* \in B^* \setminus \mathrm{Dom}(F^*)$. As above, the duality pairing between B^* and B is here exploited. This is tantamount to defining $\partial F^*(u^*)$ as a subset of B^{**}, and then to restrict it to B.

Examples. (i) Let H be a (real) Hilbert space, $1 \leq p < +\infty$ and set $F_p(u) := \|u\|^p/p$ for any $u \in H$. It is convenient to identify H with its dual space, and then to define the subdifferential as a subset of H, simply by replacing the duality pairing by the scalar product in the definition.

If $p > 1$, then $\partial F_p(u) = \{\|u\|^{p-2}u\}$ for any $u \in H$; in particular, $\partial F_1(u) = \{\|u\|^{-1}u\}$ for any $u \in H \setminus \{0\}$, and $\partial F_1(0) = \{v \in H : \|v\| \leq 1\}$.

In particular, if $H := \mathbf{R}$ then $\partial F_1 = \mathrm{sign}$, where we set

$$\mathrm{sign}(x) := \{-1\} \text{ if } x < 0, \quad \mathrm{sign}(0) := [-1, 1], \quad \mathrm{sign}(x) := \{1\} \text{ if } x > 0. \tag{4.3}$$

(ii) Let $(A, \mathcal{A}, \mu)$ be a measure space, $1 \leq p < +\infty$, $F_p : \mathbf{R} \to \mathbf{R} : v \mapsto |v|^p/p$, and set $\Phi_p(u) := \int_A F_p(u) d\mu$ for any $u \in B := L^p(A, \mathcal{A}, \mu)$. Then

$$\partial\Phi_p(u) = \{u^* \in L^{p'}(A, \mathcal{A}, \mu) : u^* \in \partial F_p(u), \mu\text{-a.e. in } A\} \qquad \forall u \in B.$$

(Here $p' := p/(p-1)$ if $p \neq 1$, $1' := +\infty$, as usual.)

(iii) Let us set $B := \mathbf{R}$ and $F_+(u) = F_-(u) := |u|$ for any $u \in \mathbf{R} \setminus \{0\}$, $F_+(0) := 1$, $F_-(0) := -1$. Then $\partial F_+(u) = \mathrm{sign}(u)$ for any $\mathbf{R} \setminus \{0\}$ and $\partial F_+(0) = \emptyset$. On the other hand, $\partial F_-(u) = \emptyset$ for any $\mathbf{R} \setminus \{0\}$ and $\partial F_-(0) = [-1, 1]$.

(iv) Let Ω be an open subset of $\mathbf{R}^N$ $(N \geq 1)$, $1 < p < +\infty$, either $B := W^{1,p}_0(\Omega)$ or $B := W^{1,p}(\Omega)$, and set

$$F(u) := \frac{1}{p}\int_\Omega |\nabla u|^p dx \qquad \forall u \in B. \tag{4.4}$$

This functional is convex and continuous on the whole B. By examples (i) and (ii), for any $u \in B$, $\xi := |\nabla u|^{p-2}\nabla u$ is the only element of $L^{p'}(\Omega)^N$ such that

$$\int_\Omega \xi \cdot \nabla(u - v) dx \geq \frac{1}{p}\int_\Omega (|\nabla u|^p - |\nabla v|^p)\, dx \qquad \forall v \in B. \tag{4.5}$$

Thus, setting $L_u : v \mapsto \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx$, $L_u \in B^*$ and $L_u \in \partial F(u)$. If $B := W^{1,p}_0(\Omega)$ then $\partial F(u) = -\nabla \cdot \big(|\nabla u|^{p-2}\nabla u\big)$ in $\mathcal{D}'(\Omega)$. ∂F is thus single-valued.

This holds also for $p = 1$, provided that we replace $|\nabla u|^{p-2}\nabla u$ by any element of $\partial g(\nabla u)$, where $g(\xi) := |\xi|$ for any $\xi \in \mathbf{R}^N$. In this case ∂F is multi-valued.

(v) Let Ω be as above, $1 < p < +\infty$, set $B := L^p(\Omega)$ and

$$\tilde{F}(u) := \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega |\nabla u|^p dx & \forall u \in W_0^{1,p}(\Omega), \\ +\infty & \forall u \in L^p(\Omega) \setminus W_0^{1,p}(\Omega). \end{cases} \tag{4.6}$$

By (4.5), $\mathrm{Dom}(\partial\tilde{F}) = \{u \in W_0^{1,p}(\Omega) : \nabla \cdot \big(|\nabla u|^{p-2}\nabla u\big) \in L^{p'}(\Omega)\}$ (otherwise the first integral of (4.5) is meaningless) and $\partial\tilde{F}(u) = -\nabla \cdot \big(|\nabla u|^{p-2}\nabla u\big)$ for any $u \in \mathrm{Dom}(\partial\tilde{F})$. For instance, for $p = 2$, $\mathrm{Dom}(\partial\tilde{F}) = H_0^1(\Omega) \cap H^2(\Omega)$ and $\partial\tilde{F}(u) = -\Delta u$ for any $u \in \mathrm{Dom}(\partial\tilde{F})$.

If in (4.6) we replace $W_0^{1,p}(\Omega)$ by $\tilde{B} := W^{1,p}(\Omega)$, the representation of $\partial\tilde{F}$ is more delicate. Actually,

$$\langle \xi, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx \qquad \forall \xi \in \partial\tilde{F}(u), \forall u, v \in W^{1,p}(\Omega), \tag{4.6$'$}$$

(so that $\langle \xi, u - v\rangle \geq \tilde{F}(u) - \tilde{F}(v) \geq 0$). By the Hahn-Banach theorem, ξ may then be extended to an element of $W^{1,p}(\Omega)'$.

(vi) These examples can easily be extended if $|\nabla u|^p$ is replaced by $\varphi(\nabla u)$, where $\varphi : \mathbf{R}^N \to \mathbf{R}$ is convex and $\varphi(\xi)$ grows at infinity at most like $|\xi|^p$ (that is, there exist $C, M > 0$ such that $\varphi(\xi) \leq C|\xi|^p + M$ for any $\xi \in \mathbf{R}^N$). □

• **Theorem 4.1** *Let $F : B \to]-\infty, +\infty]$ and assume that any function of which here we consider either the conjugate or the subdifferential is proper. Then for any $u \in B$ and any $u^* \in B^*$ we have:*

(i) $u^* \in \partial F(u) \Leftrightarrow F(u) + F^*(u^*) = \langle u^*, u\rangle$ (Fenchel's equality), *or equivalently:*

$u^* \in \partial F(u) \Leftrightarrow$ *there exist* $(u, r) \in \mathrm{epi}(F)$ *and* $(u^*, s) \in \mathrm{epi}(F^*)$ *such that* $r + s = \langle u^*, u\rangle$.

(ii) $u \in \partial F^*(u^*) \Leftrightarrow F^{**}(u) + F^*(u^*) = \langle u^*, u\rangle$ (dual Fenchel's equality), *or equivalently:*

$u \in \partial F^*(u^*) \Leftrightarrow$ *there exist* $(u, r) \in \mathrm{epi}(F^{**})$ *and* $(u^*, s) \in \mathrm{epi}(F^*)$ *such that* $r + s = \langle u^*, u\rangle$.

(iii) $u^* \in \partial F(u) \Rightarrow u \in \partial F^*(u^*)$. *The converse holds if* $F(u) = F^{**}(u)$.

(iv) $\partial F(u)$ *is convex and weakly star closed (hence strongly and weakly closed);* $\partial F^*(u^*)$ *is convex and closed.*

(v) The operator ∂F *is monotone, that is,*

$$\langle u_1^* - u_2^*, u_1 - u_2\rangle \geq 0 \qquad \forall u_i \in \mathrm{Dom}(\partial F), \forall u_i^* \in \partial F(u_i) (i = 1, 2). \tag{4.7}$$

(vi) $F(u) = \inf F \Leftrightarrow \partial F(u) \ni 0 \Leftrightarrow \big[F(u) = F^{**}(u), \partial F^{**}(u) \ni 0\big] \Rightarrow u \in \partial F^*(0)$. *If* $F \in \Gamma_0(B)$ *then the latter implication can be inverted.*

Proof. By (4.1), $u^* \in \partial F(u)$ iff $\langle u^*, v\rangle - F(v) \leq \langle u^*, u\rangle - F(u)$ for any $v \in B$, that is,

$$F^*(u^*) := \sup_{v \in B}\{\langle u^*, v\rangle - F(v)\} = \langle u^*, u\rangle - F(u).$$

(i) thus holds. (ii) can be derived similarly.

As $F^{**} \leq F$, (iii) follows from (i), (ii) and (3.6).

By (4.1) we have $\partial F(u) = \bigcap_{v \in B}\{u^* \in B^* : \langle u^*, u - v\rangle \geq F(u) - F(v)\}$; thus $\partial F(u)$ is the intersection of a (nonempty) family of weakly star closed half-spaces. This yields the first part of (iv). The remainder can be proved similarly.

(v) and (vi) easily follow from the definition of subdifferential (cf. Proposition 4.2 below). □

By the Fenchel's inequality, $F(u) + F^*(u^*) \geq \langle u^*, u\rangle$ (which holds for any $(u, u^*) \in B \times B^*$), the Fenchel's equality is equivalent to the opposite Fenchel's inequality, $F(u) + F^*(u^*) \leq \langle u^*, u\rangle$. By the same token, the dual Fenchel's equality is equivalent to the opposite dual Fenchel's inequality: $F^{**}(u) + F^*(u^*) \geq \langle u^*, u\rangle$.

By part (iii) of the latter result, for any $F \in \Gamma_0(B)$, $\partial F^* = (\partial F)^{-1}$. [5]

Corollary 4.1' *Let $F : B \to]-\infty, +\infty]$ and F^* be proper, and set*

$$\Phi(u, u^*) := F(u) + F^*(u^*) - \langle u^*, u\rangle \qquad \forall u \in \mathrm{Dom}(F), \forall u^* \in \mathrm{Dom}(F^*).$$

Then

$$u^* \in \partial F(u) \quad \text{iff} \quad \Phi(u, u^*) = \inf_{\mathrm{Dom}(F)\times\mathrm{Dom}(F^*)} \Phi. \tag{4.7'}$$

Proof. The Fenchel equality and inequality respectively read

$$u^* \in \partial F(u) \quad \text{iff} \quad \Phi(u, u^*) = 0; \qquad 0 \le \inf_{\mathrm{Dom}(F)\times\mathrm{Dom}(F^*)} \Phi.$$

This yields the "only if" part. Conversely, if $\Phi(u, u^*) = \inf_{\mathrm{Dom}(F)\times\mathrm{Dom}(F^*)} \Phi$, then $\Phi(u, u^*) \le \Phi(v, u^*)$ for any $v \in B$, whence $u^* \in \partial F(u)$. The "if" part thus holds. [6] □

Note that (4.7)' yields

$$u^* \in \partial F(u) \quad \text{iff} \quad \partial_u \Phi(u, u^*) \ni 0, \quad \partial_{u^*} \Phi(u, u^*) \ni 0. \tag{4.7'}$$

The following result can easily be proved via the Fenchel equality and inequality.

Proposition 4.2 *Under the assumptions of Theorem 4.1 we have:*
(i) If $\partial F(u) \neq \emptyset$, then $F(u) = F^{**}(u)$.
(ii) If $F(u) = F^{**}(u)$, then $\partial F(u) = \partial F^{**}(u)$ (possibly $= \emptyset$). [Ex]

The two latter implications cannot be inverted in general. As counterexamples take, e.g., $B = \mathbf{R}$ and respectively $F_1(x) := +\infty$ for any $x < 0$, $F_1(x) := -\sqrt{x}$ for any $x \ge 0$, $F_2(x) := F_1(x)$ for any $x \neq 0$, $F_2(0) := 1$. In either case consider the point $x = 0$.

• **Theorem 4.3** *(Rockafellar) Let $F_1, F_2 : B \to]-\infty, +\infty]$. Then*

$$\partial F_1(u) + \partial F_2(u) \subset \partial(F_1 + F_2)(u) \qquad \forall u \in \mathrm{Dom}(F_1) \cap \mathrm{Dom}(F_2). \tag{4.8}$$

The opposite inclusion holds if $F_1, F_2 \in \Gamma_0(B)$, and either F_1 or F_2 is continuous at some point $u_0 \in \mathrm{Dom}(F_1) \cap \mathrm{Dom}(F_2)$. [7]

Proof. To check (4.8) it suffices to write the definition of subdifferential for F_1 and F_2, and sum the two inequalities.
The opposite inclusion is less easy to be proved. □

The continuity assumption cannot be dropped. As a counterexample take $B = \mathbf{R}$, $F_1(x) := +\infty$ for any $x < 0$, $F_1(x) := -\sqrt{x}$ for any $x \ge 0$, $F_2 := I_{]-\infty,0]}$. Hence $(F_1 + F_2)(0) = 0$, $(F_1 + F_2)(x) = +\infty$ for any $x \neq 0$. Therefore $\partial(F_1 + F_2)(0) = \mathbf{R}$, whereas $\partial F_1(0) + \partial F_2(0) = \emptyset + [0, +\infty[= \emptyset$.

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[5] The inverse of any multi-valued function $f : A \to 2^B$ is defined as follows: for any $(a, b) \in A \times B$, $a \in f^{-1}(b)$ iff $b \in f(a)$. For multi-valued functions, this is not always equivalent to the property $f^{-1} \circ f = f \circ f^{-1} = Id$.

[6] This is just another way of proving the Fenchel equality.

[7] By Corollary 2.5, this hypothesis is equivalent to
$[\mathrm{int}(\mathrm{Dom}(F_1)) \cap \mathrm{Dom}(F_2)] \cup [\mathrm{Dom}(F_1) \cap \mathrm{int}(\mathrm{Dom}(F_2))] \neq \emptyset$.

Proposition 4.4 *Let B_1, B_2 be Banach spaces over* $\mathbf{R}$, *$L : B_1 \to B_2$ be linear and continuous, and $F \in \Gamma(B_2)$. Then $F \circ L \in \Gamma(B_1)$.*

Moreover, if F is continuous at some point $\bar{p} \in B_2$, then

$$\partial(F \circ L)(u) = (L^* \circ \partial F)(Lu) \qquad \forall u \in B_1. \qquad [] \tag{4.9}$$

• **Proposition 4.5** *Let $F : B \to]-\infty, +\infty]$ be lower semicontinuous at some $u \in B$, and $\{(u_n, u_n^*)\}$ be a sequence in $B \times B^*$. If*

$$u_n^* \in \partial F(u_n) \;\; \forall n \in \mathbf{N}, \quad u_n \to u \;\; \textit{weakly in } B, \quad u_n^* \to u^* \;\; \textit{weakly star in } B^*, \tag{4.10}$$

$$\liminf_{n\to\infty} \langle u_n^*, u_n\rangle \leq \langle u^*, u\rangle, \tag{4.11}$$

then $u^* \in \partial F(u)$.

Proof. For any n, $u_n^* \in \partial F(u_n)$ iff

$$\langle u_n^*, u_n - v\rangle \geq F(u_n) - F(v) \qquad \forall v \in \mathrm{Dom}(F).$$

It then suffices to pass to the inferior limit as $n \to \infty$. □

The latter result entails that the operator ∂F is strongly-weakly star and weakly-strongly sequentially closed in $B \times B^*$.

Proposition 4.6 *Let $F : B \to]-\infty, +\infty]$ be convex. If F has a point of continuity, then* $\mathrm{int}(\mathrm{Dom}(F)) \subset \mathrm{Dom}(\partial F)(\subset \mathrm{Dom}(F))$.

Proof. If F is continuous at some point, then it is continuous at any interior point of $\mathrm{Dom}(F)$, by Theorem 2.3. Let us fix any $u_0 \in \mathrm{int}(\mathrm{Dom}(F))$. As $\mathrm{int}(\mathrm{epi}(F))$ is nonempty and $(u_0, F(u_0)) \in \mathrm{int}(\mathrm{epi}(F))$, by Theorem II.3.10 there exists a closed hyperplane through $(u_0, F(u_0))$ which is tangent to $\mathrm{epi}(F)$. Thus there exists $(u^*, r) \in (B^* \times \mathbf{R}) \setminus \{(0,0)\}$ such that

$$\begin{aligned} {}_{B^*\times\mathbf{R}}\langle (u^*, r), (u - u_0, a - F(u_0))\rangle_{B\times\mathbf{R}} = {}_{B^*}\langle u^*, u - u_0\rangle_B + r[a - F(u_0)] \geq 0 \\ \forall (u,a) \in \mathrm{int}(\mathrm{epi}(F)). \end{aligned}$$

We have $r \neq 0$, since otherwise ${}_{B^*}\langle u^*, u - u_0\rangle_B \geq 0$ for any $u \in \mathrm{int}(\mathrm{Dom}(F))$, whence $u^* = 0$ as $u_0 \in \mathrm{int}(\mathrm{Dom}(F))$. Moreover $r > 0$, as a may be arbitrarily large. Taking $a := F(u)$, we get ${}_{B^*}\langle -u^*/r, u_0 - u\rangle_B \geq F(u_0) - F(u)$ for any $u \in \mathrm{Dom}(F)$, that is, $-u^*/r \in \partial F(u_0)$. □

Proposition 4.7 *If $F \in \Gamma_0(B)$ then* $\mathrm{Dom}(\partial F)$ *is dense in* $\mathrm{Dom}(F)$.

This result will be proved in Sect. XIII.4.

Inverse Fenchel equality and inequality: for any $u^* \in B^*$ and any function $F : B \to \mathbf{R} \cup \{+\infty\}$,

$$\exists \xi \in \mathbf{R} : \begin{cases} \exists u \in B : \quad \langle u^*, u\rangle = F(u) + \xi \\ \forall v \in B, \quad \langle u^*, v\rangle \leq F(v) + \xi \end{cases} \quad \Rightarrow \quad u^* \in \partial F(u), \xi = F^*(u^*). \tag{4.12}$$

(The opposite implication coincides with the usual Fenchel equality and inequality.)

In particular this entails that, for any functions $F : B \to \mathbf{R} \cup \{+\infty\}$ and $G : B^* \to \mathbf{R} \cup \{+\infty\}$ and for any $(u, u^*) \in B \times B^*$,

$$\begin{cases} \langle u^*, u\rangle = F(u) + G(u^*) \\ \langle v^*, v\rangle \le F(v) + G(v^*) \quad \forall (v, v^*) \in B \times B^* \end{cases} \Rightarrow \begin{cases} u^* \in \partial F(u), \;\; G(u^*) = F^*(u^*) \\ u \in \partial G(u^*), \;\; F(u) = G^*(u). \end{cases} \tag{4.13}$$

□

Exercises. (i) Let $F \in \Gamma_0(B)$ and $u^* \in B^*$. Check that $\partial F^*(u^*)$ coincides with the set of points which minimize the function $u \mapsto F(u) - \langle u^*, u\rangle$.

(ii) Show by a counterexample that in general $u \in \partial F^*(u^*)$ does not entail $u^* \in \partial F(u)$.

(iii) Evaluate F_i^*, F_i^{**}, ∂F_i, ∂F_i^*, ∂F_i^{**}, for any F_i defined as follows:

(a) $F_1(x) := |x|^2$ for any $x \in \mathbf{R} \setminus \{0\}$, $F_1(0) := -1$.

(b) $F_2(x) := |x|^2$ for any $x \in \mathbf{R} \setminus \{0\}$, $F_2(0) := 1$.

(c) $F_3(x) := -|x|^2$ for any $x \in \mathbf{R}$.

(d) $F_4(x) := \arctan x$ for any $x \in \mathbf{R}$.

(e) $F_5(x) := +\infty$ for any $x < 0$, $F_5(x) := -\sqrt{x}$ for any $x \ge 0$.

Hint: In order to evaluate the conjugate, one may use part (iii) of Theorem 4.1.

(iv) Let $F : B \to]-\infty, +\infty]$ ($F \not\equiv +\infty$) and $u \in K \subset B$. Check that the two following properties are equivalent:

(a) $F(u) = \inf_K F$,

(b) $0 \in \partial(F + I_K)(u)$.

Show that, if $F \in \Gamma_0(B)$, K is closed and convex, and $[\mathrm{int}(\mathrm{Dom}(F)) \cap K] \cup [\mathrm{Dom}(F) \cap \mathrm{int}(K)] \ne \emptyset$, then (a) and (b) are also equivalent to

(c) $0 \in \partial F(u) + \partial I_K(u)$, i.e., $-\partial I_K(u) \cap \partial F(u) \ne \emptyset$.

(v) Consider the elementary statements

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 \ge xy \qquad \forall x, y \in \mathbf{R},$$

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 = xy \quad \text{iff} \quad x = y \qquad \forall x, y \in \mathbf{R}.$$

Check that they respectively express the Fenchel inequality and the Fenchel equality for the function $x \mapsto \frac{1}{2}x^2$ in $\mathbf{R}$. Apply then the Fenchel inequality and the Fenchel equality to the function $x \mapsto \frac{1}{2}x^p$ in $\mathbf{R}$ for any $p \in]1, +\infty[$.

(vi) Show that the Fenchel inequality and equality entail the monotonicity of the subdifferential.

(vii) May the implication (4.13) be inverted? □