STRUCTURAL STABILITY OF RATE-INDEPENDENT NONPOTENTIAL FLOWS

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Abstract. Several phenomena may be represented by doubly-nonlinear equations of the form
\[ \alpha(D_t u) - \nabla \cdot \gamma(\nabla u) \ni h, \]
with \( \alpha \) and \( \gamma \) (possibly multivalued) maximal monotone mappings. Hysteresis effects are characterized by rate-independence, which corresponds to \( \alpha \) positively homogeneous of zero degree.

Fitzpatrick showed that any maximal monotone relation may be represented variationally. On this basis, an initial- and boundary-value problem associated to the equation above is here formulated as a null-minimization problem, without assuming \( \gamma \) to be cyclically monotone. Existence of a solution \( u \in H^1(0,T;H^1(\Omega)) \) is proved, as well as its stability with respect to variations of the data, of the mapping \( \gamma \), and of the domain \( \Omega \).

1. Introduction. Several phenomena may be represented by doubly-nonlinear equations of the form
\[ \alpha(D_t u) - \nabla \cdot \gamma(\nabla u) \ni h, \]
with \( \alpha \) and \( \gamma \) (possibly multivalued) maximal monotone mappings, see e.g. [50], [57]. The existence of a solution of associated initial- and boundary-value problems was proved in a number of works, see e.g. [1], [4], [15], [16], [25], [49], [50], [51], [52], [54], [57]. The homogenization was also studied in [31], [43].

Hysteresis corresponds to rate-independent processes; for equation (1.1) this means that \( \alpha \) is positively homogeneous of zero degree. In [29] (p. 77) it was shown that in this case \( \alpha \) is necessarily cyclically monotone, namely, it is the subdifferential of a lower semicontinuous convex function, which is known in the physical literature as a dissipation potential. We shall thus deal with the inclusion
\[ \partial \psi(D_t u) - \nabla \cdot \gamma(\nabla u) \ni h, \]
with \( \psi : \mathbb{R}^M \to \mathbb{R} \cup \{+\infty\} \) lower semicontinuous, convex and positively homogeneous of degree one.

Most of our results however apply to the more general class (1.1).

In [22] Fitzpatrick showed that any maximal monotone operator in a real Banach space may be represented as a minimization problem. Prior to Fitzpatrick’s work, Brezis and Ekeland [10] and Nayroles [44] had proposed a variational principle for a class of nonlinear evolutionary P.D.E.s of first order. In Sect. 2 we briefly review
some elements of that theory, and exhibit some examples of representative functions of monotone operators; further examples may be found e.g. in [63].

In Sect. 3 we deal with an initial- and boundary-value problem for the inclusion 
(1.2) in a space-time domain \( \Omega \times [0,T] \) (\( \Omega \subset \mathbb{R}^N \)). By using a variational representation of the mapping \( \gamma \), we provide a variational formulation of that problem; more specifically, we introduce a null-minimization principle in the sense of (2.8) ahead. We then prove existence of a solution via approximation by time-discretization, derivation of uniform estimates, and passage to the limit.

Whenever \( \gamma \) admits a convex (and lower semicontinuous) potential, under natural coerciveness assumptions, one may derive a uniform estimate for the approximate solution \( u_k \) in the space \( W^{1,1}(0,T; L^1(\Omega)) \cap L^2(0,T; H^1(\Omega)) \), by multiplying the approximate equation by \( D_t u_k \). In the limit as \( k \to \infty \), one then gets a solution \( u \in L^2(0,T; H^1(\Omega)) \) such that \( D_t u \) is a Radon measure over \( \Omega \times [0,T] \). This allows for a slip-stick dynamics, namely, the jerking motion that can occur when two objects slide over each other.

In the present work we do not assume the existence of a convex potential of \( \gamma \), but we suppose that this mapping is strongly monotone, see (3.30). Assuming a certain regularity for the time-dependence of the datum \( h \), by differentiating the approximate equation in time and then multiplying it by \( D_t u_k \), we get a uniform estimate for \( u_k \) in \( H^1(0,T; H^1(\Omega)) \). In the limit this yields the existence of a solution \( u \), with \( D_t u \) free of any Dirac-type measure. The strong monotonicity of the mapping \( \gamma \) thus excludes any discontinuity in the process – a property that had already been noticed for rate-independent potential flow, see [41]. It is not clear whether, under weaker assumptions on the data, a slip-stick dynamics might be represented via the nonpotential flow (1.2).

In Sect. 4 we then deal with the stability of an initial- and boundary-value problem for (1.2) w.r.t. variations of the data and of the operator \( \gamma \) (structural stability). More specifically, we show that, for any bounded sequences \( \{u^0_k\} \), \( \{h_k\} \) and \( \{\gamma_k\} \),

(i) for any \( k \), there exists a corresponding solution \( u_k \),
(ii) these sequences accumulate at \( u^0 \), \( h \), \( \gamma \) and \( u \) w.r.t. appropriate topologies,
(iii) \( u \) solves the corresponding limit problem.

A look at the literature. Hysteresis phenomena have been represented by several P.D.E. models, see e.g. the monographs [11], [32], [33], [39], [56]. It has been known for quite some time that doubly-nonlinear inclusions of the form (1.2) may represent hysteresis. Attention was initially devoted to the so-called (delayed) relay operator \( h \mapsto u \). For scalar fields \( h \) and \( u \), this corresponds to the inclusion

\[
\text{sign}(D_t u) + \partial I_{[a,b]}(u) \ni h \quad (a, b \in \mathbb{R}, a < b), \tag{1.3}
\]

where “sign” and \( \partial I_{[a,b]} \) respectively denote the sign graph and the subdifferential of the indicator function of the interval \( [a,b] \), see e.g. [56]; Chap. VI. These relay operators may be regarded as the atoms of a (scalar) hysteresis model. More specifically, for any distribution of admissible pairs \( (a,b) \), by averaging the solutions \( u_{(a,b)} \) associated to a same input \( h \), the large class of Preisach operators is obtained.

In the framework of space-distributed systems, the analysis of the inclusion (1.3) looks much more challenging than (1.2), since apparently (1.3) does not provide any regularity with respect to the variable \( x \), and \( I_{[a,b]} \) is not strictly convex. A variational formulation of (1.3) as a system of two variational inequalities was given in [55], [56], and then extended to vector hysteresis. This model was coupled with parabolic and hyperbolic quasilinear equations, see e.g. [56], [58], [59].
Since the late 1990s a new powerful approach to hysteresis emerged, and was applied to a multitude of physical models with a remarkable degree of generality; see e.g. [18], [23], [41], [42], the survey [40] and references therein. This research was labelled under the keywords of rate-independence and energetic formulation. Completely rate-independent (quasistationary) processes were indeed addressed, whereas in the previous research hysteretic constitutive relations had been coupled with rate-dependent dynamics.

**Potential vs. nonpotential flow.** The energetic approach may be illustrated with reference to a quasilinear differential inclusion like (1.2), although that formulation was also used in a much wider framework, e.g. in nonlinear metric spaces, see [49]. In the simplified setting of (1.2), that approach boils down to assuming that $\gamma$ has a potential. This is consistent with several mathematical-physical models, in which the inclusion (1.2) may indeed account for the balance between a dissipative force, $\partial\psi(D_t u)$, a conservative force, $-\nabla \cdot (\gamma \nabla u)$, and a load, $h$.

Two elements however suggest that nonpotential flows may be of some interest, too. First there are examples of nonpotential dynamics: these include e.g. transport phenomena and evolution of saddle points; see the Examples 2.4 and 2.6 of Sect. 2. Moreover, the class of nonpotential flows includes perturbations of potential flows; the latter actually fail to be stable under perturbations, as we briefly illustrate at the end of Sect. 2.

One of the main advantages of the Fitzpatrick theory is that it paves the way to the use of variational methods in the analysis of either stationary or evolutionary monotone problems, see e.g. [54] and [60]. In [54] Stefanelli assumed that $\gamma$ admits a convex potential, and introduced a variational formulation of the flow (1.2) based on the use of Fenchel functions. He then studied the stability of the associated Cauchy problem, assuming the Mosco-convergence of the potential $\varphi$ of $\gamma$. In the present work instead we address the stability of (1.2) for a nonpotential $\gamma$, dealing with weaker convergence notions. The variational formulation allows us to apply variational techniques, including De Giorgi’s notion of $\Gamma$-convergence, see e.g. [17], [19]. More specifically, we use $\Gamma$-convergence w.r.t. a nonlinear convergence that was introduced in [60].

2. The Fitzpatrick theory. The representation of monotone operators.
A classical result due to Fenchel [21] mutually relates a function(al) $\varphi : V \rightarrow \mathbb{R} \cup \{+\infty\}$, its convex conjugate $\varphi^*$, and the subdifferential operator $\partial \varphi$, see e.g. [20], [48]:

\[
\varphi(v) + \varphi^*(v') \geq \langle v', v \rangle \quad \forall (v, v') \in V \times V', \quad (2.1)
\]
\[
\varphi(v) + \varphi^*(v') = \langle v', v \rangle \iff v' \in \partial \varphi(v). \quad (2.2)
\]

In [22] Fitzpatrick extended this system to more general operators $\alpha : V \rightarrow \mathcal{P}(V')$. He introduced the convex and lower semicontinuous function (that nowadays is named the Fitzpatrick function of $\alpha$)

\[
f_\alpha(v, v') := \langle v', v \rangle + \sup \left\{ \langle v' - v'_0, v_0 - v \rangle : v'_0 \in \alpha(v_0) \right\}
\]
\[
= \sup \left\{ \langle v', v_0 \rangle - \langle v'_0, v_0 - v \rangle : v'_0 \in \alpha(v_0) \right\} \quad \forall (v, v') \in V \times V'. \quad (2.3)
\]
and proved that, whenever $\alpha$ is maximal monotone (in the sense e.g. of [5], [8]),

\begin{align*}
f_\alpha(v, v') &\geq \langle v', v \rangle \quad \forall (v, v') \in V \times V', \\
f_\alpha(v, v') &= \langle v', v \rangle \quad \Leftrightarrow \quad v' \in \alpha(v).
\end{align*}

This result went unnoticed for several years, until it was rediscovered by Martinez-Legaz and Théra [38] and by Burachik and Svaiter [13]. This started an intense research about relations between monotone operators and convex functions; see e.g. [14], [26], [36], [37], [45], [46], just to quote few papers, and the related notion of bipotential [12].

More generally, nowadays one says that a convex and lower semicontinuous function $f : V \times V' \rightarrow \mathbb{R} \cup \{+\infty\}$ (variationally) represents a (necessarily monotone) operator $\alpha$ whenever it fulfills the system (2.4), (2.5). Accordingly, we shall say that $f$ is a representative function of $\alpha$, and that $\alpha$ is representable. We shall denote by $\mathcal{F}(V)$ the class of these functions. For instance, whenever $\varphi$ is convex and lower semicontinuous, the Fenchel function $f(v, v') := \varphi(v) + \varphi^*(v')$ represents the operator $\partial \varphi$. The class of cyclic and maximal monotone operators is actually characterized by admitting a representative function of the additive form $f(v, v') := \varphi(v) + \psi(v')$ (whence $\psi = \varphi^*$). This issue is reviewed e.g. in Sects. 2 and 3 of [63].

### Variational formulation of monotone flows

Prior to Fitzpatrick’s [22], Brezis and Ekeland [10] and Nayroles [44] independently proposed a first example of variational formulation of a nonlinear evolutionary P.D.E.. They considered the Cauchy problem

\begin{align*}
\left\{ \begin{array}{l}
D_t u + \partial \varphi(u) \ni h \quad \text{in } V', \text{ a.e. in } [0, T[ \\
u(0) = u^0, 
\end{array} \right.
\end{align*}

(2.6)

for any $h \in L^2(0, T; V')$ and any $u^0 \in H$ ($V \subset H = H' \subset V'$ being a Gelfand triplet). They associated to this problem the functional

\begin{align*}
J_h : X_{u^0} := \{v \in L^2(0, T; V) \cap H^1(0, T; V') : v(0) = u^0 \} \rightarrow \mathbb{R} \cup \{+\infty\},
\end{align*}

\begin{align*}
J_h(v) := \int_0^T \left[ \varphi(v) + \varphi^*(h - D_t v) - \langle h, v \rangle \right] dt + \frac{1}{2} \|v(T)\|_H^2 - \frac{1}{2} \|u^0\|_H^2,
\end{align*}

(2.7)

pointed out that $J_h \geq 0$, and showed that (2.6) is equivalent to $J_h(u) = 0$, namely

\begin{align*}
J_h(u) = \inf \{J_h \leq 0 \} \quad \text{(a null-minimization problem).}
\end{align*}

The solution $u$ may thus be computed by a descent procedure, provided that it has already been established that a null-minimizer does exist – namely, $J^{-1}_h(0) \neq \emptyset$. The latter condition is nontrivial; this issue was settled by Auchmuty [3], and was then extensively studied by Ghoussoub in a series of works; in particular see e.g. [26], [28], the recent monograph [27], and references therein.

The above mentioned result of Fitzpatrick allows one to extend the null-minimization principle as follows to the Cauchy problem (2.6) with a generic maximal monotone operator $\alpha : V \rightarrow \mathcal{P}(V')$ in place of $\partial \varphi$. Whenever $\alpha$ is represented by a function $f_\alpha$, that problem is equivalent to the null-minimization principle (2.8), provided that in the definition (2.2) the term $\varphi(v) + \varphi^*(h - D_t v)$ is replaced by $f_\alpha(v, h - D_t v)$:

\begin{align*}
J_h(v) := \int_0^T \left[ f_\alpha(v, h - D_t v) - \langle h, v \rangle \right] dt + \frac{1}{2} \|v(T)\|_H^2 - \frac{1}{2} \|u^0\|_H^2 \quad \forall v \in X_{u^0},
\end{align*}

(2.9)
see e.g. [60]. The functional \((v, h) \mapsto J_h(v)\) represents the operator \(D_t + \alpha\) in the sense of (2.4) and (2.5). On the basis of this remark, one may also apply an alternative viewpoint: instead of prescribing \(h\), one may deal with the family of all pairs \((u, h)\) that fulfill the Cauchy problem (2.6). This variational approach may thus be used to study the stability of the problem, w.r.t. variations of the data \(h, u^0\) and of the mapping \(\alpha\). By means of variational techniques, including De Giorgi’s notion of \(\Gamma\)-convergence, one may then study how the set of null-minimizers \((u, h)\) (the pair solution-datum) depend on the functional (2.9) – that is, on the operator \(\alpha\), that defines the structure of the problem.

Next we provide some examples of representative functions; further instances may be found in Sect. 3 of [63].

**Example 2.1 (Nonlinear elliptic operator).** Let \(\Omega\) be a bounded domain of \(\mathbb{R}^N\) \((N > 1)\), \(p \in [2, +\infty]\), and set \(V := W_0^{1,p}(\Omega)\). Let a maximal monotone mapping \(\gamma : \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N)\) be represented by a function \(f_\gamma \in \mathcal{F}(\mathbb{R}^N)\). If

\[
\exists a_1, a_2 \in \mathbb{R}^+ : \forall w \in \mathbb{R}^N, \forall z \in \gamma(w), \quad |z| \leq a_1|w|^p + a_2,
\]

then we may define the maximal monotone operator

\[
\tilde{\gamma} : V \to \mathcal{P}(V') : v \mapsto -\nabla \cdot (\gamma(\nabla v)).
\]

Let us assume that \(f\) is coercive in the sense that

\[
\exists a, b > 0 : \forall (v, w) \in \mathbb{R}^2, \quad f(v, w) \geq a(|v|^p + |w|^p) - b.
\]

We claim that \(\tilde{\gamma}\) is then represented by the following function \(\varphi : V \times V' \to \mathbb{R}\):

\[
\varphi(v, v^*) = \inf \left\{ \int_{\Omega} f(\nabla v, \tilde{\eta} v^*) dx : \tilde{\eta} v^* \in L^{p'}(\Omega)^N, -\nabla \cdot \tilde{\eta} v^* = v^* \text{ in } \mathcal{D}'(\Omega) \right\},
\]

for any \((v, v^*) \in V \times V'\). This infimum is attained at some \(\tilde{\xi}_{v^*} \in L^{p'}(\Omega)^N\), because of the coerciveness of \(f\). The function \(\varphi\) is convex and lower semicontinuous because of (2.10); moreover,

\[
\varphi(\tilde{v}, \tilde{v}^*) = \int_{\Omega} f(\nabla v, \tilde{\xi}_{v^*}) dx \geq \int_{\Omega} \nabla v \cdot \tilde{\xi}_{v^*} dx = -\langle v, \nabla \cdot \tilde{\xi}_{v^*} \rangle = \langle v, v^* \rangle.
\]

Thus \(\varphi \in \mathcal{F}(V)\). On the other hand, as \(f(\nabla v, \tilde{\xi}_{v^*}) \geq \nabla v \cdot \tilde{\xi}_{v^*}\) pointwise in \(\Omega\), equality holds in (2.14) if and only if \(f(\nabla v, \tilde{\xi}_{v^*}) = \nabla v \cdot \tilde{\xi}_{v^*}\) a.e. in \(\Omega\). As \(f\) represents \(\tilde{\gamma}\), this is tantamount to \(\tilde{\xi}_{v^*} \in \gamma(\nabla v)\) a.e. in \(\Omega\), whence \(v^* = -\nabla \cdot \tilde{\xi}_{v^*} \in -\nabla \cdot \gamma(\nabla v)\) in \(W^{-1,p'}(\Omega)\). We thus conclude that \(\varphi\) represents the operator \(\tilde{\gamma}\).

(Notice that we could not prescribe \(\tilde{\xi}_{v^*} = -\nabla \Delta^{-1} v^*\), since \(\tilde{\xi}_{v^*} \in \gamma(\nabla v)\) need not be curl-free.)

**Example 2.2 (Time-derivative).** Let us assume that we are given a Gelfand triplet of (real) Banach spaces

\[
V \subset H = H' \subset V'
\]

and denote by \(D_t\) the time-derivative in the sense of distributions \([0, T[ \to V'\). For any \(p \in [2, +\infty]\), let us set

\[
X_0^p := \{ v \in L^p(0, T; V) \cap W^{1,p'}(0, T; V') : v(0) = 0 \},
\]

\[
\alpha(v) = D_t v \quad \text{ a.e. in } [0, T[, \forall v \in X_0^p.
\]
The bounded linear operator $\alpha : X_0^p \to (X_0^p)'$ is monotone as, denoting by $\langle D_t v, v \rangle$ the duality pairing between $V$ and $V'$,

$$\int_0^T \langle \alpha(v), v \rangle \, dt = \int_0^T \langle D_t v, v \rangle \, dt = \frac{1}{2} \| v(T) \|^2_H \geq 0 \quad \forall v \in X_0^p. \quad (2.17)$$

By the linearity this operator is then maximal monotone, although not cyclically monotone as it is not symmetric. Setting $I_{D_t}(v, v') = 0$ if $v' = D_t v$ and $I_{D_t}(v, v') = +\infty$ otherwise, $\alpha$ is represented e.g. by the functional

$$f_\alpha(v, v') = I_{D_t}(v, v') + \int_0^T \langle D_t v, v \rangle \, dt = I_{D_t}(v, v') + \frac{1}{2} \| v(T) \|^2_H \quad (2.18)$$

for any $(v, v') \in X_0^p \times (X_0^p)'$. If in (2.16) the initial condition $v(0) = 0$ is replaced by the condition of periodicity $v(0) = v(T)$, then the operator $D_t$ is also monotone and skew-symmetric. One might also deal with a nonhomogeneous initial condition $v(0) = v^0$; in this case $D_t$ would be monotone on the affine space $X_0^p + v^0$.

**Example 2.3 (Nonlinear parabolic flow).** A further example is obtained by combining the two latter ones. Under the hypothesis (2.10), for any $p \in [2, +\infty]$ the parabolic operator $\alpha : v \mapsto D_t v - \nu.\gamma(\nabla v)$ acts from the space

$$\{ v \in L^p(0, T; W_0^{1,p}(\Omega)) \cap W^{1, p'}(0, T; W^{-1,p'}(\Omega)) : v(0) = 0 \}$$

to its dual, after identifying $L^2(\Omega \times [0, T])$ with itself. This operator may be represented by the Brezis-Ekeland-Nayroles functional, see (2.9).

**Example 2.4 (Transport operator).** Let $\Omega$ be a domain of $\mathbb{R}^N$ ($N > 1$) of Lipschitz class, assume that

$$\nu$$

is the outward-oriented unit normal vector-field on $\partial \Omega$,

$$w \in C^{0,1}(\Omega)^N \cap L^\infty(\Omega)^N, \quad \nabla \cdot w \leq 0 \text{ a.e. in } \Omega. \quad (2.19)$$

Omitting the trace operator, let us set

$$\Gamma = \{ x \in \partial \Omega : \nu \cdot w < 0 \},$$

$$V_0 := \{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma \},$$

$$\theta(v) := w \cdot \nabla v \left( = \sum_{i=1}^N w_i D_i v \right) \quad \text{a.e. in } \Omega, \forall v \in V_0. \quad (2.20)$$

The bounded linear operator $\theta : V_0 \to L^2(\Omega)$ is monotone as

$$\int_\Omega \theta(v) v \, dx = \frac{1}{2} \int_\Omega w \cdot \nabla (v^2) \, dx = \frac{1}{2} \int_\Omega [\nabla \cdot (w v^2) - (\nabla \cdot w) v^2] \, dx$$

$$= \frac{1}{2} \int_{\partial \Omega} \nu \cdot w v^2 \, ds - \frac{1}{2} \int_\Omega (\nabla \cdot w) v^2 \, dx \geq 0 \quad \forall v \in V_0, \quad (2.21)$$

by (2.19)_2 and (2.20)_1. The operator $\theta$ is then maximal monotone, but not cyclically monotone. Denoting by $I_{w,\nu}$ the indicator function of the graph of the operator $v \mapsto w \cdot \nabla v$, $\theta$ is represented e.g. by

$$f_\theta(v, v') = I_{w,\nu}(v, v') + \int_\Omega (w \cdot \nabla v) v \, dx$$

$$= I_{w,\nu}(v, v') + \frac{1}{2} \int_{\partial \Omega} \nu \cdot w v^2 \, ds - \frac{1}{2} \int_\Omega (\nabla \cdot w) v^2 \, dx, \quad (2.22)$$

for any $(v, v') \in V_0 \times V_0'$. If $\Omega$ is an $N$-dimensional interval, this is easily extended to periodic boundary conditions.
Example 2.5 (Saddle operator). Let $B_1$ and $B_2$ be two real Banach spaces, and at least one of them be reflexive. Let $E_i \subseteq B_i$ ($i = 1, 2$) be nonempty, closed and convex sets, and let $L : E_1 \times E_2 \to \mathbb{R}$ be a saddle function such that
\[
L(\cdot, v_2) \text{ is convex and lower semicontinuous, } \forall v_2 \in E_2,
\]
\[
L(v_1, \cdot) \text{ is concave and upper semicontinuous, } \forall v_1 \in E_1.
\]
Let us denote by $\partial_1 L$ and $\partial_2 L$ the partial subdifferentials of $L$. The operator
\[
\partial L : E_1 \times E_2 \to \mathcal{P}(B_1') \times \mathcal{P}(B_2') : (v_1, v_2) \mapsto (\partial_1 L(v_1, v_2), -\partial_2 L(v_1, v_2))
\]
is maximal monotone, but not cyclically monotone; see e.g. [6]; p. 137], [48]; p. 396]. The definition of the corresponding Firtzpatrick function is left to the reader.

Example 2.6 (Saddle evolution). The Example 2.2 may be combined with the latter one. In an appropriate functional framework, the operator
\[
D_t + \partial L : (v_1, v_2) \mapsto (D_t v_1 + \partial_1 L(v_1, v_2), D_t v_2 - \partial_2 L(v_1, v_2))
\]
is monotone but not cyclically monotone. The dynamics $(D_t + \partial L)(v_1, v_2) = (0, 0)$ represents descent along the convex potential $L(\cdot, v_2)$, coupled with ascent along the concave potential $L(v_1, \cdot)$, and may be represented by the scheme that we outlined above.

Further examples may be built by combining the above ones.

Rate-independent saddle evolution. Rate-independent flows may also be written for any of the maximal monotone operators of the Examples 2.1, 2.4, 2.5. For instance, whenever $\psi_1, \psi_2$ are two potentials as in (1.2), the rate-independent dynamics
\[
\partial \psi_1(D_t v_1) + \partial_1 L(v_1, v_2) \ni h_1
\]
\[
\partial \psi_2(D_t v_2) - \partial_2 L(v_1, v_2) \ni h_2
\]
accounts for hysteresis in the mapping $(h_1, h_2) \mapsto (v_1, v_2)$. This may represent a system ruled by minimization of the convex potential $v_1 \mapsto L(v_1, v_2) - h_1 \cdot v_1$, coupled with maximization of the concave potential $v_2 \mapsto L(v_1, v_2) + h_2 \cdot v_2$.

On cyclic monotonicity. The operators defined in the Examples 2.2–2.6 are not cyclically monotone, that is, they are not gradients of convex potentials.

Cyclically monotone operators occur frequently in mathematical-physical models. Nevertheless this family is not stable under a class of small perturbations, as the following construction shows. Let $(V, H, V')$ be a Banach triplet as in (2.15), and $\alpha : V \to \mathcal{P}(V')$ be maximal and strongly monotone:
\[
\exists C > 0 : \forall (v_1, w_1), (v_2, w_2) \in \text{graph}(\alpha), \quad \langle w_1 - w_2, v_1 - v_2 \rangle \geq C\|v_1 - v_2\|^2. \quad (2.27)
\]
Let $\beta : V \to V'$ be Lipschitz continuous, with Lipschitz constant $L$. If $L \leq C$ the operator $\alpha + \beta : V \to \mathcal{P}(V')$ is then maximal monotone, although of course need not be cyclically monotone.
3. Rate-independent doubly-nonlinear flow. In this section we study an initial- and boundary-value problem for a doubly-nonlinear rate-independent flow of the form

\[ \alpha(D_t u) - \nabla \cdot \gamma(\nabla u) \ni h \quad \text{in} \quad Q := \Omega \times [0,T]. \]  

(3.1)

Here \( \Omega \subset \mathbb{R}^N \) (\( N \geq 1 \)) is a bounded domain of Lipschitz class, \( T \) is a fixed positive constant, the mappings

\[ \alpha : \mathbb{R}^M \to \mathcal{P}(\mathbb{R}^M), \quad \gamma : \mathbb{R}^{M \times N} \to \mathcal{P}(\mathbb{R}^{M \times N}) \]

are maximal monotone, \( h : Q \to \mathbb{R}^M \) is prescribed, and \( u : Q \to \mathbb{R}^M \) is unknown. The use of a vector state variable allows us to include systems, as they arise e.g. in rate-independent saddle evolution, cf. (2.26). We shall prove existence of a variational solution and (in the next section) discuss its structural stability, without assuming \( \gamma \) to be cyclically monotone.

Weak formulation. The inclusion (3.1) reads

\[ \alpha_i(D_t u) - \sum_{j=1}^N \nabla_j \gamma_{ij}(\nabla u) \ni h_i \quad \text{in} \quad Q, \quad \text{for} \quad i = 1, ..., M \]  

(3.3)

and is equivalent to the following system:

\[ w - \nabla \cdot z = h \quad \text{in} \quad Q, \]  

(3.4)

\[ w \in \alpha(D_t u) \quad \text{in} \quad Q, \]  

(3.5)

\[ z \in \gamma(\nabla u) \quad \text{in} \quad Q. \]  

(3.6)

This flow is rate-independent, namely invariant w.r.t. any rescaling of time, if and only if \( \alpha \) is positively homogeneous of degree zero, that is,

\[ \alpha(\lambda v) = \alpha(v) \quad \forall v \in \mathbb{R}^M, \forall \lambda > 0; \]  

(3.7)

whenever nonconstant, \( \alpha \) is then multivalued at the origin. After [29]; p. 77], this entails that the monotone mapping \( \alpha \) is cyclically monotone. Thus

\[ \exists \psi : \mathbb{R}^M \to \mathbb{R} \cup \{+\infty\} \text{ convex, lower semicontinuous and proper,} \]

such that \( \alpha = \partial \psi \),

(3.8)

and \( \psi \) is positively homogeneous of degree one, that is,

\[ \forall v \in \mathbb{R}^M \setminus \{0\}, \forall \lambda > 0, \quad \psi(\lambda v) = \lambda \psi(v). \]  

(3.9)

The convex conjugate \( \psi^* \) is then a support function; thus there exists a closed convex set \( K \subset \mathbb{R}^M \) such that

\[ 0 \in K, \quad \psi^*(v) = 0 \quad \text{if} \quad v \in K, \quad \psi^*(v) = +\infty \quad \text{if} \quad v \in \mathbb{R}^M \setminus K. \]  

(3.10)

Next we formulate an initial- and boundary-value problem for the system (3.4)–(3.6). For the sake of simplicity, we prescribe the homogeneous Dirichlet condition on \( \partial \Omega \). We set

\[ V := H_0^1(\Omega)^M \subset H := L^2(\Omega)^M = (L^2(\Omega)^M)' \subset V' = H^{-1}(\Omega)^M \]

(3.11)

(with continuous, compact and dense injections), assume that

\[ \exists c_1 > 0 : \forall v \in \mathbb{R}^M \quad |\partial \psi(v)| \leq c_1, \]  

(3.12)

\[ \exists c_2, c_3 > 0 : \forall v \in \mathbb{R}^{M \times N} \quad |\gamma(v)| \leq c_2 + c_3 |v|, \]  

(3.13)

\[ u^0 \in H, \quad h \in L^2(0,T; H), \]  

(3.14)

and introduce the following weak formulation.
Problem 3.1. Find \( u \in W^{1,1}(0,T; L^1(\Omega)^M) \cap L^2(0,T; V) \), \( w \in L^\infty(Q)^M \) and \( z \in L^2(Q)^{M \times N} \) such that

\[
\int_{\Omega} (w \cdot v + z : \nabla v - h \cdot v) \, dx = 0 \quad \forall v \in V, \text{ a.e. in } ]0,T[. \tag{3.15}
\]

Here by "\( \cdot \)" we denote the contraction over two indices: \( z : \nabla v = \sum_{i=1}^M \sum_{j=1}^N z_{ij} \nabla_j v_i \).

The equation (3.15) is clearly equivalent to (3.4) in \( V' \), a.e. in \( ]0,T[ \).

Variational representation of the monotone relations. By (3.7), the multivalued mapping \( \partial \psi \) determines an operator \( L^1(Q)^M \to \mathcal{P}(L^\infty(Q)^M) \), that is represented e.g. by the Fenchel functional

\[
L^1(Q)^M \times L^\infty(Q)^M \to \mathbb{R} : (v,v') \mapsto \iint_Q [\psi(v) + \psi^*(v')] \, dxdt. \tag{3.19}
\]

The Fenchel system (2.1), (2.2) here reads

\[
\iint_Q [\psi(v) + \psi^*(v')] \, dxdt \geq \iint_Q v' \cdot v \, dxdt \quad \forall (v,v') \in L^1(Q)^M \times L^\infty(Q)^M, \tag{3.20}
\]

\[
\iint_Q [\psi(v) + \psi^*(v')] \, dxdt = \iint_Q v' \cdot v \, dxdt \quad \Leftrightarrow \quad v' \in \partial \psi(v) \text{ a.e. in } Q. \tag{3.21}
\]

Similarly, let us fix a representative function \( \varphi : \mathbb{R}^{M \times N} \to \mathbb{R} \cup \{+\infty\} \) of the maximal monotone mapping \( \gamma \), e.g. the Fitzpatrick function, see (2.3). The associated integral functional represents an operator \( L^2(\mathbb{R}^{M \times N}) \to \mathcal{P}(L^2(\mathbb{R}^{M \times N})) \):

\[
\iint_Q \varphi(v,v') \, dxdt \geq \iint_Q v' \cdot v \, dxdt \quad \forall (v,v') \in L^2(\mathbb{R}^{M \times N})^2, \tag{3.22}
\]

\[
\iint_Q \varphi(v,v') \, dxdt = \iint_Q v' \cdot v \, dxdt \quad \Leftrightarrow \quad v' \in \gamma(v) \text{ a.e. in } Q. \tag{3.23}
\]

Next we reformulate Problem 3.1 in terms of the above representations of the mappings \( \partial \psi \) and \( \gamma \).

Problem 3.2. Find \( u \in W^{1,1}(0,T; L^1(\Omega)^M) \cap L^2(0,T; V) \) and \( z \in L^2(Q)^{M \times N} \) such that

\[
h + \nabla \cdot z \in L^\infty(\Omega)^M, \tag{3.24}
\]

\[
\iint_Q [\psi(D_t u) + \psi^*(h + \nabla \cdot z) - D_t u \cdot (h + \nabla \cdot z)] \, dxdt \leq 0, \tag{3.25}
\]

\[
\iint_Q [\varphi(\nabla u, z) - z : \nabla u] \, dxdt \leq 0, \tag{3.26}
\]

\[
u(\cdot, 0) = u^0 \quad \text{a.e. in } \Omega. \tag{3.27}
\]
As the left-hand sides of (3.25) and (3.26) are both nonnegative, this system is obviously equivalent to the null-minimization of their sum:

\[
J(u, z) := \int_Q \varphi(\nabla u, z) \, dx \, dt \\
+ \int_Q \left[ \psi(D_t u) + \psi'(h + \nabla \cdot z) - D_t u \cdot (h + \nabla \cdot z) - z \cdot \nabla u \right] \, dx \, dt \leq 0. 
\] (3.28)

It is clear that, if a triplet \((u, w, z)\) is a solution of Problem 3.1, then the pair \((u, z)\) solves Problem 3.2, and conversely for a suitable \(w \in L^\infty(Q)^M\).

**Theorem 3.1.** (Existence) Assume that (3.2), (3.8), (3.9), (3.12) and (3.13) are fulfilled, and that

\[
\exists c_4 > 0 : \forall (u, w) \in \text{graph}(\partial \psi), \quad w u \geq c_4 |u|,
\]

\[
\exists c_5 > 0 : \forall (v_i, z_i) \in \text{graph}(\gamma)(i = 1, 2), \quad (z_1 - z_2) \cdot (v_1 - v_2) \geq c_5 |v_1 - v_2|^2,
\]

\[
u_0 \in V, \quad h \in L^2(0, T; H) \cap H^1(0, T; V'),
\]

\[
\exists z^0 \in H : \quad z^0 \in \gamma(\nabla u^0), \quad h(\cdot, 0) + \nabla \cdot z^0 \in K \quad \text{a.e. in } \Omega.
\]

Then Problem 3.2 (equivalently, Problem 3.1) has at least one solution such that

\[
u \in L^\infty(0, T; V), \quad z \in H^1(0, T; L^2(\Omega)^{M \times N}).
\] (3.33)

Notice that \(u \in W^{1,1}(0, T; L^1(\Omega)^M) \cap L^\infty(0, T; V)\) may be identified to a weakly continuous mapping \([0, T] \rightarrow V\).

**Proof.** We shall approximate Problem 3.1 by discretizing the time-derivative, reformulate this problem as a minimization problem, derive a priori estimates, and pass to the limit in the variational formulation.

(i) **Approximation.** We use a somehow nontypical discretization procedure. Let us fix any \(m \in \mathbb{N}\) and set \(k = \frac{T}{m}\); for any function \(v = v(t)\) defined for \(-\infty < t \leq T\), let us set

\[
\tau_k v(t) := v(t - k), \quad D_k v := \frac{v - \tau_k v}{k},
\] (3.34)

and define the respectively piecewise-constant and piecewise-linear interpolation operators

\[
v \mapsto \tilde{v}(k)(t) = \frac{1}{k} \int_{(n - 1)k}^{nk} v(s) \, ds \quad \forall t \in [(n - 1)k, nk], \forall n \in \mathbb{N},
\] (3.35)

\[
v \mapsto \check{v}(k)(t) := \tilde{v}(k)((n - 1)k) + (t - nk) \frac{\tilde{v}(k)(nk) - \tilde{v}(k)((n - 1)k)}{k} \quad \forall t \in [(n - 1)k, nk], \forall n \in \mathbb{N}.
\] (3.36)

Notice that

\[
D_k \tilde{v}(k) = \frac{\tilde{v}(k)(nk) - \tilde{v}(k)((n - 1)k)}{k} = D_k \check{v}(k) = \frac{1}{k} \int_{(n - 1)k}^{nk} D_k v(s) \, ds
\]

\forall t \in [(n - 1)k, nk], \forall n \in \mathbb{N},
\] (3.37)

\[
\| \check{v}(k) - \tilde{v}(k) \|_{L^2(0,T)} \leq \| \check{v}(k) - \tau_k \tilde{v}(k) \|_{L^2(0,T)} \rightarrow 0 \quad \text{as } k \rightarrow 0, \forall v \in L^2(0, T).
\] (3.38)
Let us define $h_k := \tilde{h}_{(k)}$ as in (3.35). Next we approximate Problem 3.1, replacing the exact time-derivative $D_t$ by the discretized derivative $D_k$.

**Problem 3.1.** Find $u_k \in L^2(0,T; V)$, $w_k \in L^\infty(Q)^M$ and $z_k \in L^2(Q)^{M \times N}$ such that, setting $u_k(\cdot,t) = u^0$ for any $t < 0$,

\[
\begin{align*}
  w_k - \nabla \cdot z_k &= h_k \quad \text{in } V', \text{ a.e. in } [0,T], \quad (3.39) \\
  w_k &\in \partial \psi(D_k u_k) \quad \text{a.e. in } Q, \quad (3.40) \\
  z_k &\in \gamma(\nabla u_k) \quad \text{a.e. in } Q. \quad (3.41)
\end{align*}
\]

In order to prove existence of a solution of this problem step by step in time, let us first define the operator

\[
C_k : V \to P(V') : v \mapsto \partial \psi((v - \tau_k u_k)/k) - \nabla \cdot (\nabla v)
\]  

(4.2)

(here the denominator might be dropped, because of the property (3.7) of homogeneity). The system (3.39)–(3.41) then reads

\[
C_k(u_k) \ni h_k \quad \text{in } L^1((n-1)k, nk; V'), \text{ for } n = 1, \ldots, m. \quad (3.43)
\]

The operator $C_k$ is monotone and coercive by (3.30); hence it is maximal monotone. Problem 3.1 then has a solution. For any interval of the form $[(n-1)k, nk]$ ($n = 1, 2, \ldots$) $\tau_k u_k$ is known from the previous step. By the definition of the approximate time-derivative $D_k$, it is then easily seen that there exists a solution of (4.34) that is piecewise constant in time. This concludes the proof of existence of a solution of Problem 3.1.

We already set $u = u^0$ for $t < 0$. Let us recall (3.32), and set

\[
w_k = 0, \quad z_k = z^0, \quad h_k = h(\cdot, 0) \quad \text{a.e. in } \Omega, \forall t < 0.
\]

As $0 \in K$ (cf. (3.10)), $0 \in \partial \psi(0)$; the system (3.39)–(3.41) is thus fulfilled for $t < 0$, too.

**(ii) A Priori Estimates.** By taking the incremental ratio in (3.39), we get

\[
\int_{\Omega} D_k w_k \cdot (w_k - \tau_k u_k) \, dx \geq \int_{\Omega} [\psi^*(w_k) - \psi^*(\tau_k u_k)] \, dx \quad \text{a.e. in } [0,T].
\]

Let us multiply this equality by $D_k u_k$, and integrate w.r.t. space and time. Notice that (3.40) also reads $D_k u_k \in \partial \psi^*(w_k)$ a.e. in $\Omega$, whence

\[
\int_{\Omega} \nabla \cdot D_k z_k \, dx \geq c_5 \int_{\Omega} |D_k \nabla u_k|^2 \, dx \quad \text{a.e. in } [0,T].
\]

On the other hand, denoting by $\langle \cdot, \cdot \rangle$ the duality pairing between $V'$ and $V$, we have

\[
-(\nabla \cdot D_k z_k, D_k u_k) = \int_{\Omega} D_k z_k : D_k \nabla u_k \, dx \geq c_5 \int_{\Omega} |D_k \nabla u_k|^2 \, dx \quad \text{a.e. in } [0,T].
\]

We thus get

\[
\int_{\Omega} \psi^*(w_k(x,t)) \, dx + c_5 \int_{0}^{t} \int_{\Omega} |\nabla D_k u_k|^2 \, dx \, dt
\]

\[
\leq \int_{\Omega} \psi^*(w_k(x,0)) \, dx + \int_{0}^{t} \|D_k h_k\|_{V'} \|D_k u_k\|_V \, d\tau \quad \text{for a.e. } t \in [0,T]. \quad (3.45)
\]

As $w_k(\cdot, 0) = 0$ a.e. in $\Omega$, by (3.10) $\psi^*(w_k(\cdot, 0)) = 0$. By (3.31), the latter inequality then yields

\[
\|D_k u_k\|_{L^2(0,T;V)} \leq C_1. \quad (3.46)
\]
(By $C_1, C_2, \ldots$ we shall denote positive constants independent of $k$.) By (3.13) and (3.41), we then get
\[ \|D_kz_k\|_{L^2(\Omega)^{M\times N}} \leq C_2. \] (3.47)
By comparing the terms of (3.39) and by the uniform boundedness of $\partial \psi$, cf. (3.12), we conclude that
\[ \|w_k\|_{L^\infty(Q)^M}, \|\nabla \cdot z_k\|_{L^\infty(0,T;H)} \leq C_2. \] (3.48)

(iii) Passage to the Limit. By these estimates, there exist $u, z$ such that, as $k \to 0$ along a suitable sequence, We denote the strong, weak, and weak star convergence respectively by $\to, \rightharpoonup, \rightharpoonup^*$.

\[ u_k \rightharpoonup u \quad \text{in } H^1(0,T;V), \] (3.49)
\[ w_k \rightharpoonup w \quad \text{in } L^\infty(Q)^M, \] (3.50)
\[ z_k \to z \quad \text{in } H^1(0,T;L^2(\Omega)^{M\times N}), \] (3.51)
\[ \nabla \cdot z_k \rightharpoonup \nabla \cdot z \quad \text{in } L^\infty(0,T;H), \] (3.52)
\[ \nabla \cdot z_k(\cdot, T) \rightharpoonup^* \nabla \cdot z(\cdot, T) \quad \text{in } H. \] (3.53)

(Notice that $w_k = h_k + \nabla \cdot z_k$ for any $k$, and $w = h + \nabla \cdot z$.) By passing to the limit in (3.39) we then get (3.4) in $V'$, a.e. in $[0,T[$.

In order to derive (3.26), let us note that (3.41) is equivalent to
\[ \iint_Q \varphi(\nabla u_k, z_k) \, dxdt - \iint_Q z_k : \nabla u_k \, dxdt \leq 0. \] (3.54)
By (3.11) and (3.49),
\[ u_k \rightharpoonup u \quad \text{in } L^2(Q)^M, \] (3.55)
which jointly with (3.52) yields
\[ \iint_Q z_k : \nabla u_k \, dxdt = - \iint_Q (\nabla \cdot z_k) \cdot u_k \, dxdt \]
\[ \to - \iint_Q (\nabla \cdot z) \cdot u \, dxdt = \iint_Q z : \nabla u \, dxdt. \] (3.56)

By passing to the lower limit in (3.54), we then get (3.26).

Finally, we shall derive (3.25). First notice that (3.39) and (3.40) yield
\[ \iint_Q \left[ \psi(D_t u_k) + \psi^\ast(h_k + \nabla \cdot z_k) - D_t u_k \cdot (h_k + \nabla \cdot z_k) \right] \, dxdt \leq 0, \] (3.57)
and that, by (3.51) and (3.52),
\[ \nabla \cdot D_t z_k \rightharpoonup \nabla \cdot D_t z \quad \text{in } H^{-1}(0,T;H) \cap L^2(0,T;H^{-1}(\Omega)^M). \] (3.58)

By the Rellich compactness theorem and by Hilbert-space interpolation (see e.g. [35]; vol. II, Chap. 4], we have
\[ H^1(0,T;H^1(\Omega)^M) \subset H^{1/2}(0,T;H^{1/2}(\Omega)^M) \quad \text{with compact injection}, \] (3.59)
\[ H^{-1}(0,T;H) \cap L^2(0,T;H^{-1}(\Omega)^M) \subset H^{-1/2}(0,T;H^{-1/2}(\Omega)^M) = H^{1/2}(0,T;H^{1/2}(\Omega)^M)^\prime. \] (3.60)
By (3.49) and (3.58)–(3.60),
\begin{align*}
\int_Q (\nabla \cdot (\bar{k} - \bar{k}(\cdot, T))) \cdot D_t u_k \, dxdt &= - \int_Q \langle \nabla \cdot D_t \bar{k} \rangle \cdot (u_k - u^0) \, dxdt \\
&\rightarrow - \int_Q \langle \nabla \cdot D_t z \rangle \cdot (u - u^0) \, dxdt = \int_Q \langle \nabla \cdot [z - z(\cdot, T)] \rangle \cdot D_t u \, dxdt;
\end{align*}
(3.61)
on the other hand, by (3.49) and (3.53),
\begin{align*}
\int_Q (\nabla \cdot \bar{k}(\cdot, T)) \cdot D_t u_k \, dxdt &= \int_{\Omega} \langle \nabla \cdot \bar{k}(\cdot, T) \rangle \cdot (u_k(\cdot, T) - u^0) \, dx \\
&\rightarrow \int_{\Omega} \langle \nabla \cdot z(\cdot, T) \rangle \cdot (u(\cdot, T) - u^0) \, dx = \int_Q \langle \nabla \cdot z(\cdot, T) \rangle \cdot D_t u \, dxdt.
\end{align*}
The two latter formula thus yield
\begin{align*}
\int_Q \langle \nabla \cdot \bar{k} \rangle \cdot D_t u_k \, dxdt &\rightarrow \int_Q \langle \nabla \cdot z \rangle \cdot D_t u \, dxdt.
\end{align*}
(3.63)
By passing to the lower limit in (3.57), we then get (3.25).

**Remark.** (i) The strong monotonicity of the operator \( \gamma \) plays an important role in this existence result, even if \( \gamma \) is linear. This is confirmed by the next example. Let \( a > 0, b \in \mathbb{R}^N \), \( \{c_{ij}\} \) be a skew-symmetric \( N \times N \)-matrix, and consider the following dynamics for a scalar field \( u = u(x, t) \)
\begin{equation}
\text{sign}(D_t u) - a \Delta u + \sum_{i,j=1}^N c_{ij} \nabla_j u + \sum_{j=1}^N b_j \nabla_j u = h \quad \text{in } Q.
\end{equation}
(3.64)
If \( h \in H^1(0, T; H^{-1}(\Omega)) \) and \( u^0 \in H^1(0, T; H_0^1(\Omega)) \), then the corresponding Cauchy problem has a solution \( u \in H^1(0, T; H_0^1(\Omega)) \). But the asymptotic behavior of \( u = u_a \) as \( a \rightarrow 0^+ \) is not clear to this author.

(ii) Theorem 3.1 may be extended to rate-dependent doubly-nonlinear equations of the form (1.1).

\[ \Box \]

4. **Structural stability.** In this section we deal with the dependence of the solution(s) \((u, z)\) of Problem 3.2 on the data \( u^0, h \), on the mapping \( \gamma \), and on the domain \( \Omega \). The variational formulation allows us to apply some classical variational techniques, including De Giorgi’s notion of \( \Gamma \)-convergence, see e.g. [17], [19].

\( \Gamma \)-**Convergence of representative functions.** First, after [60] and along the lines of the parallel work [63], we introduce and study a nonlinear convergence, that looks rather natural in the class of representative functions. Let \( B \) be a real Banach space, and denote by \( \langle \cdot, \cdot \rangle \) the duality pairing between \( B' \) and \( B \).

We shall denote by \( \bar{\pi} \) the coarsest among the topologies of \( B \times B' \) that are finer than the product of the weak topology of \( B \) by the weak star topology of \( B' \), and that make the duality pairing \( \langle v, v' \rangle \mapsto \langle v', v \rangle \) continuous. Thus, for any sequence \( \{(v_n, v'_n)\} \) in \( B \times B' \),
\begin{align*}
(v_n, v'_n) &\rightarrow (v, v') \quad \text{in } B \times B' \quad \Leftrightarrow \\
v_n &\rightarrow v \quad \text{in } B, \quad v'_n &\rightharpoonup^* v' \quad \text{in } B', \quad \langle v'_n, v_n \rangle \rightarrow \langle v', v \rangle,
\end{align*}
(4.1)
and similarly for any net.

**Lemma 4.1.** [63] (Local metrizability) If \( B' \) is separable, then there exists a metric on \( B \times B' \) that induces a topology, whose restriction to any bounded subset of \( B \times B' \) coincides with the restriction of the topology \( \bar{\pi} \).
The next statement rests upon the previous lemma, and may be proved along the lines of Proposition 8.10 and Corollary 8.12 of [17].

We remind the reader that we denote by $\mathcal{F}(B)$ the set of the convex and lower semicontinuous functions $f : B \times B' \to \mathbb{R} \cup \{+\infty\}$ such that $f(v,v') \geq \langle v',v \rangle$ for any $(v,v') \in B \times B'$.

**Proposition 4.2.** [63] (Compactness) Let $B'$ be separable, and let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of functions $B \times B' \to \mathbb{R} \cup \{+\infty\}$ such that

$$\forall C \in \mathbb{R}, \sup_{n \in \mathbb{N}} \left\{ \|v\|_B + \|v'\|_{B'} : f_n(v,v') \leq C \right\} < +\infty.$$  

Then:

(i) up to a subsequence, $f_n \Gamma$-converges to some function $f$ w.r.t. the topology $\Gamma$.

(ii) $f_n \Gamma$-converges w.r.t. the topology $\Gamma$ if and only if it $\Gamma$-converges sequentially w.r.t. the same topology.

The next statement provides the stability of the class of representative functionals, and that of the represented operators.

**Proposition 4.3.** [63] (Stability) Let $B'$ be separable, and let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathcal{F}(B)$ that $\Gamma$-converges to some function $f$ w.r.t. the topology $\Gamma$. Then:

(i) $f \in \mathcal{F}(B)$.

(ii) Let us denote by $\beta_n$ ($\beta$, resp.) the operator $B \to \mathcal{P}(B')$ that is represented by $f_n$ ($f$, resp.). For any sequence $\{(v_n,v'_n)\}$ in $B \times B'$,

$$v'_n \in \beta_n(v_n) \quad \forall n, \quad (v_n,v'_n) \Gamma (v,v') \Rightarrow v' \in \beta(v);$$  

(i.e., the upper limit of $\beta_n$ in the sense of Kuratowski w.r.t. the topology $\Gamma$ is included into $\beta$.

**Structural stability.** Our program reads as follows. Let us first denote

by $\mathcal{D}$ the set of the admissible data (here $\Omega$, $u^0$ and $h$);

by $\mathcal{O}$ the set of the admissible operators (here $D_t$, $\nabla$, and those associated to $\partial \psi$ and $\gamma$);

by $\mathcal{S}$ the set of the admissible solutions.

These three sets must be equipped with appropriate notions of convergence. By the existence Theorem 3.1, there exists a (possibly multivalued) resolution operator, $\mathcal{R} : \mathcal{D} \times \mathcal{O} \to \mathcal{S}$. We would like to show that $\mathcal{D}$, $\mathcal{O}$ and $\mathcal{R}$ are sequentially compact and structurally stable, in the sense of the following three points:

(i) any bounded sequence $\{(d_n,o_n)\}$ in $\mathcal{D} \times \mathcal{O}$ has an accumulation point $(d,o)$, and, for any corresponding sequence of solutions $\{s_n \in \mathcal{R}(d_n,o_n)\}$,

(ii) $s_n$ accumulates at some point $s \in \mathcal{S}$,

(iii) this entails that $s \in \mathcal{R}(d,o)$.

However, here we shall be content with a more modest result: we shall just variate the data $u^0$, $h$, and the mapping $\gamma$; we shall also outline variations of the differential operators $D_t$, $\nabla$ and of the domain $\Omega$. More specifically, we shall apply the Proposition 4.2 and 4.3 to the perturbations of a representative function $\psi$ of the mapping $\gamma$. We cannot proceed similarly for the function $\psi + \psi^*$ (that represents $\partial \psi$), since the associated integral functional $(v,v') \mapsto \int_Q [\psi(v) + \psi^*(v')] \, dx \, dt$ is coercive in $L^1(Q)^M \times L^\infty(Q)^M$, and there the separability assumption fails.
Theorem 4.4. Let $\psi : \mathbb{R}^M \to \mathbb{R} \cup \{+\infty\}$ be convex, lower semicontinuous and proper, and fulfill (3.9), (3.12) and (3.29). For any $k$, let $\gamma_k : \mathbb{R}^{M \times N} \to \mathcal{P}(\mathbb{R}^{M \times N})$ be maximal monotone, and be represented by a mapping $\varphi_k \in \mathcal{F}(\mathbb{R}^{M \times N})$. Let $h_k, u^0_k$ be as in (3.31); let (3.13), (3.30), (3.32) be fulfilled uniformly w.r.t. $k$; finally, let the sequence $\{\varphi_k\}$ be equicoercive in $\mathbb{R}^{M \times N}$ in the sense of (4.2). Then:

(i) There exist $u^0, h$ and $\varphi$ such that

$$u^0_k \to u^0 \quad \text{in } V, \quad h_k \rightharpoonup h \quad \text{in } L^2(0,T;H) \cap H^1(0,T;V'),$$

and $\int_Q v_k : v'_k \, dxdt \to \int_Q v : v' \, dxdt$, then

$$\liminf_{m \to \infty} \int_Q \varphi_k(v_k, v'_k) \, dxdt \geq \int_Q \varphi(v, v') \, dxdt,$$  \quad \varphi \in \mathcal{F}(\mathbb{R}^{M \times N}). \tag{4.6}$$

(ii) For any $k$, let $(u_k, z_k)$ be a solution of the corresponding Problem 3.2, (that exists by Theorem 3.1). Then there exists a pair $(u, z)$ such that, as $k \to 0$ along a suitable sequence,

$$u_k \rightharpoonup u \quad \text{ in } H^1(0,T;V),$$

$$z_k \rightharpoonup z \quad \text{ in } H^1(0,T;L^2(\Omega)^{M \times N}),$$

$$\nabla \cdot z_k \rightharpoonup \nabla \cdot z \quad \text{ in } L^\infty(0,T;H).$$

(iii) $(u, z)$ is then a solution of Problem 3.2.

Proof. (i) By the boundedness of the sequences $\{u^0_k\}$ and $\{h_k\}$, there exist $u^0$ and $h$ as in (4.4).

Let us denote by $\Phi_k \in \mathcal{F}(L^2(Q)^{M \times N})$ the integral functional

$$\Phi_k(v,v') = \int_Q \varphi_k(v,v') \, dxdt \quad \forall (v,v') \in (L^2(Q)^{M \times N})^2, \forall k.$$ 

By the assumed equicoerciveness of the mappings $\varphi_k$, this sequence of functionals is also equicoercive in $(L^2(Q)^{M \times N})^2$. By Proposition 4.2, $\Phi_k$ then $\Gamma$-converges w.r.t. the topology $\tilde{\tau}$ to some functional $\Phi \in \mathcal{F}(L^2(Q)^{M \times N})$, up to extracting a subsequence. By Theorem 20.4 of [17], there exists a Borel function $\varphi : \mathbb{R}^{M \times N} \to \mathbb{R}$ such that

$$\Phi(v,v') = \int_Q \varphi(v,v') \, dxdt \quad \forall (v,v') \in (L^2(Q)^{M \times N})^2, \forall k.$$  

The two latter statements yield (4.5). By restricting $\Phi$ to constant functions, it is easily seen that $\varphi$ is convex and lower semicontinuous, and that $\varphi(v,v') \geq \langle v', v \rangle$ for any $(v,v') \in B \times B'$. (4.6) is thus established.

(ii) Let us label by the index $k$ the solution of Problem 3.2 written in terms of $u^0_k, h_k, \gamma_k$. As we saw, for any $k$ the pair $(u_k, z_k)$ solves Problem 3.2; we thus have

$$\int_Q \left[ \psi(D_t u_k) + \psi^*(h_k + \nabla \cdot z_k v) - D_t u_k : (h_k + \nabla \cdot z_k) \right] \, dxdt \leq 0,$$

$$\int_Q \left[ \varphi(\nabla u_k, z_k) - z_k : \nabla u_k \right] \, dxdt \leq 0.$$
Uniform estimates like (3.46)–(3.48) may be derived by mimicking the procedure of Sect. 3, so that (4.7)–(4.9) hold up to subsequences.

(iii) The statements (3.56) and (3.63) may be derived as we did in Sect. 3. This allows one to pass to the lower limit as \( k \to \infty \) in (4.10) and (4.11). The pair \( (u, z) \) therefore solves Problem 3.2.

\[ \square \]

**Domain variation.** Let us replace the domain \( \Omega \) by a sequence of Lipschitz domains \( \{\Omega_k\} \), each one corresponding to the space \( V_k = H^1_0(\Omega_k) \). Let us extend any function \( \Omega_k \to \mathbb{R} \) to the whole \( \mathbb{R}^N \) with vanishing value; this clearly preserves the \( L^p \)- and \( H^1_0 \)-regularities. We may thus identify each \( V_k \) with the space of these extensions. Denoting the indicator function of \( V_k \) by \( I_{V_k} \), we may thus reformulate (1.1) in \( \Omega_k \) as

\[ D_t \partial \psi (u_k) - \nabla \cdot \gamma_k (\nabla u_k) + \partial I_{V_k} (u_k) \ni \mu_k \quad \text{in} \; D'(\mathbb{R}^N). \tag{4.12} \]

The extension of the associated Problems 3.1 and 3.2 is then fairly obvious. As the operator \( H^1(\mathbb{R}^N) \to H^{-1}(\mathbb{R}^N) : v \mapsto -\nabla \cdot \gamma_k (\nabla v) \) is maximal monotone and is defined in the whole space, and \( I_{V_k} \) is convex and lower semicontinuous as a functional on \( H^1(\mathbb{R}^N) \), the sum \(-\nabla \cdot \gamma_k (\nabla u_k) + \partial I_{V_k} (u_k) \) is also maximal monotone in this space. Theorem 4.1 then takes over to this setting.

The issue of domain variation is relevant for several applications. For instance, a classical problem of homogenization consists in considering an (e.g. periodic) medium with holes, and then passing to the limit as the periodicity length-scale vanishes. Here we shall not dwell on this issue.

**Remark.** (i) The above theorem might also be extended to include the stability of Problem 3.2 w.r.t. the operators \( D_t, \nabla \). For instance, the previous approximation of Sect. 3 was based on the discretization of the time-derivative. The operator \( \nabla \) might also be discretized.

(ii) The results of Sects. 3 and 4 might easily be extended to the case of space-dependent \( \psi \) and \( \gamma \), and to abstract equations of the form

\[ \partial \psi (D_t u) + \beta (u) \ni h \quad \text{in} \; V', \; \text{a.e. in} \; ]0, T[, \tag{4.13} \]

for \( \psi \) as in (1.2), and for a maximal monotone operator \( \beta : V \to \mathcal{P}(V') \), which might also explicitly depend on time.

(iii) More generally, for any convex lower semicontinuous mapping \( \tilde{\psi} : \mathbb{R}^M \to \mathbb{R} \cup \{+\infty\} \), the inclusion

\[ \partial \tilde{\psi} (D_t u) + \partial \psi (D_t u) + \beta (u) \ni h \quad \text{in} \; V', \; \text{a.e. in} \; ]0, T[, \tag{4.14} \]

combines rate-dependent and rate-independent evolutions. This class of equations has already been studied in a number of works, see the Introduction.

(iv) Instead of a single inclusion, one might fix a measure space \( (S, \mathcal{A}, \mu) \), with \( \mu \) a nonnegative finite measure, prescribe two families \( \{\psi _\rho \}_\rho \in S \) and \( \{\beta _\rho \}_\rho \in S \) as above, and then deal with a family of inclusions of the form

\[ \partial \psi _\rho (D_t u_\rho) + \beta _\rho (u_\rho) \ni h \quad \text{in} \; V', \; \text{a.e. in} \; ]0, T[, \forall \rho \in S. \tag{4.15} \]

This defines a (possibly multivalued) operator \( f_\rho : h \mapsto u_\rho \). In analogy with the classical models of hysteresis due to Preisach and Prandtl-Ishlinskii (see e.g. [11], [32], [33], [39], [56]), under suitable assumptions, one may thus define a (possibly multivalued) operator

\[ F_\mu : h \mapsto u := \int _S u_\rho \, d\mu (\rho) \tag{4.16} \]
between suitable spaces of vector functions of time; see e.g. [56]; Chaps. IV, VI.

(v) Instead of assuming that the input function $h$ is prescribed, one might couple either the inclusion (4.13) or the system (4.15), (4.16) with a P.D.E. relating $u$ and $h$: for instance, this might be either of the equations

$$D_t(u + h) - \Delta h = g, \quad D_t h - \Delta h + u = g,$$

(4.17)

for a known field $g$. One might prove existence of a solution for an associated initial-and boundary-value problem; see e.g. [56].

(vi) In general the uniqueness of the solution of the inclusion (4.13) is hard to be established. If the monotone operator $\beta$ is linear and self-adjoint, then one may prove uniqueness via a standard procedure: one writes the equation for two solutions $u_1, u_2 \in H^1(0,T; V)$, multiplies the difference of these equations by $D_t(u_1 - u_2)$, and integrates in time; this easily yields $u_1 = u_2$. This technique however fails whenever $\beta$ is not self-adjoint. ☐

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