1 Distributions

The theory of distributions was introduced in the 1940s by Laurent Schwartz, who provided a thorough functional formulation on the basis of previous ideas of Heaviside, Dirac and others, and forged a powerful tool of calculus. Distributions offer also a solid basis for the construction of Sobolev spaces, that had been introduced by Sobolev in the 1930s using the notion of *weak derivative.* These spaces play a fundamental role in the modern analysis of either linear or nonlinear partial differential equations. In this section we just outline some basic tenets of this theory, and provide some tool that we used ahead.

Test Functions. Let Ω be a domain of $\mathbb{R}^N$. By $\mathcal{D}(\Omega)$ we denote the space of infinitely differentiable functions $\Omega \to \mathbb{C}$ whose support is a compact $K \subset \Omega$. [1] These are named **test functions.** For any compact subset K of Ω, let us denote by $\mathcal{D}_K(\Omega)$ the space of infinitely differentiable functions $\Omega \to \mathbb{C}$ whose support is contained in K. Obviously $\mathcal{D}(\Omega) = \bigcup_{K\subset\subset\Omega} \mathcal{D}_K(\Omega)$.

The null function is the only analytic function of $\mathcal{D}(\Omega)$. The *bell-shaped* function

$$\rho(x) := \begin{cases} \exp\left[\left(|x|^2 - 1\right)^{-1}\right] & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1 \end{cases} \tag{1.1}$$

also belongs to $\mathcal{D}(\mathbb{R})$. By suitably translating ρ and by rescaling w.r.t. x, further nontrivial elements of $\mathcal{D}(\Omega)$ are easily constructed for any Ω.

$\mathcal{D}(\Omega)$ is equipped with the finest topology among those that make all injections $\mathcal{D}_K(\Omega) \to \mathcal{D}(\Omega)$ continuous (so-called *inductive limit topology).* [This topology makes $\mathcal{D}(\Omega)$ a nonmetrizable locally convex Hausdorff space.] []

A sequence $\{u_n\}$ in $\mathcal{D}(\Omega)$ converges to $u \in \mathcal{D}(\Omega)$ in this topology iff, for some compact set $K \subset \Omega$, $u \in \mathcal{D}_K(\Omega)$ and $u_n \to u$ in $\mathcal{D}_K(\Omega)$. This means that

(i) there exists a $K \subset\subset \Omega$ that contains the support of any u_n, and

(ii) $\sup_\Omega |D^\alpha u_n| \to 0$ for any $\alpha \in \mathbb{N}^N$. []

Distributions. All linear and continuous functionals $\mathcal{D}(\Omega) \to \mathbb{C}$ are called **distributions.** These functionals thus form the dual space $\mathcal{D}'(\Omega)$. For any $T \in \mathcal{D}'(\Omega)$ and any $v \in \mathcal{D}(\Omega)$ we also write $\langle T, v\rangle := T(v)$. Here are some examples:

(i) For any $f \in L^1_{loc}(\Omega)$, the integral functional $T_f : v \mapsto \int_\Omega f(x)\, v(x)\, dx$ is a distribution. (The sequential continuity is easily checked on the basis of the above definition of convergence in $\mathcal{D}(\Omega)$. []) The mapping $f \mapsto T_f$ is injective, so that we can identify $L^1_{loc}(\Omega)$ with a subspace of $\mathcal{D}'(\Omega)$.

(ii) Although the real function $x \mapsto 1/x$ is not locally integrable in $\mathbb{R}$, its *principal value* (p.v.), that is defined by

$$\langle \text{p.v. } \frac{1}{x}, v\rangle := \lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} \frac{v(x)}{x}\, dx \qquad \forall v \in \mathcal{D}(\mathbb{R}), \tag{1.2}$$

is a distribution. For any $v \in \mathcal{D}(\mathbb{R})$ and for any $a > 0$ such that $\text{supp}(v) \subset [-a, a]$, by the oddness of the function $1/x$, we have

$$\begin{aligned} \langle \text{p.v. } \tfrac{1}{x}, v\rangle &= \lim_{\varepsilon\to 0^+} \left(\int_{\varepsilon<|x|<a} \tfrac{v(x)-v(0)}{x}\, dx + \int_{\varepsilon<|x|<a} \tfrac{v(0)}{x}\, dx \right) \\ &= \lim_{\varepsilon\to 0^+} \int_{\varepsilon<|x|<a} \tfrac{v(x)-v(0)}{x}\, dx; \end{aligned} \tag{1.3}$$

this limit exists finite, for by the mean value theorem

$$\left| \int_{\varepsilon<|x|<a} \frac{v(x) - v(0)}{x}\, dx \right| \leq 2a \max_{\mathbb{R}} |v'| \qquad \forall \varepsilon > 0. \tag{1.4}$$

[1] We shall write $K \subset\subset \Omega$ to mean that K is a compact subset of Ω.

Notice that the principal value is quite different from other notions of *generalized integral.*

(iii) For any $x_0 \in \Omega$ the **Dirac mass** $\delta_{x_0} : v \mapsto v(x_0)$ is a distribution. [Ex]

(iv) The series of Dirac masses $\sum_{n=1}^{\infty} \delta_{x_n} : v \mapsto \sum_{n=1}^{\infty} v(x_n)$ is also a distribution iff $\{x_n\}$ is a sequence in Ω that intersects any compact subset of Ω only in a finite number of points. [Ex]

Theorem 1.1 . *(Characterization of Distributions) For any linear functional $T : \mathcal{D}(\Omega) \to \mathbb{C}$ the following properties are mutually equivalent:*

(i) $T \in \mathcal{D}'(\Omega)$;

(ii) T is sequentially continuous; i.e., $T(v_n) \to 0$ whenever $v_n \to 0$ in $\mathcal{D}(\Omega)$;

(iii)

$$\begin{array}{l} \forall K \subset\subset \Omega, \exists m \in \mathbb{N}, \exists C > 0 : \forall v \in \mathcal{D}(\Omega), \\ \text{supp}(v) \subset K \quad \Rightarrow \quad |T(v)| \le C \max_{|\alpha| \le m} \sup_K |D^\alpha v|. \qquad [] \end{array} \tag{1.5}$$

We equip the space $\mathcal{D}'(\Omega)$ with the sequential (weak) convergence: for any sequence $\{T_n\}$ in $\mathcal{D}'(\Omega)$,

$$T_n \to 0 \quad \text{in} \quad \mathcal{D}'(\Omega) \quad \Leftrightarrow \quad T_n(v) \to 0 \quad \forall v \in \mathcal{D}(\Omega). \tag{1.6}$$

This makes $\mathcal{D}'(\Omega)$ a nonmetrizable locally convex Hausdorff space. []

Proposition 1.2 *If $T_n \to T$ in $\mathcal{D}'(\Omega)$ and $v_n \to v$ in $\mathcal{D}(\Omega)$, then $T_n(v_n) \to T(v)$. []*

Differentiation of Distributions. We define the multiplication of a distribution by a C^∞-function and the differentiation of a distribution via **transposition:** [2]

$$\langle fT, v\rangle := \langle T, fv\rangle \qquad \forall T \in \mathcal{D}'(\Omega), \forall f \in C^\infty(\Omega), \forall v \in \mathcal{D}(\Omega), \tag{1.7}$$

$$\langle \tilde{D}^\alpha T, v\rangle := (-1)^{|\alpha|} \langle T, D^\alpha v\rangle \qquad \forall T \in \mathcal{D}'(\Omega), \forall v \in \mathcal{D}(\Omega), \forall \alpha \in \mathbb{N}^N. \tag{1.8}$$

It may be checked that $\tilde{D}^\alpha T$ is a distributions via the characterization (**??**)0.2). [Ex] Thus any distribution has derivatives of any order.

For any $f \in L^1_{loc}(\Omega)$ this definition of the derivative is clearly consistent with partial integration: setting $T = T_f$, (1.8) indeed reads

$$\int_\Omega [D^\alpha f(x)] v(x)\, dx = (-1)^{|\alpha|} \int_\Omega f(x) D^\alpha v(x)\, dx \qquad \forall v \in \mathcal{D}(\Omega), \forall \alpha \in \mathbb{N}^N. \tag{1.9}$$

The operator $\tilde{D}^\alpha$ is linear and continuous in $\mathcal{D}'(\Omega)$. [Ex] As derivatives commute in $\mathcal{D}(\Omega)$, the same applies to $\mathcal{D}'(\Omega)$, that is, $\tilde{D}^\alpha \circ \tilde{D}^\beta = \tilde{D}^{\alpha+\beta} = \tilde{D}^\beta \circ \tilde{D}^\alpha$ for any multi-indices α, β. [Ex]

The formula of differentiation of the product is extended as follows:

$$\tilde{D}_i(fT) = (D_i f)T + f \tilde{D}_i T \qquad \forall f \in C^\infty(\Omega), \forall T \in \mathcal{D}'(\Omega), \forall i; \tag{1.10}$$

in fact

$$\begin{aligned} \langle \tilde{D}_i(fT), v\rangle &= -\langle fT, D_i v\rangle = -\langle T, f D_i v\rangle = \langle T, (D_i f)v\rangle - \langle T, D_i(fv)\rangle \\ &= \langle (D_i f)T, v\rangle + \langle \tilde{D}_i T, fv\rangle = \langle (D_i f)T + f\tilde{D}_i T, v\rangle \quad \forall v \in \mathcal{D}(\Omega). \end{aligned} \tag{1.11}$$

A recursive procedure then yields the extension of the classic Leibniz rule:

$$\tilde{D}^\alpha(fT) = \sum_{\beta \le \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} (D^{\alpha-\beta} f) \tilde{D}^\beta T \qquad \forall f \in C^\infty(\Omega), \forall T \in \mathcal{D}'(\Omega), \forall \alpha \in \mathbb{N}^N. \; [Ex] \tag{1.12}$$

The translation, the conjugation and other linear operations on functions are also easily extended to distributions via transposition. [Ex]

Comparison with Classic Derivatives.

[2] In this section we denote the distributional derivative by $\tilde{D}^\alpha$, and the classic derivative, i.e. the pointwise limit of the incremental ratio, by D^α, whenever the latter exists.

Theorem 1.3 . *(Du-Bois Reymond) Let $f \in C^0(\Omega)$ and $i \in \{1, \dots, N\}$. Then $\tilde{D}_i f \in C^0(\Omega)$ (possibly after modification in a set of vanishing measure) iff f is classically differentiable w.r.t. x_i and $D_i f \in C^0(\Omega)$. In this case $\tilde{D}_i f = D_i f$ in Ω. []*

The next statement applies to $N = 1$, with $\Omega :=]a, b[$ and $-\infty \le a < b \le +\infty$. A multidimensional extension will be provided in the next section.

First we remind the reader that a function $f \in L^1(a, b)$ is absolutely continuous iff there exists $g \in L^1(a, b)$ such that $f(x) = f(y) + \int_y^x g(\xi)\, d\xi$ for any $x, y \in]a, b[$. This entails that $f' = g$ a.e. in $]a, b[$. Thus if $f \in L^1(a, b)$ is absolutely continuous, then it is a.e. differentiable and $f' \in L^1(a, b)$. The converse may fail: the Heaviside function H is a counterexample, for $H' = 0$ a.e. in **R**, but of course H is not absolutely continuous (and is not a.e. equal to any absolutely continuous function). [3]

Theorem 1.4 . *A function $f \in L^1(a, b)$ is a.e. equal to an absolutely continuous function, $\hat{f}$, iff $\tilde{D} f \in L^1(a, b)$. In this case $\tilde{D} f = D \hat{f}$ a.e. in $]a, b[$.*

If $f \in L^1_{loc}(a, b)$ then Df may exist a.e. in $]a, b[$ and even be locally integrable without coinciding with the distributional derivative $\tilde{D} f$, which need not be an element of $L^1_{loc}(a, b)$. For instance, let H be the Heaviside function: $H(x) := 0$ if $x < 0$, $H(x) := 1$ if $x \ge 0$. We have $DH = 0$ a.e., but $\tilde{D} H = \delta_0$ for

$$\langle \tilde{D} H, v \rangle = - \int_{\mathbb{R}} H(x)\, Dv(x)\, dx = - \int_{\mathbb{R}^+} Dv(x)\, dx = v(0) = \langle \delta_0, v \rangle \qquad \forall v \in \mathcal{D}(\mathbb{R}). \tag{1.13}$$

Support and Order of Distributions. A distribution $T \in \mathcal{D}'(\Omega)$ is said to vanish in an open subset $\tilde{\Omega}$ of Ω iff it vanishes on any function of $\mathcal{D}(\Omega)$ supported in $\tilde{\Omega}$. There exists a (possibly empty) largest open set $A \subset \Omega$ in which T vanishes; [] [4] its complement in Ω is called the **support** of T, and is denoted by $\text{supp}(T)$.

If m is the smallest integer that fulfills (1.5), we say that T has *order m in K*. The supremum of these orders is called the **order** of T; thus distributions may be of either finite or infinite order. For instance the (distributions that can be identified with) functions of $L^1_{loc}(\Omega)$ and the Dirac mass are of order zero. On the other hand $\tilde{D}^\alpha \delta_0$ is of order $|\alpha|$ for any $\alpha \in \mathbb{N}^N$, and p.v. $(1/x)$ is of order one. [Ex]

The next statement establishes a strict relation between support and order.

Theorem 1.5 . *Any compactly supported distribution is of finite order. (The converse may fail.)*

In view of the analysis of the Fourier transform via distributions, next we define two further important spaces of distributions.

The Space $\mathcal{E}(\Omega)$ and its Dual. In the framework of the theory of distributions the space $C^\infty(\Omega)$ is usually denoted by $\mathcal{E}(\Omega)$. We shall equip the dual space $\mathcal{E}'(\Omega)$ ($\subset \mathcal{D}'(\Omega)$) with the sequential (weak) convergence induced by $\mathcal{D}'(\Omega)$: for any sequence $\{T_n\}$ in $\mathcal{D}'(\Omega)$,

$$T_n \to 0 \quad \text{in} \quad \mathcal{E}'(\Omega) \quad \Leftrightarrow \quad T_n(v) \to 0 \quad \forall v \in \mathcal{E}(\Omega). \tag{1.14}$$

[This makes $\mathcal{D}'(\Omega)$ a Fréchet space and a dense subspace of $\mathcal{D}'(\Omega)$.] []

[3] Functions that are a.e. equal to an absolutely continuous function are sometimes said absolutely continuous themselves, as they are identified with their equivalence class.

[4] This is the union of all the open sets in which T vanishes, since if a distribution vanishes on a family of open sets then it also vanishes on their union. This property is less trivial than it may look: the argument is based on a classic tool, named *partition of unity.*

Theorem 1.6 . *$\mathcal{E}'(\Omega)$ can be identified with the subspace of distributions having compact support. []*

Henceforth all derivatives will be meant in the sense of distributions, if not otherwise stated. We shall denote them by D^α, dropping the tilde.

The Space $\mathcal{S}$ of Rapidly Decreasing Functions. In view of the analysis of the Fourier transform via distributions, following Schwartz we define the space of *rapidly decreasing functions* (at ∞): [5]

$$\begin{aligned}\mathcal{S} := \;& \big\{v \in C^\infty : \forall \alpha, \beta \in \mathbb{N}_0^N, x^\beta D_x^\alpha v \in L^\infty\big\} \\ = \;& \big\{v \in C^\infty : \forall \alpha \in \mathbb{N}_0^N, \forall m \in \mathbb{N}, |x|^m D_x^\alpha v(x) \to 0 \text{ as } |x| \to +\infty\big\} \\ = \;& \big\{v \in C^\infty : \forall \alpha \in \mathbb{N}_0^N, \forall m \in \mathbb{N}, |x|^m D_x^\alpha v(x) \\ & \text{is bounded in a neighbourhood of } |x| = \infty\big\}.\end{aligned} \tag{1.15}$$

Here we shall just deal with functions defined on $\mathbb{R}^N$, and shall not display this set; e.g., we shall write $\mathcal{S}$ in place of $\mathcal{S}(\mathbb{R}^N)$. This is a locally convex Fréchet space equipped with either of the following equivalent families of seminorms

$$|v|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^N} |x^\beta D_x^\alpha v(x)| \qquad \forall \alpha, \beta \in \mathbb{N}_0^N, \tag{1.16}$$

$$|v|_{m,\alpha} := \sup_{x \in \mathbb{R}^N} \big(1 + |x|^2\big)^m |D_x^\alpha v(x)| \qquad \forall m \in \mathbb{N}, \forall \alpha \in \mathbb{N}_0^N. \tag{1.17}$$

For instance, the function $x \mapsto e^{-|x|^2}$, $x \mapsto e^{-\sqrt{1+|x|^2}}$ and all elements of $\mathcal{D}(\mathbb{R}^N)$ are in $\mathcal{S}$.

By the Leibniz rule, for any polynomials P and Q, the operators $u \mapsto P(x)Q(D)u$ and $u \mapsto P(D)[Q(x)u]$ map $\mathcal{S}$ to itself and are continuous. [Ex]

The Space $\mathcal{S}'$ of Temperate Distributions. We denote the topological dual of $\mathcal{S}$ by $\mathcal{S}'$, and equip it with the sequential weak convergence: for any sequence $\{T_n\}$ in $\mathcal{S}'(\Omega)$,

$$T_n \to 0 \quad \text{in} \quad \mathcal{S}' \quad \Leftrightarrow \quad T_n(v) \to 0 \quad \forall v \in \mathcal{S}. \tag{1.18}$$

For any $T \in \mathcal{S}'$ and any $v \in \mathcal{S}$, we shall also write $\langle T, v\rangle$ in place of $T(v)$. This notation is consistent with that we used for distributions, because of the following result.

Proposition 1.7 . *$\mathcal{S}$ is a dense subset of $\mathcal{S}'$. Moreover,*

$$\mathcal{D} \subset \mathcal{S} \subset \mathcal{E}, \quad \mathcal{E}' \subset \mathcal{S}' \subset \mathcal{D}' \quad \textit{with continuous and dense injections.} \qquad [] \tag{1.19}$$

Notice that

$$\mathcal{D} \subset \mathcal{D}', \quad \mathcal{S} \subset \mathcal{S}' \quad \text{but} \quad \mathcal{E} \not\subset \mathcal{E}', \tag{1.20}$$

since any smooth non-compactly-supported function belongs to $\mathcal{E} \setminus \mathcal{E}'$.

Overview of Distribution Spaces. We introduced the spaces $\mathcal{D}(\Omega), \mathcal{E}(\Omega)$, with (up to identifications)

$$\mathcal{D}(\Omega) \subset \mathcal{E}(\Omega) \quad \text{with continuous and dense injection.} \tag{1.21}$$

[5] Schwartz founded the theory of distributions upon three main spaces: $\mathcal{D}(\Omega)$, $\mathcal{E}(\Omega)$ and $\mathcal{S}(\mathbb{R}^N)$.

For $\Omega = \mathbb{R}^N$ (which is not displayed), we also defined $\mathcal{S}$, which is such that

$$\mathcal{D} \subset \mathcal{S} \subset \mathcal{E} \quad \text{with continuous and dense injection.} \tag{1.22}$$

We equipped the respective dual spaces with the weak star convergence. (1.21) and (1.22) respectively yield

$$\mathcal{E}'(\Omega) \subset \mathcal{D}'(\Omega) \quad \text{with continuous and dense injection,} \tag{1.23}$$

and, for $\Omega = \mathbb{R}^N$,

$$\mathcal{E}' \subset \mathcal{S}' \subset \mathcal{D}' \quad \text{with continuous and dense injection.} \tag{1.24}$$

(The density of the inclusions $\mathcal{S} \subset \mathcal{E}$ and $\mathcal{E}' \subset \mathcal{S}'$ was not mentioned above. However the former one is obvious, and the second one directly follows from it.)

L. Schwartz also introduced spaces of slowly increasing functions and rapidly decreasing distributions. But we shall not delve on them.

Exercises. (i) Check that the two families of seminorms (1.15) are equivalent.

(ii) Check that $v \in \mathcal{S}$ iff, for any polynomial P, Q of N variables, $P(x)Q(D)v \in \mathcal{S}$.

(iii) Check that $x^\beta D_x^\alpha v \in L^1$, for any $v \in \mathcal{S}$ and any $\alpha, \beta \in \mathbb{N}_0^N$.

— Check that $\exp(-x^2) \in \mathcal{S}$ but $\exp(-x^2)\sin\{\exp(-x^2)\} \notin \mathcal{S}$.

(iv) Show that, for any $p \in [1, +\infty]$, the functions of L^p are *slowly increasing.*

Hint: The statement is obvious for $p = 1$ or $p = +\infty$. Let $p \in]1, +\infty[$ and set $p' := p/(p-1)$. For any $f \in L^p$, by the Hölder inequality, for any $k \in \mathbb{N}$ we have

$$\int (1 + |x|^2)^{-k} |f(x)|\, dx \le \left(\int (1 + |x|^2)^{-kp'}\, dx \right)^{1/p'} \|f\|_{L^p}. \tag{1.25}$$

(v) Let S be the class of the measurable function $f : \mathbb{R}^N \to \mathbb{C}$ such that there exist $k \in \mathbb{N}$ and $C > 0$ such that $|f(x)| \le C|x|^k$ for any x. Compare S with the class of *slowly increasing* functions (see Sect. VIII.6).

2 Convolution

Convolution of L^1-Functions. For any measurable functions $f, g : \mathbb{R}^N \to \mathbb{C}$, we call **convolution product** (or just **convolution)** of f and g the function

$$(f * g)(x) := \int f(x - y) g(y)\, dy \qquad \text{for a.e. } x \in \mathbb{R}^N, \tag{2.26}$$

whenever this integral converges (absolutely) for a.e. x. (We write $\int ...dy$ in place of $\int ... \int_{\mathbb{R}^N} ... dy_1 ... dy_N$, and omit to display the domain $\mathbb{R}^N$.) Note that

$$\text{supp}(f * g) \subset \text{supp}(f) + \text{supp}(g). \quad [Ex] \tag{2.27}$$

If A and B are two topological vector spaces of functions for which the convolution makes sense, we set $A * B := \{f * g : f \in A, g \in B\}$, and define $A \cdot B$ similarly.

Proposition 2.1 *(i) $L^1 * L^1 \subset L^1$, and*

$$\|f * g\|_{L^1} \le \|f\|_{L^1} \|g\|_{L^1} \qquad \forall f, g \in L^1. \tag{2.28}$$

(ii) $L^1_{\rm loc} * L^1_{\rm comp} \subset L^1_{\rm loc}$, *and* [6]

$$\begin{aligned} &\|f * g\|_{L^1(K)} \le \|f\|_{L^1(K-{\rm supp}(g))}\|g\|_{L^1} \\ &\forall K \subset\subset \mathbb{R}^N, \forall f \in L^1_{\rm loc}, \forall g \in L^1_{\rm comp}\,. \end{aligned} \tag{2.29}$$

Moreover $L^1_{\rm comp} * L^1_{\rm comp} \subset L^1_{\rm comp}$.

(iii) For $N = 1$, $L^1_{\rm loc}(\mathbb{R}^+) * L^1_{\rm loc}(\mathbb{R}^+) \subset L^1_{\rm loc}(\mathbb{R}^+)$. [7] *For any* $f, g \in L^1_{\rm loc}(\mathbb{R}^+)$,

$$(f * g)(x) = \begin{cases} \int_0^x f(x-y)g(y)\,dy & \textit{for a.e. } x \ge 0 \\ 0 & \textit{for a.e. } x < 0, \end{cases} \tag{2.30}$$

$$\|f * g\|_{L^1(0,M)} \le \|f\|_{L^1(0,M)}\|g\|_{L^1(0,M)} \qquad \forall M > 0\,. \tag{2.31}$$

The mapping $(f, g) \mapsto f * g$ is thus continuous in each of these three cases.

* *Proof.* (i) For any $f, g \in L^1$, the function $(\mathbb{R}^N)^2 \to \mathbb{C} : (z, y) \mapsto f(z)g(y)$ is (absolutely) integrable, and by changing integration variable we get

$$\iint f(z)g(y)\,dz\,dy = \iint f(x-y)g(y)\,dy\,dx.$$

By Fubini's theorem the function $f * g : x \mapsto \int f(x-y)g(y)\,dy$ is then integrable. Moreover

$$\begin{aligned} \|f * g\|_{L^1} &= \int dx \left| \int f(x-y)g(y)\,dy \right| \\ &\le \iint |f(x-y)||g(y)|\,dx\,dy = \iint |f(z)||g(y)|\,dz\,dy = \|f\|_{L^1}\|g\|_{L^1}\,. \end{aligned}$$

(ii) For any $f \in L^1_{\rm loc}$ and $g \in L^1_{\rm comp}$, setting $S_g := {\rm supp}(g)$,

$$(f * g)(x) = \int_{S_g} f(x-y)g(y)\,dy \qquad \text{for a.e. } x \in \mathbb{R}^N.$$

Moreover, for any $K \subset\subset \mathbb{R}^N$,

$$\begin{aligned} \|f * g\|_{L^1(K)} &\le \int_K dx \int_{S_g} |f(x-y)g(y)|\,dy = \int_{S_g} dy \int_K |f(x-y)g(y)|\,dx \\ &= \int_{S_g} dy \int_{K-S_g} |f(z)g(y)|\,dz \le \|f\|_{L^1(K-S_g)}\|g\|_{L^1}\,. \end{aligned}$$

The proof of the inclusion $L^1_{\rm comp} * L^1_{\rm comp} \subset L^1_{\rm comp}$ is based on (2.27), and is left to the Reader.

(iii) Part (iii) may be proved by means of an argument similar to that of part (ii), that we also leave to the reader. □

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[6] By $L^1_{\rm comp}$ we denote the space of integral functions that have compact support.

[7] Any function or distribution defined on $\mathbb{R}^+$ will be automatically extended to the whole $\mathbb{R}$ with value 0. (In signal theory, the functions of time that vanish for any $t < 0$ are said *causal).*

Proposition 2.2 *L^1, $L^1_{\rm comp}$ and $L^1_{\rm loc}(\mathbb{R}^+)$, equipped with the convolution product, are commutative algebras (without unit).* [8] *In particular,*

$$\begin{array}{l} f * g = g * f, \qquad (f * g) * h = f * (g * h) \qquad a.e.\ in\ \mathbb{R}^N \\ \forall (f,g,h) \in (L^1)^3 \cup \left(L^1_{\rm loc} \times L^1_{\rm comp} \times L^1_{\rm comp}\right). \end{array} \tag{2.32}$$

If $N = 1$, the same holds for any $(f,g,h) \in L^1_{\rm loc}(\mathbb{R}^+)^3$, too.

* *Proof.* For any $(f,g,h) \in (L^1)^3$ and a.e. $x \in \mathbb{R}^N$,

$$\begin{aligned} (f * g)(x) &= \int f(x-y)g(y)\,dy = \int f(z)g(x-z)dz = (g * f)(x), \\ [(f * g) * h](x) &= \int [(f * g)](z)\,h(x-z)\,dz = \int dz \int f(y)g(z-y)\,dy\ h(x-z) \\ &= \iint f(y)g(t)h((x-y)-t)\,dt\,dy \\ &= \int dy\, f(y) \int g(t)h(x-y-t)\,dt \\ &= \int f(y)[(g * h)](x-y)\,dy = [f * (g * h)](x)\,. \end{aligned}$$

The cases of $(f,g,h) \in \left(L^1_{\rm loc} \times L^1_{\rm comp} \times L^1_{\rm comp}\right)$ and $(f,g,h) \in L^1_{\rm loc}(\mathbb{R}^+)^3$ are similarly checked. □

It is easily seen that $(L^1, *)$ and $(L^\infty, \cdot)$ (here "$\cdot$" stands for the pointwise product) are commutative Banach algebras; $(L^\infty, \cdot)$ has the unit element $e \equiv 1$.

Convolution of L^p-Functions. The following result generalizes Proposition 2.1. [9]

• **Theorem 2.3 (Young)** *Let*

$$p, q, r \in [1, +\infty], \qquad p^{-1} + q^{-1} = 1 + r^{-1}.\,^{10} \tag{2.34}$$

Then: (i) $L^p * L^q \subset L^r$ and

$$\|f * g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q} \qquad \forall f \in L^p, \forall g \in L^q. \tag{2.35}$$

[8] Let a vector space X over a field $\mathbb{K}$ ($= \mathbb{C}$ or $\mathbb{R}$) be equipped with a product $* : X \times X \to X$. This is called an **algebra** iff, for any $u, v, z \in X$ and any $\lambda \in \mathbb{K}$:

(i) $u * (v * z) = (u * v) * z$,

(ii) $(u + v) * z = u * z + v * z, \quad z * (u + v) = z * u + z * v$,

(iii) $\lambda(u * v) = (\lambda u) * v = u * (\lambda v)$.

The algebra is said **commutative** iff the product $*$ is commutative.

X is called a **Banach algebra** iff it is both an algebra and a Banach space (over the same field), and, denoting the norm by $\|\cdot\|$,

(iv) $\|u * v\| \le \|u\| \|v\|$ for any $u, v \in X$.

X is called a **Banach algebra with unit** iff

(v) there exists (a necessarily unique) $e \in X$ such that $\|e\| = 1$, and $e * u = u * e = u$ for any $u \in X$.

[9] This theorem may be compared with the following result, that easily follows from the Hölder inequality: If $p, q, r \in [1, +\infty[$ are such that $p^{-1} + q^{-1} = r^{-1}$, then

$$uv \in L^r(\Omega), \quad \|uv\|_r \le \|u\|_p \|v\|_q \qquad \forall u \in L^p(\Omega), \forall v \in L^q(\Omega).[Ex] \tag{2.33}$$

[10] Here we set $(+\infty)^{-1} := 0$.

(ii) $L^p_{\rm loc} * L^q_{\rm comp} \subset L^r_{\rm loc}$ *and*

$$\begin{gathered}\|f*g\|_{L^r(K)} \le \|f\|_{L^p(K-{\rm supp}(g))}\|g\|_{L^q}\\ \forall K \subset\subset \mathbb{R}^N, \forall f \in L^p_{\rm loc}, \forall g \in L^q_{\rm comp}.\end{gathered} \tag{2.36}$$

Moreover $L^p_{\rm comp} * L^q_{\rm comp} \subset L^r_{\rm comp}$.

(iii) For $N = 1$, $L^p_{\rm loc}(\mathbb{R}^+) * L^q_{\rm loc}(\mathbb{R}^+) \subset L^r_{\rm loc}(\mathbb{R}^+)$, *and*

$$\begin{gathered}\|f*g\|_{L^r(0,M)} \le \|f\|_{L^p(0,M)}\|g\|_{L^q(0,M)}\\ \forall M > 0, \forall f \in L^p_{\rm loc}(\mathbb{R}^+), \forall g \in L^q_{\rm loc}(\mathbb{R}^+).\end{gathered} \tag{2.37}$$

The mapping $(f, g) \mapsto f * g$ is thus continuous in each of these three cases.

By this result, we may regard $L^{p(\theta)}(\Omega)$ as an *interpolate space* between $L^{p_1}(\Omega)$ and $L^{p_2}(\Omega)$. This theorem is actually the prototypical example of the theory of Banach spaces interpolation.

For any $f : \mathbb{R}^N \to \mathbb{C}$, let us set $\check{f}(x) = f(-x)$.

Corollary 2.4 *Let*

$$p, q, s \in [1, +\infty], \qquad p^{-1} + q^{-1} + s^{-1} = 2. \tag{2.38}$$

Then:

$$\begin{aligned}&\forall (f, g, h) \in L^p \times L^q \times L^s,\\ &(f*g)\cdot h,\ g\cdot(\check{f}*h),\ f\cdot(\check{g}*h) \in L^1, \quad \textit{and}\\ &\int (f*g)\cdot h = \int g\cdot(\check{f}*h) = \int f\cdot(\check{g}*h).\end{aligned} \tag{2.39}$$

The same holds also

$$\begin{aligned}&\forall (f, g, h) \in \big(L^p_{\rm comp} \times L^q_{\rm loc} \times L^s_{\rm comp}\big),\\ &\forall (f, g, h) \in L^p_{\rm loc}(\mathbb{R}^+) \times L^q_{\rm loc}(\mathbb{R}^+) \times L^s_{\rm comp}(\mathbb{R}^+)\,.\end{aligned} \tag{2.40}$$

Proof. For any $(f, g, h) \in L^p \times L^q \times L^s$, by the Young Theorem 2.34 $f * g \in L^r$ for r as in (2.34). By (2.38) then $r^{-1} + s^{-1} = 1$, and (2.39) follows.

The remainder is similarly checked. □

Let us next set $\tau_h f(x) := f(x + h)$ for any $f : \mathbb{R}^N \to \mathbb{C}$ and any $x, h \in \mathbb{R}^N$.

Let us denote by $C^0(\mathbb{R}^N)$ the space of continuous functions $\mathbb{R}^N \to \mathbb{C}$ (which is a Fréchet space equipped with the family of sup-norms on compact subsets of $\mathbb{R}^N$), and by $C^0_0(\mathbb{R}^N)$ the subspace of $C^0(\mathbb{R}^N)$ of functions that vanish at infinity (this is a Banach space equipped with the sup-norm).

Lemma 2.5 *As* $h \to 0$,

$$\tau_h f \to f \qquad \textit{in } C^0,\ \forall f \in C^0, \tag{2.41}$$

$$\tau_h f \to f \qquad \textit{in } L^p,\ \forall f \in L^p,\ \forall p \in [1, +\infty[. \tag{2.42}$$

Proof. As any $f \in C^0$ is locally uniformly continuous, $\tau_h f \to f$ uniformly in any $K \subset\subset \mathbb{R}^N$; (2.41) thus holds. This yields (2.42), as C^0 is dense in L^p for any $p \in [1, +\infty[$. □

By the next result, in the Young theorem the space L^∞ may be replaced by $L^\infty \cap C^0$, and in part (i) also by $L^\infty \cap C^0_0$.

Proposition 2.6 *Let $p, q \in [1, +\infty]$ be such that $p^{-1} + q^{-1} = 1$. Then:*

$$f * g \in C^0 \qquad \forall (f, g) \in (L^p \times L^q) \cup (L^p_{\rm loc} \times L^q_{\rm comp}), \tag{2.43}$$

$$f * g \in C^0 \qquad \forall (f, g) \in L^p_{\rm loc}(\mathbb{R}^+) \times L^q_{\rm loc}(\mathbb{R}^+) \quad \textit{if } N = 1, \tag{2.44}$$

$$(f * g)(x) \to 0 \quad \textit{as} \quad |x| \to +\infty \quad \forall (f, g) \in L^p \times L^q, \forall p, q \in [1, +\infty[. \tag{2.45}$$

Proof. For instance, let $p \neq +\infty$ and $(f, g) \in L^p \times L^q$; the other cases may be dealt with similarly. By Lemma 2.5,

$$\begin{aligned} \|\tau_h(f * g) - (f * g)\|_{L^\infty} &= \Big\| \int [f(x + h - y) - f(x - y)] g(y)]\, dy \Big\|_{L^\infty} \\ &\le \|\tau_h f - f\|_{L^p} \|g\|_{L^q} \to 0 \qquad \text{as } h \to 0; \end{aligned} \tag{2.46}$$

the function $f * g$ may then be identified with a uniformly continuous function.

Let $\{f_n\} \subset L^p_{\rm comp}$ and $\{g_n\} \subset L^q_{\rm comp}$ be such that $f_n \to f$ in L^p and $g_n \to g$ in L^q. Hence $f_n * g_n$ has compact support, and $f_n * g_n \to f * g$ uniformly. This yields the final statement of the theorem. □

It is easily seen that if either p or $q = +\infty$ then (2.45) fails.

**** Convolution of Distributions.** By part (ii) of Proposition 2.1, for any

$$(f, g) \in (L^1_{\rm loc} \times L^1_{\rm comp}) \cup (L^1_{\rm comp} \times L^1_{\rm loc}),$$

$f * g \in L^1_{\rm loc}$. For any $\varphi \in \mathcal{D}$, then

$$\int (f * g)(x) \varphi(x)\, dx = \iint f(x - y) g(y) \varphi(x)\, dx dy = \iint f(z) g(y) \varphi(z + y)\, dz dy, \tag{2.47}$$

and of course each of these double integrals equals the corresponding iterated integrals, by Fubini's theorem. This formula allows one to extend the operation of convolution to two distributions, under analogous restrictions on the supports. Let either $(T, S) \in (\mathcal{D}' \times \mathcal{E}') \cup (\mathcal{E}' \times \mathcal{D}')$, and define

$$\langle T * S, v \rangle := \langle T_x, \langle S_y, \varphi(x + y) \rangle \rangle. \tag{2.48}$$

(In $\langle S_y, \varphi(x + y) \rangle$ the variable x is just a parameter; if this pairing is reduced to an integration, then y is the integration variable.) This is meaningful, since

$$S \in \mathcal{E}' \;\; (S \in \mathcal{D}', \text{ resp.}) \qquad \Rightarrow \qquad \langle S_y, \varphi(x + y) \rangle \in \mathcal{D} \;\; (\in \mathcal{E}, \text{ resp.}).\ [Ex] \tag{2.49}$$

For $N = 1$, if $T \in \mathcal{D}'(\mathbb{R}^+)$, then (2.48) still makes sense.

On the other hand, one cannot write $\langle T_x S_y, \varphi(x + y) \rangle$ in the duality between $\mathcal{D}'(\mathbb{R}^N \times \mathbb{R}^N)$ and $\mathcal{D}(\mathbb{R}^N \times \mathbb{R}^N)$, since the support of the mapping $(x, y) \mapsto \varphi(x + y)$ is compact only if $\varphi \equiv 0$.

In $\mathcal{E}'$ the convolution commutes and is associative. Thus $(\mathcal{E}', *)$ is a convolution algebra, with unit element δ_0. Here are some further properties:

$$\mathcal{D}' * \mathcal{E}' \subset \mathcal{D}', \qquad \mathcal{E}' * \mathcal{E}' \subset \mathcal{E}', \tag{2.50}$$

$$\mathcal{S}' * \mathcal{E}' \subset \mathcal{S}', \qquad \mathcal{S} * \mathcal{S}' \subset \mathcal{E} \cap \mathcal{S}', \tag{2.51}$$

$$\mathcal{S} * \mathcal{E}' \subset \mathcal{S}, \qquad \mathcal{S} * \mathcal{S}' \subset \mathcal{E}, \tag{2.52}$$

and in all of these cases the convolution is separately continuous w.r.t. each of the two factors.

For instance, the inclusion $\mathcal{D}' * \mathcal{E}' \subset \mathcal{D}'$ is an extension of $L^1_{\rm loc} * L^1_{\rm comp} \subset L^1_{\rm loc}$, and actually may be proved by approximating distributions by $L^1_{\rm loc}$- or $L^1_{\rm comp}$-functions, by using the latter property, and then passing to the limit. This procedure may also be used to prove $\mathcal{E}' * \mathcal{E}' \subset \mathcal{E}'$, too. The other inclusions may similarly be justified by approximation and passage to the limit.

3 The Fourier Transform in L^1

Integral Transforms. These are linear integral operators $\mathcal{T}$ that typically act on functions $\mathbb{R} \to \mathbb{C}$, and have the form

$$\big(\widehat{u}(\xi) := \big)\ (\mathcal{T}u)(\xi) = \int_{\mathbb{R}} K(\xi, x)u(x)\,dx \qquad \forall \xi \in \mathbb{R}, \tag{3.1}$$

for a prescribed kernel $K : \mathbb{R}^2 \to \mathbb{C}$, and for any transformable function u. [11] The main properties of this class of transforms include the following:

(i) Inverse Transform. Under appropriate restrictions, there exists another kernel $\widetilde{K} : \mathbb{R}^2 \to \mathbb{C}$ such that (formally)

$$\int_{\mathbb{R}} \widetilde{K}(x, \xi)K(\xi, y)\,d\xi = \delta_0(x - y) \qquad \forall x, y \in \mathbb{R}. \tag{3.2}$$

Denoting by R the integral operator associated to $\widetilde{K}$, we thus have $R\mathcal{T}u = u$ for any transformable u.

(ii) Commutation Formula. Any integral transform is associated to a class of linear operators (typically of differential type), that act on functions of time. For any such operator, L, there exists a function, $\widetilde{L}(s)$, such that

$$\mathcal{T}L\mathcal{T}^{-1} = \widetilde{L}(s) \quad \text{(this is a multiplicative operator).} \tag{3.3}$$

By applying $\mathcal{T}$, an equation of the form $Lu = f$ (for a prescribed function $f = f(t)$) is then transformed into $\widetilde{L}(\xi)\widehat{u}(\xi) = \widehat{f}(\xi)$. Thus $\widehat{u} = \widehat{f}/\widetilde{L}$, whence $u = R(\widehat{f}/\widetilde{L})$. This procedure is at the basis of so-called *symbolic (or operational) calculus,* that was introduced by O. Heaviside at the end of the 19th century.

The first of the transforms that we illustrate is named after J. Fourier, who introduced it at the beginning of the 19th century, and is the keystone of all integral transforms. In the 1950s Laurent Schwartz introduced the space of *tempered distributions,* and extended the transform to this class. This transform allows one to reduce linear ordinary differential equations with constant coefficients to algebraic equations, and this has found many uses in the study of stationary problems. This is useful for applications, and is also an important tool in functional analysis, as we shall see when dealing with Sobolev spaces.

The Fourier Transform in L^1. We shall systematically deal with spaces of functions from the whole $\mathbb{R}^N$ to $\mathbb{C}$. We shall then write L^1 in place of $L^1(\mathbb{R}^N)$, C^0 in place of $C^0(\mathbb{R}^N)$, and so on. For any $u \in L^1$, we define the **Fourier transform** (also called **Fourier integral**) $\widehat{u}$ of u by [12]

$$\widehat{u}(\xi) := (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-i\xi\cdot x}u(x)\,dx \qquad \forall \xi \in \mathbb{R}^N, \tag{3.4}$$

here $\xi \cdot x := \sum_{i=1}^N \xi_i x_i$.

Proposition 3.1 *Formula (3.5) defines a linear and continuous operator*

$$\begin{aligned} &\mathcal{F} : L^1 \to C^0_b : u \mapsto \widehat{u}; \\ &\|\widehat{u}\|_{L^\infty} \le (2\pi)^{-N/2}\|u\|_{L^1} \quad \forall u \in L^1.[Ex] \end{aligned} \tag{3.5}$$

[11] To devise hypotheses that encompass a large number of integral transforms is not easy may not be convenient. In this brief overview we then refer to the Fourier transform. We are intentionally sloppy drop the regularity properties, that however are specified ahead.

[12] Some authors introduce a factor 2π in the exponent under the integral, others omit the factor in front of the integral. Our definition is maybe the most frequently used. Each of these modifications simplifies some formulas, but none is able to simplify all of them.

(By C_b^0 we denote the Banach space $C_b^0 \cap L^\infty$.)

Thus $\widehat{u}_n \to \widehat{u}$ uniformly in $\mathbb{R}^N$ whenever $u_n \to u$ in L^1. In passing notice that $\|\widehat{u}\|_{L^\infty} = (2\pi)^{-N/2}\|u\|_{L^1}$ for any nonnegative $u \in L^1$, as in this case

$$\|\widehat{u}\|_{L^\infty} \le (2\pi)^{-N/2}\|u\|_{L^1} = \widehat{u}(0) \le \|\widehat{u}\|_{L^\infty}.$$

Apparently, no simple condition characterizes the image set $\mathcal{F}(L^1)$.

Proposition 3.2 *For any $u \in L^1$,* [13]

$$v(x) = u(x-y) \quad\Rightarrow\quad \widehat{v}(\xi) = e^{-i\xi\cdot y}\widehat{u}(\xi) \qquad \forall y \in \mathbb{R}^N, \tag{3.6}$$

$$v(x) = e^{ix\cdot\eta}u(x) \quad\Rightarrow\quad \widehat{v}(\xi) = \widehat{u}(\xi-\eta) \qquad \forall \eta \in \mathbb{R}^N, \tag{3.7}$$

$$v(x) = u\big(A^{-1}x\big) \quad\Rightarrow\quad \widehat{v}(\xi) = |\det A|\widehat{u}(A^*\xi) \quad \forall A \in \mathbb{R}^{N^2}, \det A \neq 0, \tag{3.8}$$

$$v(x) = \overline{u(x)} \quad\Rightarrow\quad \widehat{v}(\xi) = \overline{\widehat{u}(-\xi)}, \tag{3.9}$$

$$u \text{ is even (odd, resp.)} \quad\Rightarrow\quad \widehat{u} \text{ is even (odd, resp.)}, \tag{3.10}$$

$$u \text{ is real and even} \quad\Rightarrow\quad \widehat{u} \text{ is real (and even)}, \tag{3.11}$$

$$u \text{ is real and odd} \quad\Rightarrow\quad \widehat{u} \text{ is imaginary (and odd)}, \tag{3.12}$$

$$u \text{ is radial} \quad\Rightarrow\quad \widehat{u} \text{ is radial.} \tag{3.13}$$

[Ex]

Henceforth by D (or D_j or D^α) we shall denote the derivative operator in the sense of distributions.

• **Proposition 3.3** *For any multi-index $\alpha \in \mathbb{N}^N$,*

$$u, D_x^\alpha u \in L^1 \quad\Rightarrow\quad (i\xi)^\alpha\widehat{u} = (D_x^\alpha u)\widehat{\ } \in C_b^0, \tag{3.14}$$

$$u, x^\alpha u \in L^1 \quad\Rightarrow\quad D_\xi^\alpha\widehat{u} = [(-ix)^\alpha u]\widehat{\ } \in C_b^0. \tag{3.15}$$

Corollary 3.4 *Let $m \in \mathbb{N}_0$.*
(i) If $D_x^\alpha u \in L^1$ for any $\alpha \in \mathbb{N}_0^N$ with $|\alpha| \le m$, then $(1+|\xi|)^m\widehat{u}(\xi) \in L^\infty$.
(ii) If $(1+|x|)^m u \in L^1$, then $\widehat{u} \in C^m$. [Ex]

In other terms:
(i) the faster u decreases at infinity, the greater is the regularity of $\widehat{u}$;
(ii) the greater is the regularity of u, the faster $\widehat{u}$ decreases at infinity.

Proposition 3.5 (Riemann-Lebesgue) *For any $u \in L^1$, $\widehat{u}(\xi) \to 0$ as $|\xi| \to +\infty$, and $\widehat{u}$ is uniformly continuous in $\mathbb{R}^N$.*

Theorem 3.6 (Parseval)
The formal adjoint of $\mathcal{F}$ coincides with $\mathcal{F}$ itself, that is,

$$\int_{\mathbb{R}^N} \widehat{u}\, v\, dx = \int_{\mathbb{R}^N} u\, \widehat{v}\, dx \qquad \forall u, v \in L^1. \tag{3.16}$$

Moreover,

$$u*v \in L^1, \quad \text{and} \quad (u*v)\widehat{\ } = (2\pi)^{N/2}\widehat{u}\,\widehat{v} \qquad \forall u, v \in L^1. \tag{3.17}$$

[13] For any $A \in \mathbb{R}^{N^2}$, we set $(A^*)_{ij} := A_{ji}$ for any i, j. For any $z \in \mathbb{C}$, we denote its complex conjugate by $\overline{z}$. We say that u is **radial** iff $u(Ax) = u(x)$ for any x and any orthonormal matrix $A \in \mathbb{R}^{N^2}$ (i.e., with $A^* = A^{-1}$).

We now present the inversion formula for the Fourier transform. First let us introduce the so-called *conjugate Fourier transform:*

$$\widetilde{\mathcal{F}}(v)(x) := (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{2\pi i\xi\cdot x} v(\xi)d\xi \qquad \forall v \in L^1. \tag{3.18}$$

Clearly this has analogous properties to the operator $\mathcal{F}$.

Theorem 3.7 *For any $u \in L^1 \cap C^0 \cap L^\infty$, if $\widehat{u} \in L^1$ then*

$$u(x) = \widetilde{\mathcal{F}}(\widehat{u})(x) \qquad \forall x \in \mathbb{R}^N. \tag{3.19}$$

By Proposition 3.1, the regularity assumptions of Theorem 3.7 are actually needed, as $\bar{u} = \mathcal{F}\big(\overline{\widehat{u}}\big)$. However, by a more refined argument one might show that (3.19) holds under the only hypotheses that $u, \widehat{u} \in L^1$. (Of course, a posteriori one then gets that $u, \widehat{u} \in C^0_b$.)

By Theorem 3.7, $\mathcal{F}(u) \equiv 0$ only if $u \equiv 0$; hence the Fourier transform $L^1 \to C^0_b$ is injective. Under the assumptions of this theorem, we also have

$$\overline{\widehat{\widehat{u}}(x)} = \bar{u}(-x) \qquad \forall x \in \mathbb{R}^N. \tag{3.20}$$

We recall that $B(0, R)$ denotes the closed ball in $\mathbb{R}^N$ with center at the origin and radius R.

Theorem 3.8 (Paley-Wiener)
For any $u \in C^\infty\big(\mathbb{R}^N\big)$ and any $R > 0$, $\operatorname{supp} u \subset B(0, R)$ iff $\mathcal{F}(u)$ can be extended to an analytic function $\mathbb{C}^N \to \mathbb{C}$ (also denoted by $\mathcal{F}(u)$). [14]

This extended function $\mathcal{F}(u) : \mathbb{C}^N \to \mathbb{C}$ is called the **Fourier-Laplace transform** of u.

Overview of the Fourier Transform in L^1. We defined the classic Fourier transform $\mathcal{F} : L^1 \to C^0_b$ and derived its basic properties. In particular, we saw that

(i) $\mathcal{F}$ transforms partial derivatives to multiplication by powers of the independent variable (up to a multiplicative constant) and conversely. This is at the basis of the application of the Fourier transform to the study of linear partial differential equations with constant coefficients, that we shall outline ahead.

(ii) $\mathcal{F}$ establishes a correspondence between the regularity of u and the order of decay of $\widehat{u}$ at ∞, and conversely between the order of decay of u at ∞ and the regularity of $\widehat{u}$. In the limit case of a compactly supported function, its Fourier transform may be extended to an entire analytic function $\mathbb{C}^N \to \mathbb{C}$.

(iii) $\mathcal{F}$ transforms the convolution of two functions to the product of their transforms (the converse statement may fail, because of summability restrictions).

(iv) Under suitable regularity restrictions, the inverse transform exists, and has an integral representation analogous to that of the direct transform.

The properties of the two transforms are then similar; this accounts for the duality of the statements (i) and (ii). However the assumptions are not perfectly symmetric; in the next section we shall see a different functional framework where this is remedied.

The inversion formula (3.19) also provides an interpretation of the Fourier transform. (3.19) represents u as a weighted average of the *harmonic components* $x \mapsto e^{i\xi\cdot x}$. For any $\xi \in \mathbb{R}^N$, $\widehat{u}(\xi)$ is the *amplitude* of the component having *vector frequency* ξ (that is, frequency ξ_i in each

[14] A function $\mathbb{C}^N \to \mathbb{C}$ is called analytic iff it is separately analytic with respect to each variable. For any $z \in \mathbb{C}^N$, we set $|z| = \big(\sum_{i=1}^N |z_i|^2\big)^{1/2}$ and $\mathcal{I}(z) = (\mathcal{I}(z_1), ..., \mathcal{I}(z_N))$ (the vector of the imaginary parts).

direction x_i). Therefore any function which fulfills (3.19) may equivalently be represented by specifying either the value $u(x)$ at a.a. points $x \in \mathbb{R}^N$, or the amplitude $\widehat{u}(\xi)$ for a.a. frequencies $\xi \in \mathbb{R}^N$. (Loosely speaking, any non-identically vanishing $u \in \mathcal{D}$ has *harmonic components* of arbitrarly large frequencies.)

The analogy between the Fourier transform and the Fourier series is obvious, and will be briefly discussed at the end of the next section.

4 Extensions of the Fourier Transform

Fourier Transform in $\boldsymbol{\mathcal{S}}$. For any $u \in \mathcal{D}$, by Theorem 3.8 $\widehat{u}$ is analytic; hence $\widehat{u} \in \mathcal{D}$ only if $\widehat{u} \equiv 0$, namely $u \equiv 0$. Thus $\mathcal{D}$ is not stable by Fourier transform. This means that the set of the frequencies of the harmonic components of any non-identically vanishing $u \in \mathcal{D}$ is unbounded. This situation induced L. Schwartz to introduce the space of rapidly decreasing functions $\mathcal{S}$, cf. Sect. 1, and to extend the Fourier transform to this space and to its (topological) dual. Next we review the tenets of that theory.

Proposition 4.1 *(The restriction of) $\mathcal{F}$ operates in $\mathcal{S}$ and is continuous. Moreover, the formulae of Proposition 3.3 and Theorem 3.6 hold in $\mathcal{S}$ without any restriction, $\mathcal{F}$ is invertible in $\mathcal{S}$, and $\mathcal{F}^{-1} = \widetilde{\mathcal{F}}$ (cf. (3.19)). [Ex]*

The first part is easily checked via repeated use of the Leibniz rule, because of the stability of $\mathcal{S}$ w.r.t. multiplication by any polynomial and w.r.t. application of any differential operator (with constant coefficients). Actually, $\mathcal{S}$ is the smallest space that contains L^1 and has these properties. [Ex]

The next statement extends and also completes (3.17).

Proposition 4.2 *For any $u, v \in \mathcal{S}$,*

$$u{*}v \in \mathcal{S}, \qquad (u{*}v)\widehat{} = (2\pi)^{N/2}\widehat{u}\,\widehat{v} \quad \text{in } \mathcal{S}, \tag{4.1}$$

$$uv \in \mathcal{S}\,, \qquad (uv)\widehat{} = (2\pi)^{-N/2}\widehat{u}{*}\widehat{v} \quad \text{in } \mathcal{S}. \tag{4.2}$$

Proof. The first statement is a direct extension of (3.17). Let us prove the second one.

It is easily checked that $uv \in \mathcal{S}$. By writing (3.17) with $\widehat{u}$ and $\widehat{v}$ in place of u and v, and $\widetilde{\mathcal{F}}$ in place of $\mathcal{F}$, we have

$$\widetilde{\mathcal{F}}(\widehat{u}{*}\widehat{v}) = (2\pi)^{N/2}\widetilde{\mathcal{F}}(\widehat{u})\,\widetilde{\mathcal{F}}(\widehat{v}) = (2\pi)^{N/2}u\,v.$$

By applying $\mathcal{F}$ to both members of this equality, (4.2) follows. □

Fourier Transform in $\boldsymbol{\mathcal{S}'}$. Denoting by $\mathcal{F}^\tau$ the transposed of $\mathcal{F}$, we set [15]

$$\bar{\mathcal{F}} := [\mathcal{F}^\tau]^* : \mathcal{S}' \to \mathcal{S}'. \tag{4.3}$$

By the Parseval Theorem 3.6, $\mathcal{F}^\tau = \mathcal{F}$; hence $\bar{\mathcal{F}} = \mathcal{F}^*$, that is,

$$\langle \bar{\mathcal{F}}(T), v\rangle := \langle T, \mathcal{F}(v)\rangle \qquad \forall v \in \mathcal{S}, \forall T \in \mathcal{S}'. \tag{4.4}$$

As $\mathcal{S}$ is dense in $\mathcal{S}'$, cf. Proposition VIII.6.2, $\bar{\mathcal{F}}$ is the unique continuous extension of the Fourier transform from $\mathcal{S}$ to $\mathcal{S}'$.

Henceforth we shall use the same symbols $\mathcal{F}$ or $\widehat{}$ for the many restrictions and extensions of the Fourier transform. We shall thus write $\mathcal{F}(T)$, or $\widehat{T}$, in place of $\bar{\mathcal{F}}(T)$.

[15] Notice that $\mathcal{F} : \mathcal{S} \to \mathcal{S}$, $\mathcal{F}^\tau : \mathcal{S} \to \mathcal{S}$, $\mathcal{F}^* : \mathcal{S}' \to \mathcal{S}'$.

Proposition 4.3 *$\mathcal{F}$ may be uniquely extended to an operator which acts in $\mathcal{S}'$ and is continuous. Moreover, the formulae of Proposition 3.3 and Theorem 3.6 hold in $\mathcal{S}'$ without any restriction, $\mathcal{F}$ is invertible in $\mathcal{S}'$, and $\mathcal{F}^{-1} = \widetilde{\mathcal{F}}$. [Ex]*