Distributions — A. Visintin (March 2010)

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Note. The bullet • and the asterisk ∗ are respectively used to indicate the most relevant results and complements. The symbol [] follows statements the proof of which has been omitted, whereas [Ex] is used to propose the reader to fill in the argument as an exercise.

Here are some abbreviations that are used throughout:

a.a. = almost any; resp. = respectively; w.r.t. = with respect to.

p': conjugate exponent of p, that is, $p' := p/(p-1)$ if $1 < p < +\infty$, $1' := \infty$, $\infty' := 1$.

0. Two Relevant Topologies

Inductive Topology. Here we review two standard methods of constructing a topology on a given set, in such a way that certain continuity properties are fulfilled.

Let A be any index set. Let $\{X_\alpha\}_{\alpha\in A}$ be a family of topological spaces, and $\{\lambda_\alpha\}_{\alpha\in A}$ be a family of mappings $X_\alpha \to X$. Defining a set $B \subset X$ to be open in X iff $\lambda_\alpha^{-1}(B)$ is open in X_α for any α, we obtain a topology, which is called the *inductive* (or *final)* topology on X generated by the family $\{X_\alpha, \lambda_\alpha\}$. By construction, this is the finest topology on X such that λ_α is continuous for any α.

Proposition 0.1 *Let X be equipped with the inductive topology generated by a family $\{X_\alpha, \lambda_\alpha\}_{\alpha\in A}$. Then:*

(i) For any topological space H, a mapping $f : X \to H$ is continuous iff $f \circ \lambda_\alpha : X_\alpha \to H$ is continuous for any $\alpha \in A$.

(ii) The property (i) characterizes the inductive topology.

Proof. (i) For any open subset U of H, $f^{-1}(U)$ is open in X iff $(f \circ \lambda_\alpha)^{-1}(U) = \lambda_\alpha^{-1}(f^{-1}(U))$ is open in X_α for any α. This yields the first statement.

(ii) Let us now denote by $\hat{X}$ the set X equipped with another topology which fulfils (i). By applying the "if" part of (i) to the identity mapping $j : X \to \hat{X}$ and to $j^{-1} : X \to \hat{X}$, we see that j is a homeomorphism. □

For instance, let $\sim$ be an equivalence relation in a topological space E, denote by $X := E/\sim$ the corresponding quotient set, and let $\pi : E \to X$ map any $x \in E$ to its equivalence class. X is canonically equipped with the inductive topology of $\{E, \pi\}$; this means that a set $B \subset X$ is open iff $\pi^{-1}(B)$ is open. This is called the *quotient topology.*

Projective Topology. The following construction may be regarded as dual of the previous one.

Let us consider a set Y, a family $\{Y_\alpha\}_{\alpha\in A}$ of topological spaces, and a family $\{\gamma_\alpha\}_{\alpha\in A}$ of mappings $Y \to Y_\alpha$. We define a set to be open in Y iff it is a union of finite intersections of sets of the form $\gamma_\alpha^{-1}(B_\alpha)$, as α ranges in A and B_α varies among the open subsets of Y_α. This means that the sets of the form $\gamma_\alpha^{-1}(B_\alpha)$ constitue a sub-basis for this topology, that we call the *projective* (or *initial)* topology on Y generated by the family $\{Y_\alpha, \gamma_\alpha\}$. By construction, this is the coarsest topology on Y such that γ_α is continuous for any α.

Proposition 0.2 *Let Y be equipped with the projective topology generated by a family $\{Y_\alpha, \gamma_\alpha\}_{\alpha\in A}$. Then:*

(i) For any topological space H, a mapping $f : H \to Y$ is continuous iff $\gamma_\alpha \circ f : H \to Y_\alpha$ is

continuous for any $\alpha \in A$.

(ii) The property (i) characterizes the projective topology.

Proof. (i) We only have to prove the "if" part. Let $\gamma_\alpha \circ f$ be continuous for any $\alpha \in A$; then $f^{-1}(\gamma_\alpha^{-1}(B_\alpha)) = (\gamma_\alpha \circ f)^{-1}(B_\alpha)$ is open in H for any open set $B_\alpha \subset Y_\alpha$. By the above construction, if B is open in Y it follows that $f^{-1}(B)$ is open in H.

(ii) Let us now denote by $\hat{Y}$ the set Y equipped with another topology which fulfils (i). By applying the "if" part of (i) to the identity mapping $j : Y \to \hat{Y}$ and to $j^{-1} : \hat{Y} \to Y$, we see that j is a homeomorphism. □

For instance, the Cartesian product $Y := \prod_{\alpha \in A} Y_\alpha$ of a family of topological spaces $\{Y_\alpha\}$ is canonically equipped with the projective topology generated by the family of the canonic projections $\pi_\alpha : Y \to Y_\alpha$. This is called the *product topology.*

The topology induced by a topological space T on a subset M is another example of projective topology. In this case $Y = M$, the index set A is a singleton, $Y_\alpha = T$, and $\gamma_\alpha : M \to T : u \mapsto u$.

1. Test Functions

The theory of distributions was introduced in the 1940s by Laurent Schwartz, who provided a precise functional formulation to previous ideas of Heaviside, Dirac and others, and forged a powerful tool of calculus. Distributions also offered a solid basis for the construction of Sobolev spaces, that had been introduced by Sobolev in the 1930s using the notion of *weak derivative* (which had been pioneered by Beppo Levi). These spaces play a fundamental role in the modern analysis of either linear or nonlinear partial differential equations.

Let Ω be a domain of $\mathbf{R}^N$ ($N \geq 1$). For any $m \in \mathbf{N}$, and any compact subset K of Ω, let us denote by $\mathcal{D}^m_K(\Omega)$ the space of m-times differentiable functions $\Omega \to \mathbf{C}$ whose support is contained in K. Let us also denote by $\mathcal{D}^m(\Omega)$ the space of m-times differentiable functions $\Omega \to \mathbf{C}$ whose support is a compact subset of Ω; we then set $\mathcal{D}(\Omega) := \mathcal{D}^\infty(\Omega)$ — the space of infinitely differentiable compactly supported functions $\Omega \to \mathbf{C}$, also named **test functions.** Thus $\mathcal{D}^m_K(\Omega) \subset C^m(\Omega)$ and

$$\mathcal{D}^m(\Omega) = \bigcup_{K \subset\subset \Omega} \mathcal{D}^m_K(\Omega) \quad \forall m \in \mathbf{N}, \qquad \mathcal{D}(\Omega) = \bigcap_{m \in \mathbf{N}} \mathcal{D}^m(\Omega) \ \ \text{(set-wise)}.$$

The null function is the only analytical function of $\mathcal{D}(\Omega)$, as any function of this space vanishes in some open set. The *bell-shaped* function

$$\rho(x) := \exp\left[\left(|x|^2 - 1\right)^{-1}\right] \quad \text{if } |x| < 1, \qquad \rho(x) := 0 \quad \text{if } |x| \geq 1 \tag{1.1}$$

also belongs to $\mathcal{D}(\mathbf{R}^N)$. By suitably translating and rescaling the variable x, further nontrivial elements of $\mathcal{D}(\Omega)$ are easily constructed, with support included in an arbitrary open subset of Ω.

For any $m \in \mathbf{N} \cup \{\infty\}$, $\mathcal{D}^m_K(\Omega)$ may be identified with a subspace of $C^m(K)$, by which we denote the space of functions $\Omega \to \mathbf{C}$ whose restriction to K is of class C^m. The latter is a Banach space only for finite m. On the other hand, we shall equip the space $C^\infty(K) = \bigcap_{m \in \mathbf{N}} C^m(K)$ with the projective topology generated by the injections $C^\infty(K) \to C^m(K)$, as m ranges through $\mathbf{N}$.

We shall equip the space $\mathcal{D}^m(\Omega)$ with the *inductive topology* generated by the injections $\mathcal{D}^m_K(\Omega) \to \mathcal{D}^m(\Omega)$, as K ranges through the compact subsets of Ω. This topology is thus generated by the union of the topologies of the spaces $\mathcal{D}^m_K(\Omega)$, as $K \subset\subset \Omega$. This means that a set $A \subset \mathcal{D}^m(\Omega)$ is open in this space iff $A \cap \mathcal{D}^m_K(\Omega)$ is open in $\mathcal{D}^m_K(\Omega)$ for all $K \subset\subset \Omega$.

A subset $B \subset \mathcal{D}^m(\Omega)$ is said to be bounded iff it is contained and bounded in $\mathcal{D}^m_K(\Omega)$, for some compact subset K of Ω. [(1)] This holds iff

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[(1)] This is consistent with a more general notion of boundedness. Any subset of a topological vector space V is said to be bounded iff, for any neighbourhood U of the origin, there exists $\lambda > 0$ such that $\lambda B := \{\lambda u : u \in B\} \subset U$.

(i) there exists a $K \subset\subset \Omega$ that contains the support of all the functions of B, and
(ii) $\sup_{v \in B} \sup_{\Omega} |D^\alpha v| < +\infty$ for any $\alpha \in \mathbf{N}^N$ with $|\alpha| \le m$, if m is finite. [Ex]

As any convergent sequence is bounded, the next statement is easily established. (Note that, by the linearity of $\mathcal{D}(\Omega)$, it suffices to define vanishing sequences.)

Proposition 1.1 *(Sequential Convergence) For any $m \in \mathbf{N} \cup \{\infty\}$, $u_n \to 0$ in $\mathcal{D}^m(\Omega)$ iff*
(i) there exists a compact set $K \subset \Omega$ such that $\{u_n\} \subset \mathcal{D}^m_K(\Omega)$, and
(ii) $u_n \to 0$ vanishes in $\mathcal{D}^m_K(\Omega)$. [Ex]

By the next result, this notion of sequential convergence does not determine the topology of $\mathcal{D}^m(\Omega)$. (2)

Theorem 1.2 *For any $m \in \mathbf{N} \cup \{\infty\}$, the space $\mathcal{D}^m(\Omega)$ is not metrizable.*

Proof. Let $\{K_n\}$ be a strictly increasing sequence of compact subsets of Ω such that $\bigcup_n K_n = \Omega$, and for any n let us select any $v_n \in \mathcal{D}^m_{K_{n+1}}(\Omega) \setminus \mathcal{D}^m_{K_n}(\Omega)$. For any n obviously $\lambda v_n \to 0$ in $\mathcal{D}^m(\Omega)$ as $\lambda \to 0$.

By contradiction let us assume that the topology of $\mathcal{D}^m(\Omega)$ is induced by a metric d. For any $n > m$ then there exists $\lambda_n > 0$ such that $d(\lambda_n v_n, 0) \le 1/n$; that is, $d(\lambda_n v_n, 0) \to 0$. Thus $\lambda_n v_n \to 0$ in $\mathcal{D}^m(\Omega)$, although the supports of these functions are not all included in any $K \subset\subset \Omega$, at variance with Proposition 1.1. □

Theorem 1.3 *(Sequential Completeness) For any $m \in \mathbf{N} \cup \{\infty\}$, the space $\mathcal{D}^m(\Omega)$ is sequentially complete.*

Proof. We prove this statement for $m = \infty$; for finite m it is even simpler. Any Cauchy sequence (3) $\{u_n\}$ in $\mathcal{D}(\Omega)$ is clearly bounded. It is then also a Cauchy sequence in $\mathcal{D}_K(\Omega)$ for some $K \subset\subset \Omega$. One may then extract a subsequence $\{u_n^{(0)}\}$ that converges in $\mathcal{D}^0_K(\Omega)$. From it one may extract a subsequence $\{u_n^{(1)}\}$ that converges in $\mathcal{D}^1_K(\Omega)$. For $\ell = 1, 2, ...$, one may similarly extract nested subsequences $\{u_n^{(\ell)}\}$ that converge in $\mathcal{D}^\ell_K(\Omega)$. The diagonal subsequence $\{u_\ell^{(\ell)}\}$ then converges in $\mathcal{D}(\Omega)$. □

Theorem 1.4 *(Compactness) For any $m \in \mathbf{N} \cup \{\infty\}$, any subset of $\mathcal{D}^m(\Omega)$ is sequentially relatively compact iff it is bounded.*

Proof. If B is a bounded subset of $\mathcal{D}^m(\Omega)$, then it is bounded in $\mathcal{D}^m_K(\Omega)$ for some $K \subset\subset \Omega$. By applying the Ascoli-Arzelà theorem to all the derivatives of the functions of B (up to order m, if m is finite), we see that B is then relatively compact in $\mathcal{D}^m_K(\Omega)$, hence also in $\mathcal{D}^m(\Omega)$.

Conversely, by the characterization of sequential convergence, any sequentially relatively compact subset of $\mathcal{D}^m(\Omega)$ is contained in $\mathcal{D}^m_K(\Omega)$ for some $K \subset\subset \Omega$. As it is sequentially relatively compact in the latter space, it is bounded in $\mathcal{D}^m_K(\Omega)$, hence also in $\mathcal{D}^m(\Omega)$. □

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(2) The family of sequentially closed subsets of a topological space X defines a new topology, the *sequential topology* of the original topology, that may be strictly coarser than the latter. However, in metric spaces the two topologies coincide.

Note that, if one extends a subset A of a topological space by including the limits of all convergent sequences of elements of the given set, the resulting set may not be sequentially closed (!). The same may occur if one reiterates this procedure, by successively including the sequential limits. Anyway there exists a smallest sequentially closed set that includes A.

Several *sequential* topological notions are equivalent to the corresponding notion referred to the sequential topology. For instance, a functional $j : X \to \mathbf{R}$ is sequentially lower semicontinuous iff it is lower semicontinuous w.r.t. sequential topology.

(3) A sequence $\{u_n\}$ in the topological vector space $\mathcal{D}(\Omega)$ is called a Cauchy sequence iff, for any neighbourhood U of the origin, there exists $M \in \mathbf{N}$ such that $u_m - u_n \in U$ for any $m, n \ge M$.

The Space $\mathcal{E}(\Omega)$. [4] This is the space of infinitely differentiable functions $\Omega \to \mathbf{C}$, namely [5]

$$\mathcal{E}(\Omega) = C^\infty(\Omega) = \bigcap_{m\in\mathbf{N}} C^m(\Omega), \qquad C^m(\Omega) = \big\{v : \Omega \to \mathbf{C} : v|_K \in C^m(K), \forall K \subset\subset \Omega\big\}.$$

$\mathcal{E}(\Omega)$ may thus be equipped with the topology generated by the intersection of the topologies of the spaces $C^m(K)$, as m varies in $\mathbf{N}$ and $K \subset\subset \Omega$. This is the coarsest among all the topologies such that the injections $C^m(K) \to \mathcal{E}(\Omega)$ are continuous, namely, it is the *projective topology* generated by these injections. [6]

This entails that a subset B of $\mathcal{E}(\Omega)$ is bounded iff it is bounded in any $C^m(\Omega)$, and that a sequence vanishes in $\mathcal{E}(\Omega)$ iff it vanishes in any $C^m(\Omega)$. [Ex]

2. Distributions

The linear and continuous functionals $\mathcal{D}(\Omega) \to \mathbf{C}$ are called **distributions.** [7] They form the topological dual space of $\mathcal{D}(\Omega)$, denoted $\mathcal{D}'(\Omega)$ For any $T \in \mathcal{D}'(\Omega)$ and any $v \in \mathcal{D}(\Omega)$, we also write $\langle T, v\rangle := T(v)$. Here are some examples:

(i) For any $f \in L^1_{\rm loc}(\Omega)$, the integral functional $T_f : v \mapsto \int_\Omega f(x)\, v(x)\, dx$ is a distribution. As the mapping $f \mapsto T_f$ is injective, [Ex] we may identify $L^1_{\rm loc}(\Omega)$ with a subspace of $\mathcal{D}'(\Omega)$. This subspace is dense in $\mathcal{D}'(\Omega)$. [] For this reason, the distributions are also named *generalized functions.* The distributions of the form T_f are also named *regular distributions,* whereas the other ones are called *singular distributions.*

(ii) Although the real function $x \mapsto 1/x$ is not locally integrable in $\mathbf{R}$, its *principal value*

$$\text{p.v. } \frac{1}{x} : v \mapsto \lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} \frac{v(x)}{x}\, dx \qquad \forall v \in \mathcal{D}(\mathbf{R}) \tag{2.1}$$

is a distribution. [8] For any $v \in \mathcal{D}(\mathbf{R})$ and for any $a > 0$ such that $\operatorname{supp} v \subset [-a, a]$, by the oddness of the function $1/x$, we have

$$\begin{aligned} \langle \text{p.v. } \frac{1}{x}, v\rangle &= \lim_{\varepsilon\to 0^+} \left(\int_{\varepsilon<|x|<a} \frac{v(x) - v(0)}{x}\, dx + \int_{\varepsilon<|x|<a} \frac{v(0)}{x}\, dx \right) \\ &= \lim_{\varepsilon\to 0^+} \int_{\varepsilon<|x|<a} \frac{v(x) - v(0)}{x}\, dx = \int_{\mathbf{R}} \frac{v(x) - v(0)}{x}\, dx. \end{aligned} \tag{2.1$'$}$$

This limit exists finite, as by the mean value theorem

$$\left| \int_{\varepsilon<|x|<a} \frac{v(x) - v(0)}{x}\, dx \right| \le 2a \max_{\mathbf{R}} |v'| \qquad \forall \varepsilon > 0.$$

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[4] Laurent Schwartz founded the theory of distributions upon three spaces: $\mathcal{D}(\Omega)$, $\mathcal{E}(\Omega)$, and the space of *rapidly decreasing functions:* $\mathcal{S}(\mathbf{R}^N)$ (that is also known as the Schwartz space) , see Sect. 4. The latter space allowed him to extend the Fourier transform to distributions.

[5] We defined $C^m(K)$ as the space of functions $\Omega \to \mathbf{C}$ whose restriction to K is of class C^m.

[6] This topology may be induced by a metric, that makes $C^\infty(K)$ a Fréchet space.

[7] One might also consider the space of the linear and continuous functionals $\mathcal{D}^m(\Omega) \to \mathbf{C}$ for any finite m. Ahead we shall deal with the latter space for $m = 0$.

[8] This is called a *regularization* of the (non-locally-integrable) function $1/x$, since its restriction to $\mathcal{D}(\mathbf{R}\setminus\{0\})$ coincides with this function. More generally, one says that $T \in \mathcal{D}(\Omega)$ is a regularization of a function $f \in L^1_{\rm loc}(\Omega \setminus \{x_0\})$ iff its restriction to $\Omega \setminus \{x_0\}$ coincides with f. (Of course, this is of interest only if $f \notin L^1_{\rm loc}(\Omega)$.) Several other singular functions admit a similar regularization in the class of distributions (see ahead).

Note that the principal value is quite different from other notions of *generalized integral.*

Other distributions may also be associated to the function $1/x$.

(iii) The **Dirac mass** $\delta_0 : v \mapsto v(0)$ is a distribution (also named the *Dirac distribution*). [Ex] Obviously, the same applies to the translated Dirac mass $\delta_x : v \mapsto v(x)$ for any $x \in \mathbf{R}^N$.

(iv) A series of (translated) Dirac masses $\sum_{n=1}^{\infty} \delta_{x_n} : v \mapsto \sum_{n=1}^{\infty} v(x_n)$ is also a distribution iff $\{x_n\}$ is a sequence in Ω that intersects any compact subset of Ω only in a finite number of points. [Ex]

(v) For any sequence $\{x_n\}$ in Ω, the series $\sum_{n=1}^{\infty} n^{-2}\delta_{x_n}$ is a distribution. [Ex]

(vi) For any Borel measure μ over Ω, the functional $v \mapsto \int_\Omega v(x)\, d\mu(x)$ is a distribution.

Proposition 2.0 *Let us denote by $\mathcal{T} : \mathcal{D}(\Omega) \to \mathcal{D}'(\Omega)$ the canonic injection. $\mathcal{T}(\mathcal{D}(\Omega))$ is dense in $\mathcal{D}'(\Omega)$. []*

• **Theorem 2.1** *(Characterization of Distributions) For any linear functional $T : \mathcal{D}(\Omega) \to \mathbf{C}$ the following properties are mutually equivalent:*

(i) $T \in \mathcal{D}'(\Omega)$;

(ii) T is sequentially continuous; i.e., $T(v_n) \to 0$ whenever $v_n \to 0$ in $\mathcal{D}(\Omega)$;

(iii) T is of finite order on any $K \subset\subset \Omega$, that is,

$$\begin{aligned} &\forall K \subset\subset \Omega, \exists m \in \mathbf{N}, \exists C > 0 : \forall v \in \mathcal{D}(\Omega),\\ &\operatorname{supp} v \subset K \quad \Rightarrow \quad |T(v)| \le C \max_{|\alpha|\le m} \sup_K |D^\alpha v|. \end{aligned} \tag{2.2}$$

(iv) T is bounded, i.e., it maps bounded subsets of $\mathcal{D}(\Omega)$ to bounded subsets of $\mathbf{C}$.

Proof. Obviously, (i) ⇒ (ii). Let us prove that (ii) ⇒ (iii). By contradiction, let us assume that

$$\exists K \subset\subset \Omega : \forall m \in \mathbf{N}, \exists v_m \in \mathcal{D}_K(\Omega) : \quad |T(v_m)| > m \max_{|\alpha|\le m} \sup_K |D^\alpha v_m|.$$

Possibly dividing v_m by $T(v_m)$, we can assume that $T(v_m) = 1$ for any m. Hence $\sup_K |D^\alpha v_m| < 1/m$ for any $\alpha \in \mathbf{N}^N$ and any $m \ge |\alpha|$. Thus $v_m \to 0$ in $\mathcal{D}_K(\Omega)$, whence in $\mathcal{D}(\Omega)$, although $T(v_m)$ does not vanish. Therefore (iii) does not hold.

The implication (iii) ⇒ (iv) easily follows from (2.2) and the characterization of bounded subsets of $\mathcal{D}(\Omega)$. [Ex]

We shall not prove that (iv) ⇒ (i). □

We equip the space $\mathcal{D}'(\Omega)$ with the sequential topology induced by the pointwise convergence: for any sequence $\{T_n\}$ in $\mathcal{D}'(\Omega)$,

$$T_n \to 0 \quad \text{in} \quad \mathcal{D}'(\Omega) \quad \Leftrightarrow \quad T_n(v) \to 0 \quad \forall v \in \mathcal{D}(\Omega). \tag{2.3}$$

(Because of the linear structure of $\mathcal{D}'(\Omega)$, it suffices to define vanishing sequences.) The convergence of a series of distributions is defined as the convergence of the sequence of the partial sums; that is, a series $\sum_{n=0}^{\infty} T_n$ converges to T in $\mathcal{D}'(\Omega)$ whenever $\{\sum_{n=0}^{m} T_n\}$ converges to T in $\mathcal{D}'(\Omega)$ as $m \to \infty$. Thus

$$T = \sum_{n=0}^{\infty} T_n \quad \text{in } \mathcal{D}'(\Omega) \quad \Leftrightarrow \quad \langle T, \varphi\rangle = \sum_{n=0}^{\infty} \langle T_n, \varphi\rangle \qquad \forall \varphi \in \mathcal{D}(\Omega). \tag{2.4}$$

Examples. (i) The very definition of the principal value is based on the approximation (by regularization in this case) in the sense of distributions. Setting

$$f_\varepsilon(x) = 1/x \quad \text{if } |x| \ge \varepsilon, \qquad f_\varepsilon(x) = 0 \quad \text{if } |x| < \varepsilon,$$

by (2.1') we have $T_{f_\varepsilon} \to$ p.v. $(1/x)$ in $\mathcal{D}'(\mathbf{R})$.

(ii) The Dirac mass may be approximated in several ways. For instance, both sequences

$$f_n(x) := \begin{cases} n/2 & \text{if } |x| < 1/n \\ 0 & \text{if } |x| \geq 1/n, \end{cases} \qquad g_n(x) := \frac{n}{\sqrt{\pi}} e^{-n^2x^2} \qquad \forall x \in \mathbf{R}, \forall n \in \mathbf{N}$$

converge to δ in $\mathcal{D}'(\mathbf{R})$. Indeed, by the mean value theorem, for any $n \in \mathbf{N}$ there exists $x_n \in]-1/n, 1/n[$ such that

$$\langle T_{f_n}, \varphi\rangle = \frac{n}{2}\int_{-1/n}^{1/n} \varphi(x)\,dx = \varphi(x_n) \to \varphi(0) = \langle \delta, \varphi\rangle \qquad \forall \varphi \in \mathcal{D}(\mathbf{R}).$$

On the other hand, recalling the classical Poisson identity $\int_{\mathbf{R}} e^{-y^2}\,dy = \sqrt{\pi}$,

$$\begin{aligned}\langle T_{g_n}, \varphi\rangle &= \frac{n}{\sqrt{\pi}}\int_{\mathbf{R}} e^{-n^2x^2}\varphi(x)\,dx = \frac{1}{\sqrt{\pi}}\int_{\mathbf{R}} e^{-y^2}\varphi(y/n)\,dy \\ &\to \frac{\varphi(0)}{\sqrt{\pi}}\int_{\mathbf{R}} e^{-y^2}\,dy = \varphi(0) = \langle \delta, \varphi\rangle \qquad \forall \varphi \in \mathcal{D}(\mathbf{R}).\end{aligned}$$

$\{f_n\}$ and $\{g_n\}$ are accordingly said two *regularizing sequences* of the Dirac mass.

Proposition 2.2 *If $T_n \to T$ in $\mathcal{D}'(\Omega)$ and $v_n \to v$ in $\mathcal{D}(\Omega)$, then $T_n(v_n) \to T(v)$. []*

Theorem 2.3 *The space $\mathcal{D}'(\Omega)$ is sequentially complete. []*

Theorem 2.4 *Any subset of $\mathcal{D}'(\Omega)$ is sequentially relatively compact iff it is bounded. []*

Exercises. (i) Check that a sequence $\{f_n\}$ in $L^1(\mathbf{R})$ approximates δ in $\mathcal{D}'(\mathbf{R})$ whenever

$$\exists M > 0 : \forall n \in \mathbf{N}, \;\; f_n \geq -M \quad \text{a.e. in } \mathbf{R},$$

$$\forall \varepsilon > 0, \exists C > 0 : \forall n \in \mathbf{N}, \;\; |x| > \varepsilon \;\;\Rightarrow\;\; f_n(x) \leq C,$$

$$f_n \to 0 \quad \text{a.e. in } \mathbf{R}, \qquad \int_{-1/n}^{1/n} f_n(x)\,dx \to 1.$$

Show that no one of these four conditions is needed for the convergence: actually they may fail simultaneously.

(ii) For which complex sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are $T_1 := \sum_n a_n D^n \delta_0$, $T_2 := \sum_n b_n D^n \delta_{1/n}$ and $T_3 := \sum_n c_n D^n \delta_n$ distributions over $\mathbf{R}$?

3. Distributional Calculus

Some operations may be defined in the space $\mathcal{D}'(\Omega)$ by transposition. For $\Omega = \mathbf{R}^N$ this applies, e.g., to the traslations (denoted by τ_h for any $h \in \mathbf{R}^N$) and to the rescalings of x:

$$\langle \tau_h T, v\rangle := \langle T, \tau_{-h} v\rangle \qquad \forall v \in \mathcal{D}(\mathbf{R}^N), \forall T \in \mathcal{D}'(\mathbf{R}^N), \forall h \in \mathbf{R}^N, \tag{3.1}$$

$$\begin{aligned}&\langle T(A\cdot), v\rangle := \operatorname{Det}(A)^{-1}\,\langle T, v(A^{-1}\cdot)\rangle \\ &\forall v \in \mathcal{D}(\mathbf{R}^N), \forall T \in \mathcal{D}'(\mathbf{R}^N), \forall \text{ nonsingular } A \in \mathbf{R}^{N^2}.\end{aligned} \tag{3.2}$$

The multipication of distributions by C^∞-functions may also be defined by transposition: for any open set Ω,

$$\langle fT, v\rangle := \langle T, fv\rangle \qquad \forall T \in \mathcal{D}'(\Omega), \forall f \in C^\infty(\Omega), \forall v \in \mathcal{D}(\Omega). \tag{3.3}$$

It is easily checked that $\tau_h T, T(A\cdot), fT$ are distributions, and that for regular distributions these definitions are consistent with known properties. (9)

The differentiation is also extended to $\mathcal{D}'(\Omega)$ by transposition: (10)

$$\langle \tilde{D}^\alpha T, v\rangle := (-1)^{|\alpha|}\langle T, D^\alpha v\rangle \qquad \forall T \in \mathcal{D}'(\Omega), \forall v \in \mathcal{D}(\Omega), \forall \alpha \in \mathbf{N}^N. \tag{3.4}$$

Whenever T and $\tilde{D}^\alpha T$ are regular distributions, this definition is consistent with the properties of ordinary derivatives.

Any distribution thus has derivatives of any order. For any $f \in L^1_{\text{loc}}(\Omega)$ such that $D^\alpha f \in L^1_{\text{loc}}(\Omega)$, this notion of derivative is consistent with partial integration: for $T = T_f$, (3.4) indeed reads (11)

$$\int_\Omega [D^\alpha f(x)] v(x)\, dx = (-1)^{|\alpha|} \int_\Omega f(x) D^\alpha v(x)\, dx \qquad \forall v \in \mathcal{D}(\Omega), \forall \alpha \in \mathbf{N}^N. \tag{3.5}$$

The operator $\tilde{D}^\alpha$ is linear and continuous in $\mathcal{D}'(\Omega)$. [Ex] As derivatives commute in $\mathcal{D}(\Omega)$, the same applies in $\mathcal{D}'(\Omega)$, that is, $\tilde{D}^\alpha \circ \tilde{D}^\beta = \tilde{D}^{\alpha+\beta} = \tilde{D}^\beta \circ \tilde{D}^\alpha$ for any multi-indices α, β. [Ex]

The formula of differentiation of the product is extended as follows:

$$\tilde{D}_i(fT) = (D_i f)T + f\tilde{D}_i T \qquad \forall f \in C^\infty(\Omega), \forall T \in \mathcal{D}'(\Omega), \forall i;$$

in fact

$$\begin{aligned} \langle \tilde{D}_i(fT), v\rangle &= -\langle fT, D_i v\rangle = -\langle T, fD_i v\rangle = \langle T, (D_i f)v\rangle - \langle T, D_i(fv)\rangle \\ &= \langle (D_i f)T, v\rangle + \langle \tilde{D}_i T, fv\rangle = \langle (D_i f)T + f\tilde{D}_i T, v\rangle \quad \forall v \in \mathcal{D}(\Omega). \end{aligned}$$

A recursive procedure then yields the extension of the classical Leibniz rule:

$$\begin{aligned} &\tilde{D}^\alpha(fT) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} (D^{\alpha-\beta} f)\tilde{D}^\beta T \qquad \forall f \in C^\infty(\Omega), \forall T \in \mathcal{D}'(\Omega), \forall \alpha \in \mathbf{N}^N, \\ &\text{where} \quad \binom{\alpha}{\beta} := \prod_{i=1}^N \binom{\alpha_i}{\beta_i} = \prod_{i=1}^N \frac{\alpha_i!}{(\alpha_i - \beta_i)!\beta_i!}\,. \quad [Ex] \end{aligned} \tag{3.6}$$

Proposition 3.1 *$\mathcal{D}(\Omega) \subset \mathcal{E}(\Omega)$ with continuous injection and density (by an obvious identification). Moreover, $\mathcal{E}'(\Omega) \subset \mathcal{D}'(\Omega)$ with continuous injection.*

Proof. It is obvious that $\mathcal{D}(\Omega) \subset \mathcal{E}(\Omega)$ with continuous injection. Let ρ be as in (1.1), and define the mollifier $\rho_\varepsilon(x) := \varepsilon^{-N}\rho(x/\varepsilon)$ for any $x \in \mathbf{R}^N$ and any $\varepsilon > 0$.

For any $v \in \mathcal{E}(\Omega)$, if K is a nondecreasing family of compact subsets that converges to Ω and $\varepsilon \to 0$, then $(v1_K) * \rho_\varepsilon \to v$ in $\mathcal{E}(\Omega)$. [Ex] Hence $\mathcal{D}(\Omega)$ is dense in $\mathcal{E}(\Omega)$. This entails the final statement. □

Comparison with Classical Derivatives.

(9) For any $m \in \mathbf{N}$, the same properties hold for the topological dual of $\mathcal{D}^m(\Omega)$, in this case for any $f \in C^m(\Omega)$.

(10) In this section we denote the distributional derivative by $\tilde{D}^\alpha$, and the classical derivative, i.e. the pointwise limit of the incremental ratio, by D^α, whenever the latter exists.

(11) This coincides with the definition of *weak derivative* due to Sobolev.

• **Theorem 3.2** *(Du-Bois Reymond) Let $f \in C^0(\Omega)$ and $i \in \{1, \dots, N\}$. Then $\tilde{D}_i f \in C^0(\Omega)$ (possibly after modification in a set of vanishing measure) iff f is classically differentiable w.r.t. x_i and $D_i f \in C^0(\Omega)$. In this case $\tilde{D}_i f = D_i f$ in Ω. []*

The next statement applies to $N = 1$, with $\Omega :=]a, b[$ and $-\infty \le a < b \le +\infty$. A multidimensional extension will be provided in the next section.

First we remind the reader that a function $f \in L^1(a, b)$ is absolutely continuous iff there exists $g \in L^1(a, b)$ such that $f(x) = f(y) + \int_y^x g(\xi)\, d\xi$ for any $x, y \in]a, b[$. This entails that $f' = g$ a.e. in $]a, b[$. Thus if $f \in L^1(a, b)$ is absolutely continuous, then it is a.e. differentiable and $f' \in L^1(a, b)$. The converse may fail: the Heaviside function H ($H(x) = 0$ for any $x < 0$, $H(x) = 1$ for any $x \ge 0$) is a counterexample, as $H' = 0$ a.e. in $\mathbf{R}$, but H is not a.e. equal to any absolutely continuous function. (12)

• **Theorem 3.3** *A function $f \in L^1(a, b)$ is a.e. equal to an absolutely continuous function, $\hat{f}$, iff $\tilde{D} f \in L^1(a, b)$. In this case $\tilde{D} f = D\hat{f}$ a.e. in $]a, b[$.*

If $f \in L^1_{\rm loc}(a, b)$ then Df may exist a.e. in $]a, b[$ and even be locally integrable without coinciding with the distributional derivative $\tilde{D} f$, which need not be an element of $L^1_{\rm loc}(a, b)$. For instance, still denoting by H the Heaviside function, we have $DH = 0$ a.e. in $\mathbf{R}$, but $\tilde{D} H = \delta_0$ as

$$\langle \tilde{D} H, v\rangle = -\int_{\mathbf{R}} H(x)\, Dv(x)\, dx = -\int_{\mathbf{R}^+} Dv(x)\, dx = v(0) = \langle \delta_0, v\rangle \qquad \forall v \in \mathcal{D}(\mathbf{R}).$$

Support and Order of Distributions. A distribution $T \in \mathcal{D}'(\Omega)$ is said to vanish in an open subset $\tilde{\Omega}$ of Ω iff it vanishes on any function of $\mathcal{D}(\Omega)$ supported in $\tilde{\Omega}$. There exists then a (possibly empty) largest open set $A \subset \Omega$ in which T vanishes; [] (13) its complement in Ω is called the **support** of T, and is denoted by $\operatorname{supp} T$. This extends the usual notion, since the support (in the usual sense) of any $f \in C^0(\Omega)$ coincides with the support of the associated distribution T_f. [Ex]

If m is the smallest integer that fulfills (2.2), we say that T has *order m in K*. The supremum of these orders as K varies among the compact subsets of Ω is called the **order** of T; thus distributions may be of either finite of infinite order. For instance,

the (distributions that are identified with) functions of $L^1_{\rm loc}(\Omega)$ and the Dirac mass are of order zero;

p.v. $(1/x)$ is of order one;

$\tilde{D}^\alpha \delta_0$ is of order $|\alpha|$ for any $\alpha \in \mathbf{N}^N$;

$\sum_{n=0}^\infty a_n \tilde{D}^n \delta_n$ is a distribution over $\mathbf{R}$ of infinite order for any complex sequence $\{a_n\}$. [Ex]

The next statement establishes a strict relation between support and order.

Theorem 3.4 *Any compactly supported distribution is of finite order.*

This is easily established recalling the characterization (2.2). [Ex] The converse implication obviously fails.

The Space $\mathcal{E}'(\Omega)$. We saw that $\mathcal{D}(\Omega)$ is a dense subspace of $\mathcal{E}(\Omega)$. Moreover, $\mathcal{E}'(\Omega)$ is a dense subspace of $\mathcal{D}'(\Omega)$. []

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(12) Functions that are a.e. equal to an absolutely continuous function are often said absolutely continuous themselves, for they are identified with their equivalence class.

(13) This is the union of all the open sets in which T vanishes, for if a distribution vanishes on a family of open sets then it also vanishes on their union. This property is less trivial than it might look at first sight: the argument is based on the classical tool, that is named *partition of unity*.

Theorem 3.5 *$\mathcal{E}'(\Omega)$ may be identified with the subspace of compactly supported distributions. []*

Examples. (i) As the support of any $v \in \mathcal{D}(\mathbf{R})$ is contained in some interval of the form $[-a, a]$, we have

$$\begin{aligned} \langle D \text{ (p.v. } \frac{1}{x}), v\rangle &= -\langle \text{ (p.v. } \frac{1}{x}), v'\rangle \overset{(2.1')}{=} - \int_{\mathbf{R}} \frac{1}{x}[v'(x) - v'(0)]\, dx \\ &= - \int_{\mathbf{R}} \frac{1}{x}[v(x) - v(0) - xv'(0)]'\, dx = - \int_{\mathbf{R}} \frac{1}{x^2}[v(x) - v(0) - xv'(0)]\, dx. \end{aligned} \tag{3.7}$$

The latter integral converges, since v has compact support and (by the mean-value theorem) the integrand equals $v''(\xi_x)$, for some ξ_x between 0 and x.

(ii) The function $f(x) = [\sin(1/|x|)]/|x|$ is defined a.e., but is not locally (Lebesgue-)integrable in $\mathbf{R}$; hence it cannot be identified with a distribution. On the other hand, setting

$$g(x) := \lim_{\varepsilon \to 0+} \int_{\varepsilon}^{x} f(t)\, dt \in L^1_{loc}(\mathbf{R}) \subset \mathcal{D}'(\mathbf{R}) \tag{3.8}$$

(i.e., $g(x) := \int_0^x f(t)\, dt$, this being meant as a *generalized Riemann-integral),* we have $Dg \in \mathcal{D}'(\mathbf{R})$; thus Dg cannot be identified with f ($\notin \mathcal{D}'(\mathbf{R})$). Actually, as g is odd and has a finite limit (denoted $g(+\infty)$) at $+\infty$, for any $v \in \mathcal{D}(\mathbf{R})$ we have (as $v(0) = 0$ for $|a|$ large enough)

$$\begin{aligned} \langle Dg, v\rangle &= -\langle g, Dv\rangle = - \int_{\mathbf{R}} g(x)[v(x) - v(0)]'\, dx = - \lim_{a \to +\infty} \int_{-a}^{a} g(x)[v(x) - v(0)]'\, dx \\ &= \lim_{a \to +\infty} \int_{-a}^{a} f(x)[v(x) - v(0)]\, dx + \lim_{a \to +\infty} [g(a) - g(-a)]v(0) \\ &= \int_{\mathbf{R}} f(x)[v(x) - v(0)]\, dx + 2g(+\infty)v(0) \qquad \forall v \in \mathcal{D}(\mathbf{R}), \end{aligned} \tag{3.9}$$

the latter being a generalized Riemann-integral.

(iii) The modifications for $\tilde{f}(x) = [\sin(1/|x|)]/x$ and $\tilde{g}(x) := \int_0^x \tilde{f}(t)\, dt$ are left to the reader. □

Problems of Division. For any $f \in C^\infty(\mathbf{R}^N)$ and $S \in \mathcal{D}'(\mathbf{R}^N)$, let us consider the problem of determining $T \in \mathcal{D}'(\mathbf{R}^N)$ such that

$$fT = S \tag{3.10}$$

(This is named a *problem of division,* since formally $T = S/f$.) The general solution may be represented as the sum of a particular solution of (3.10) and the general solution of the homogeneous equation $fT_0 = 0$. The latter may depend on a number of arbitrary constants.

If f does not vanish in $\mathbf{R}^N$, then $1/f \in C^\infty(\mathbf{R}^N)$ and (3.10) has one and only one solution: $T = (1/f)S$. On the other hand, if f vanish at some points of $\mathbf{R}^N$, the situation is less trivial. Let us see the case of $N = 1$, along the lines of [Gilardi: Analisi 3]. For instance, if $f(x) = x^m$ (with $m \in \mathbf{N}$), then the homogeneous equation $x^m T = 0$ has the general solution $T_0 = \sum_{n=0}^{m-1} c_n D^n \delta_0$, with $c_n \in \mathbf{C}$ for any n. [Ex] On the other hand, even the simple-looking equation $x^m T = 1$ is more demanding.

Proposition 3.6 *Let $f \in C^\infty(\mathbf{R})$ have just isolated and finite-order zeroes. Then, denoting by Z the set of these zeroes and by ν_z the order of any $z \in Z$,*

$$T = \sum_{z \in Z} \sum_{n=0}^{\nu_z} c_{z,n} D^n \delta_z, \qquad \text{with } c_{z,n} \in \mathbf{C}, \forall z, \forall n \tag{3.11}$$

is the general solution in $\mathcal{D}'(\mathbf{R})$ *of the equation* $fT = 0$.

Corollary 3.7 *For any polynomial* P *and any* $f \in \mathcal{D}'(\mathbf{R})$, *there exists* $T \in \mathcal{D}'(\mathbf{R})$ *such that* $PT = f$.

Exercises.

— Define the conjugate, the real and the imaginary part of distributions via transposition. [Ex]

— Let the function ρ be defined as in (1.0) above, and set $v_n(x) := \rho(nx)/n$ for any $x \in \mathbf{R}$ and any $n \in \mathbf{N}$. Is the sequence $\{v_n\}$ bounded in $\mathcal{D}(\mathbf{R})$?

— Let $u \in \mathcal{D}(\mathbf{R})$ be such that $u = 1$ in $[-1, 1]$, $u = 0$ outside $]-2, 2[$; set $u_n(x) := u(nx)$ for any $x \in \mathbf{R}$ and any $n \in \mathbf{N}$. Study the possible convergence of the sequence $\{u_n\}$ in the spaces $\mathcal{D}(\mathbf{R})$, $\mathcal{E}(\mathbf{R})$, $L^1_{\text{loc}}(\mathbf{R})$.

— Show that, if $\{x_n\} \subset \Omega$ is a sequence of points that do not accumulate in Ω and $\{\alpha_n\} \subset \mathbf{N}^N$, then $T : v \mapsto \sum_n \tilde{D}^{\alpha_n} v(x_n)$ is a distribution of order equal to $\sup_n |\alpha_n|$ $(\le +\infty)$. Show also that, if instead the sequence accumulates in Ω, then T is not a distribution.

— Let $\{T_n\}$ be a sequence of distributions over an open set Ω, and assume that $T(v) := \lim_{n\to\infty} T_n(v)$ exists in $\mathbf{C}$ for any $v \in \mathcal{D}(\Omega)$. Show that then T is a distribution.

— Check that $\tilde{D} \log |x| = \text{p.v. } 1/x$ in $\mathcal{D}'(\mathbf{R})$, cf. (2.5), and $\tilde{D} \log |x| = 1/x$ in $\mathcal{D}'(\mathbf{R} \setminus \{0\})$.

— Show that if $T \in \mathcal{D}'(\mathbf{R})$ is such that $\tilde{D}^2 T = 0$ in $\mathcal{D}'(\mathbf{R})$ then T is a first-degree polynomial.

— For any convergent sequence $\{T_n\}$ in $\mathcal{D}'(\mathbf{R})$, $\lim_{n\to\infty} \tilde{D}T_n = \tilde{D} \lim_{n\to\infty} T_n$, without any restriction. In particular this applies to elements of $L^1_{\text{loc}}(\Omega)$. On the other hand, some classical counterexamples show that the passage to the limit in the Lebesgue integral requires some restrictions. Why those counterexamples do not contradict the above statement?

— In classical analysis it is known that the existence of the mixed derivatives may not commute, even if they exist. On the other hand in the theory of distributions mixed derivatives do commute? Is there any contradiction?

$\lim_{n\to\infty} \tilde{D}T_n = \tilde{D} \lim_{n\to\infty} T_n$, without any restriction. In particular this applies to elements of $L^1_{\text{loc}}(\Omega)$. On the other hand, some classical counterexamples show that the passage to the limit in the Lebesgue integral requires some restrictions. Why those counterexamples do not contradict the above statement?

— Are the following linear functionals $\mathcal{D}(\mathbf{R}) \to \mathbf{C}$ distributions?

$$T_1 = \sum_{k=1}^{\infty} e^k \delta_{\log k}\,, \quad T_2 = \sum_{k=1}^{\infty} D^k \delta_{1/k}\,, \quad T_3 = \sum_{k=1}^{\infty} k^{-1} \delta_{1/k}\,, \quad T_4 = \sum_{k=1}^{\infty} k^{-2} \delta_{1/k}\,.$$

— For any $\alpha \in C^\infty(\mathbf{R})$ and $x_0 \in \mathbf{R}^N$, evaluate $\langle \alpha\delta_{x_0} - \alpha(x_0)\delta_{x_0}, \varphi \rangle$ for any $\varphi \in \mathcal{D}'(\mathbf{R}^N)$.

— Show that $\delta(\lambda \cdot) = \delta/|\lambda|$ for any $\lambda \in \mathbf{C} \setminus \{0\}$, and compare this formula with (3.2).

— Check that if T is an even distribution then T' is odd, and that if T is odd then T' is even.

— Let $a \in \mathbf{R}$, $k \in \mathbf{N}$. Is $\tilde{D}^k \delta_a$ either even or odd?

— For any integer $m \ge 1$ and any $\lambda_1, ..., \lambda_m \in \mathbf{C}$, let us set $T := \sum_{k=1}^m \lambda_k \tilde{D}^k \delta$. Check that $x^{m+1} T = 0$ in $\mathcal{D}'(\mathbf{R})$. (In particular, $x\delta = 0$ in $\mathcal{D}'(\mathbf{R})$.)

— Show that $n\delta - n^2 I_{]0,1/n[} \to \delta'/2$ in $\mathcal{D}'$. Find an approximation of δ'' in $\mathcal{D}'$.

— Show that $D^2(H(x) \sin x) = \delta - H(x) \sin x$ in $\mathcal{D}'(\mathbf{R})$.

Exercises. (i) May the differentiation also be defined as the limit of the incremental ratios in $\mathcal{D}'(\mathbf{R})$?

(ii) May a convergent series in $\mathcal{D}'(\Omega)$ be differentiated term by term?

(iii) Prove that $\text{sign}(\sin(nx)) \to 0$ in $\mathcal{D}'(\mathbf{R})$ as $n \to \infty$.

(iv) For any $k \in \mathbf{N}$, study the limit of $n^k \sin(nx)$ in $\mathcal{D}'(\mathbf{R})$ as $n \to \infty$.

(v) Let $\{\lambda_k\}$ and $\{a_k\}$ be two sequences in $\mathbf{R}$. Provide a condition on $\{\lambda_k\}$ ($\{a_k\}$, resp.) such that the series $\sum_{k=0}^{\infty} \lambda_k D^k \delta_{a_k}$ converges in $\mathcal{D}'(\mathbf{R})$ for any sequence $\{a_k\}$ ($\{\lambda_k\}$, resp.).

(vi) Show that, if T is a distribution of order $n > 0$, then DT has order $n+1$.
(vii) Check that

$$xD^n\delta = -nD^{n-1}\delta, \qquad x^nD^n\delta = (-1)^n n!\delta, \qquad x^mD^n\delta = 0 \quad \text{if } m > n > 0.$$

(viii) Solve the equations $x^2T = 0$, $x^2T = 1$, $T' + aT = H$, $T' + aT = \delta$ (with $a \in \mathbf{R}$) in $\mathcal{D}'(\mathbf{R})$.

4. Temperate Distributions

The Schwartz Space $\mathcal{S}$ of Rapidly Decreasing Functions. In view of the analysis of the Fourier transform via distributions, we define the Schwartz space of *rapidly decreasing functions* (as $|x| \to +\infty$):

$$\begin{aligned}\mathcal{S} :=& \big\{v \in C^\infty : \forall\alpha, \beta \in \mathbf{N}^N, x^\beta D_x^\alpha v \in L^\infty\big\}\\ =& \big\{v \in C^\infty : \forall\alpha \in \mathbf{N}^N, \forall m \in \mathbf{N}, |x|^m D_x^\alpha v(x) \to 0 \text{ as } |x| \to +\infty\big\}.\end{aligned} \tag{4.1}$$

(We still omit the domain $\mathbf{R}^N$.) By the Leibniz rule, this space is invariant by application of operators of the form $u \mapsto P(x)Q(D)u$, for any complex polynomials P and Q of N real variables. It is actually the smallest space that includes L^1 and has this stability property. $\mathcal{S}$ is a locally convex Fréchet space equipped with either of the following equivalent families of seminorms

$$|v|_{\alpha,\beta} := \sup_{x\in\mathbf{R}^N} |x^\beta D_x^\alpha v(x)| \qquad \forall\alpha, \beta \in \mathbf{N}^N, \tag{4.2}$$

$$|v|_{m,\alpha} := \sup_{x\in\mathbf{R}^N} \big(1 + |x|^2\big)^m |D_x^\alpha v(x)| \qquad \forall m \in \mathbf{N}, \forall\alpha \in \mathbf{N}^N. \tag{4.3}$$

For instance, the function $x \mapsto \exp(-|x|^2)$ and all functions of $\mathcal{D}$ are elements of $\mathcal{S}$, indeed $\mathcal{D} \subset \mathcal{S}$ (with continuous injection). As these families of seminorms are countable, $\mathcal{S}$ is a Frechét space.

The Space $\mathcal{S}'$ of Temperate Distributions. We denote the topological dual of $\mathcal{S}$ by $\mathcal{S}'$. In analogy with $\mathcal{D}'$, we equip $\mathcal{S}'$ with the sequential topology of pointwise convergence: for any sequence $\{T_n\}$ in $\mathcal{S}'$,

$$T_n \to 0 \quad \text{in } \mathcal{S}' \quad \Leftrightarrow \quad T_n(v) \to 0 \quad \forall v \in \mathcal{S}. \tag{4.4}$$

Proposition 4.1 *A linear functional $L : \mathcal{S} \to \mathbf{C}$ is an element of $\mathcal{S}'$ (if and) only if*

$$\exists C >, \exists m \in \mathbf{N} : \forall v \in \mathcal{S} \qquad |L(v)| \le C \sum_{|\alpha|,|\beta|\le m} |v|_{\alpha,\beta}. \tag{4.5}$$

Proof. For any $m \in \mathbf{N}$, let us set

$$\mathcal{S}^m := \left\{v \in C^m : x^\beta D_x^\alpha v \in L^\infty, \forall\alpha, \beta \in \mathbf{N}^N \text{ such that } |\alpha|, |\beta| \le m\right\},$$

and notice that $\mathcal{S} = \bigcap_{m\ge1} \mathcal{S}^m$ is equipped with the projective limit topology of the sequence $\{\mathcal{S}^m\}$. This yields $\mathcal{S}' = \bigcup_{m\ge1}\{(\mathcal{S}^m)'\}$. [] □

For any $T \in \mathcal{S}'$ and any $v \in \mathcal{S}$, we shall also write $\langle T, v\rangle$ in place of $T(v)$. This notation is consistent with that we used for distributions, because of the following result.

Proposition 4.2

$$\mathcal{S} \subset \mathcal{S}', \quad \mathcal{D} \subset \mathcal{S} \subset \mathcal{E}, \quad \mathcal{E}' \subset \mathcal{S}' \subset \mathcal{D}' \quad \textit{with continuous and dense injections.} \tag{4.6}$$

Proof. The density of $\mathcal{S}$ in $\mathcal{S}'$ can be proved as for the density of $\mathcal{D}$ in $\mathcal{D}'$ [that argument was omitted].

It is clear that the injection $\mathcal{D} \subset \mathcal{S}$ is continuous. In order to prove the density, let us fix any $\varphi \in \mathcal{D}(\mathbf{R})$ such that $\varphi = 1$ in $[0,1]$, set $\varphi_n(x) := \varphi(|x|/n)$ for any $x \in \mathbf{R}^N$ and any $n \in \mathbf{N}$, and notice that $\varphi_n v \in \mathcal{D} \cap \mathcal{S}$ for any $v \in \mathcal{S}$. By means of the Leibniz rule one can then see that $\varphi_n v \to v$ in $\mathcal{S}$. [Ex]

The continuity of the injection $\mathcal{S} \subset \mathcal{E}$ is obvious. As $\mathcal{D}$ is dense in $\mathcal{E}$ and $\mathcal{D} \subset \mathcal{S}$, $\mathcal{S}$ is also dense in $\mathcal{E}$. The inclusions among the dual spaces and the continuity of the corresponding injections then follow from a general result. [] □

In passing notice that

$$\mathcal{D} \subset \mathcal{D}', \quad \mathcal{S} \subset \mathcal{S}', \quad \text{but} \quad \mathcal{E} \not\subset \mathcal{E}', \tag{4.7}$$

since there exists smooth non-compactly-supported functions.

$\mathcal{S}'$ is the smallest space that includes L^1 and is invariant by differentiation and multiplication by a polynomial. Any function $u \in L^1_{\rm loc}$ is said *slowly increasing* iff it is of the form $u(x) = P(x)w(x)$, with P polynomial of N variables and $w \in L^1$. The space $L^1_{\rm loc}$ is not included in $\mathcal{S}'$, but any slowly increasing function and any compactly supported distribution are temperate distributions (by the usual identifications).

For instance, $\psi : \mathbf{R} \to \mathbf{R} : x \mapsto e^x$ is (identified with) a nontemperate distribution. In fact, any $f \in C^\infty(\mathbf{R})$ such that $f(x) = e^{-|x|/2}$ if $|x| > 1$ is rapidly decreasing, but $\psi f \notin L^1$. On the other hand, the real function $x \mapsto \sin(e^x)$ is slowly increasing, hence temperate. Therefore its derivative $e^x \cos(e^x)$ is also temperate, although its absolute value is not so. [Ex]

For any $\alpha \in \mathbf{N}^N$, the linear and continuous operators $u \mapsto x^\alpha u$ and $u \mapsto D^\alpha u$ are extended from $\mathcal{S}$ to $\mathcal{S}'$ by transposition, that is,

$$\langle x^\alpha T, v\rangle := \langle T, x^\alpha v\rangle, \qquad \langle D^\alpha T, v\rangle := \langle T, (-D)^\alpha v\rangle \qquad \forall v \in \mathcal{S}, \forall T \in \mathcal{S}', \forall \alpha \in \mathbf{N}^N.$$

These operators are linear and continuous in $\mathcal{S}'$, and this is the smallest space containing L^1 in which this occurs. This is easily generalized to the operators $T \mapsto P(x)T$ and $T \mapsto Q(D)T$, whence to $T \mapsto P(x)Q(D)T$, for any polynomials P and Q of N variables.

Exercises.

— Check that the two families of seminorms (4.2) and (4.3) are equivalent.

— Check that $x^\beta D_x^\alpha v \in L^1$, for any $v \in \mathcal{S}$ and any $\alpha, \beta \in \mathbf{N}^N$.

— Show that, for any $p \in [1, +\infty]$, the functions of L^p are *slowly increasing.*

Hint: The statement is obvious for $p = 1$ or $p = +\infty$. For any $p \in\,]1, +\infty[$, set $p' := p/(p-1)$; then notice that by the Hölder inequality

$$\int (1+|x|^2)^{-k} |f(x)|\, dx \le \left(\int (1+|x|^2)^{-kp'}\, dx \right)^{1/p'} \|f\|_{L^p} \qquad \forall f \in L^p, \forall k \in \mathbf{N},$$

and that for k large enough $\int (1+|x|^2)^{-kp'}\, dx$ converges...

— Let $u(x) = P(x)w(x)$, with P polynomial and $w \in L^p$, for some $p \in [1, +\infty]$. Is then u slowly increasing?

— Compare the class of the *slowly increasing* functions with that of the measurable functions $f : \mathbf{R}^N \to \mathbf{C}$ such that $|f(x)| \le C|x|^k$ for any x, for some $k \in \mathbf{N}$ and $C > 0$.

— Let $\{T_n\}$ be a sequence of $\mathcal{E}'$. Establish implications among the following properties:

$$T_n \to 0 \ \text{ in } \mathcal{E}', \quad T_n \to 0 \ \text{ in } \mathcal{S}', \quad T_n \to 0 \ \text{ in } \mathcal{D}'.$$

— Does the topology of the spaces $\mathcal{D}, \mathcal{E}, \mathcal{S}$ and of their duals coincide with the corresponding sequential topology? □