Fourier and Laplace Transforms, Convolution

Contents: 1. The Fourier transform in L^1. 2. Extensions of the Fourier transform. 3. The Laplace transform. 4. Convolution. 5. Fourier Transform and Differential Equations

The Fourier transform was introduced by Fourier at the beginning of the XIX century. In the 1940s Laurent Schwartz introduced the *temperate distributions,* and extended the transform to this class. This transform is an important theoretical tool in many branches of analysis, and is also very useful for applications. In particular it allows one to reduce linear ordinary differential equations with constant coefficients to algebraic equations.

1. The Fourier Transform in L^1

In this section we define the Fourier transform in the space of integrable functions, and study its basic properties, in view of the extensions of the next section.

The Fourier Transform in L^1. Dealing with this transform, we shall always use spaces of functions from $\mathbf{R}^N$ to $\mathbf{C}$; we shall then write L^1 in place of $L^1(\mathbf{R}^N)$, C^0 in place of $C^0(\mathbf{R}^N)$, and so on. We shall also denote by C^0_b the space of continuous and bounded functions $\mathbf{R}^N \to \mathbf{R}$, equipped with the sup-norm. For any $u \in L^1$, we define the *Fourier transform* (also called *Fourier-integral)* $\widehat{u}$ of u: [1]

$$\widehat{u}(\xi) := \int_{\mathbf{R}^N} e^{-2\pi i\xi\cdot x} u(x)\, dx \qquad \forall \xi \in \mathbf{R}^N \ \left(\xi\cdot x := \textstyle\sum_{i=1}^N \xi_i x_i\right). \tag{1.1}$$

Proposition 1.1 *(1.1) defines a linear and continuous operator $\mathcal{F} : L^1 \to C^0_b : u \mapsto \widehat{u}$, with $\|\widehat{u}\|_{L^\infty} \le \|u\|_{L^1}$ for any $u \in L^1$. [Ex]*

Notice that $\|\widehat{u}\|_{L^\infty} = \widehat{u}(0) = \|u\|_{L^1}$ for any nonnegative $u \in L^1$, as

$$\|\widehat{u}\|_{L^\infty} \le \int_{\mathbf{R}^N} |u(x)|\, dx = \int_{\mathbf{R}^N} u(x)\, dx = \widehat{u}(0) \le \|\widehat{u}\|_{L^\infty}.$$

Let us denote the adjoint of any $A \in \mathbf{R}^{N^2}$ by A^*, and the complex conjugate of any $z \in \mathbf{C}$ by $\bar{z}$. We shall say that u is *radial* iff $u(Ax) = u(x)$ for any x and any orthonormal matrix $A \in \mathbf{R}^{N^2}$ (i.e., with $A^* = A^{-1}$).

Proposition 1.2 *For any $u \in L^1$,*

$$v(x) = u(x-y) \quad \Longrightarrow \quad \widehat{v}(\xi) = e^{-2\pi i\xi\cdot y}\widehat{u}(\xi) \qquad \forall y \in \mathbf{R}^N, \tag{1.2}$$

$$v(x) = e^{2\pi i x\cdot\eta}u(x) \quad \Longrightarrow \quad \widehat{v}(\xi) = \widehat{u}(\xi-\eta) \qquad \forall \eta \in \mathbf{R}^N, \tag{1.3}$$

$$v(x) = u\left(A^{-1}x\right) \quad \Longrightarrow \quad \widehat{v}(\xi) = |\det A|\widehat{u}(A^*\xi) \qquad \forall A \in \mathbf{R}^{N^2}, \det A \ne 0, \tag{1.4}$$

$$v(x) = \overline{u(x)} \quad \Longrightarrow \quad \widehat{v}(\xi) = \overline{\widehat{u}(-\xi)}, \tag{1.5}$$

u is even (odd, resp.) $\Rightarrow$ $\widehat{u}$ is even (odd, resp.), (1.6)

u is real $\Rightarrow$ Re($\widehat{u}$) is even, Im($\widehat{u}$) is odd, (1.7)

u is imaginary $\Rightarrow$ Re($\widehat{u}$) is odd, Im($\widehat{u}$) is even, (1.8)

u is radial $\Rightarrow$ $\widehat{u}$ is radial. [Ex] (1.9)

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(1) Some authors omit the factor 2π in the exponent, others omit it and include a denominator $(2\pi)^{-N/2}$ for the integral (maybe the latter definition is the most usual one). Each of these modifications simplifies some formulas, but none is able to simplify all of them.

Henceforth by D (or D_j or D^α) we shall denote the derivative in the sense of distributions.

Lemma 1.3 *Let $j \in \{1, ..., N\}$. If $\varphi, D_j\varphi \in L^1$ then $\int_{\mathbf{R}^N} D_j\varphi(x)\,dx = 0$.*

Proof. Let us set

$$\begin{aligned} &\rho(x) := \exp\left[\left(|x|^2 - 1\right)^{-1}\right] \quad \text{if} \quad |x| < 1, \quad \rho(x) := 0 \quad \text{if} \quad |x| \geq 1, \\ &\rho_n(x) := \rho(x/n) \quad \forall x \in \mathbf{R}^N, \forall n \in \mathbf{N}. \end{aligned}$$

Hence $\rho_n(x) \to 1$ a.e. in $\mathbf{R}$, and

$$\Big|\int \big[D_j\varphi(x)\big]\rho_n(x)\,dx\Big| = \Big|\int_{\mathbf{R}^N} \varphi(x) D_j\rho_n(x)\,dx\Big| \leq \frac{1}{n}\int_{\mathbf{R}^N} \big|\varphi(x)\big|\,dx \sup|D_j\rho| \to 0.$$

Therefore, by the dominated convergence theorem,

$$\int D_j\varphi(x)\,dx = \int D_j\varphi(x) \lim_{n\to\infty} \rho_n(x)\,dx = \lim_{n\to\infty} \int \big[D_j\varphi(x)\big]\rho_n(x)\,dx = 0. \qquad \square$$

• **Proposition 1.4** *For any $\alpha \in \mathbf{N}^N$,*

$$u, D_x^\alpha u \in L^1 \quad \Rightarrow \quad (2\pi i)^{|\alpha|}\xi^\alpha \widehat{u} = \widehat{(D_x^\alpha u)} \qquad in\ C_b^0, \tag{1.10}$$

$$u, x^\alpha u \in L^1 \quad \Rightarrow \quad D_\xi^\alpha \widehat{u} = (-2\pi i)^{|\alpha|}\widehat{(x^\alpha u)} \qquad in\ C_b^0. \tag{1.11}$$

Proof. In both cases it suffices to prove the equality for any first-order derivative $D_j (:= \partial/\partial x_j)$; the general case will then follow by induction. As

$$D_j[e^{-2\pi i\xi\cdot x}u(x)] = -2\pi i\xi_j e^{-2\pi i\xi\cdot x}u(x) + e^{-2\pi i\xi\cdot x}D_j u(x),$$

and $u, D_x^\alpha u \in L^1$, we infer that $D_j[e^{-2\pi i\xi\cdot x}u(x)] \in L^1$. Integrating the latter equality over $\mathbf{R}^N$, and noticing that $\int_{\mathbf{R}^N} D_j[e^{-2\pi i\xi\cdot x}u(x)]\,dx = 0$ by Lemma 1.3, we get $(2\pi i)^{|\alpha|}\xi^\alpha\widehat{u} = \widehat{(D_x^\alpha u)}$. Moreover, $\widehat{(D_x^\alpha u)} \in C_b^0$, by Proposition 1.1.

Denoting by e_j the unit vector in the jth direction, we have

$$\frac{\widehat{u}(\xi + te_j) - \widehat{u}(\xi)}{t} = \int_{\mathbf{R}^N} \frac{e^{-2\pi i(\xi+te_j)\cdot x} - e^{-2\pi i\xi\cdot x}}{t} u(x)\,dx \qquad \forall t \neq 0,$$

and the absolute value of the integrand is bounded by $2|t|^{-1}|\sin(\pi t x_j)u(x)| \leq 2\pi|x_j u(x)|$ for any t. Passing to the limit as $t \to 0$, by the dominated convergence theorem we then get $D_j\widehat{u} = -2\pi i\widehat{(x_j u)}$. By Proposition 1.1, this is an element of C_b^0. $\square$

Corollary 1.5 *Let $m \in \mathbf{N}$. For any polynomial $P(\xi) := \sum_{|\alpha|\leq m} a_\alpha\xi^\alpha$ (with $a_\alpha \in \mathbf{C}$, $\forall\alpha$),*

$$\begin{aligned} &D_x^\alpha u \in L^1 \quad \forall\alpha \in \mathbf{N}^N, |\alpha| \leq m \quad \Rightarrow \\ &(1 + |\xi|)^m\widehat{u}(\xi) \in C_b^0, \quad P(2\pi i\xi)\widehat{u} = \widehat{(P(D)u)} \ \ in\ C_b^0, \end{aligned} \tag{1.12}$$

$$\begin{aligned} &(1 + |x|)^m u \in L^1 \quad \Rightarrow \quad D^\alpha\widehat{u} \in C_b^0 \quad \forall\alpha \in \mathbf{N}^N, |\alpha| \leq m, \\ &P(D)\widehat{u} = \widehat{[P(-2\pi i x)u]} \ \ in\ C_b^0. \qquad [Ex] \end{aligned} \tag{1.13}$$

By the latter statement, the integrability of any derivative of $u \in L^1$ provides information on the decay of $\widehat{u}$ as $|\xi| \to \infty$; and conversely a suitable decay of $u \in L^1$ as $|x| \to \infty$ entails the differentiability of certain derivatives of $\widehat{u}$. The regularity of u is thus related to the decay of $\widehat{u}$, and the decay of u is related to the regularity of $\widehat{u}$.

Proposition 1.6 *(Riemann-Lebesgue) For any $u \in L^1$, $\widehat{u}(\xi) \to 0$ as $|\xi| \to +\infty$, and $\widehat{u}$ is uniformly continuous in $\mathbf{R}^N$.*

Proof. For any $u \in L^1$, there exists a sequence $\{u_n\}$ in $\mathcal{D}$ such that $u_n \to u$ in L^1. By part (i) of Corollary 1.5, $\widehat{u}_n(\xi) \to 0$ as $|\xi| \to +\infty$. This holds also for $\widehat{u}$, as $\widehat{u}_n \to \widehat{u}$ uniformly in $\mathbf{R}^N$.

(In alternative one may check that the thesis holds for the characteristic function of any N-dimensional interval $[a_1, b_1] \times \cdots \times [a_N, b_N]$. It then suffices to approximate u in L^1 by a sequence of finite linear combinations of characteristic functions of N-dimensional intervals, and then apply Proposition 1.1 to pass to the limit.)

As $\widehat{u} \in C^0_b$, the uniform continuity follows from the asymptotic behaviour. [Ex] □

Theorem 1.7 *(Parseval)*

$$\int_{\mathbf{R}^N} \widehat{u}\, v\, dx = \int_{\mathbf{R}^N} u\, \widehat{v}\, dx \qquad \forall u, v \in L^1, \tag{1.14}$$

$$u*v \in L^1, \quad \text{and} \quad \widehat{(u*v)} = \widehat{u}\,\widehat{v} \qquad \forall u, v \in L^1. \tag{1.15}$$

Proof. By the theorems of Tonelli and Fubini, for any $u, v \in L^1$ we have

$$\int_{\mathbf{R}^N} \widehat{u}(y)v(y)\, dy = \iint_{\mathbf{R}^N \times \mathbf{R}^N} e^{-2\pi i y\cdot x} u(x)v(y)\, dxdy = \int_{\mathbf{R}^N} u(y)\widehat{v}(y)\, dy,$$

$$\begin{aligned}\widehat{(u*v)}(\xi) &= \iint_{\mathbf{R}^N \times \mathbf{R}^N} e^{-2\pi i \xi\cdot x} u(x-y)v(y)\, dxdy \\ &= \int_{\mathbf{R}^N} e^{-2\pi i \xi\cdot(x-y)} u(x-y)\, dx \int_{\mathbf{R}^N} e^{-2\pi i \xi\cdot y} v(y)\, dy = \widehat{u}(\xi)\widehat{v}(\xi).\end{aligned}$$ □

Theorem 1.8 *For any $u \in L^1 \cap C^0 \cap L^\infty$, if $\widehat{u} \in L^1$ then*

$$u(x) = \int_{\mathbf{R}^N} e^{2\pi i \xi\cdot x} \widehat{u}(\xi) d\xi \;\; (=: \tilde{\mathcal{F}}(\widehat{u})) \qquad \forall x \in \mathbf{R}^N. \tag{1.16}$$

Proof. Let us set $v(x) := \exp(-\pi x^2)$ for any $x \in \mathbf{R}^N$. A calculation based on integration along paths in the complex plane shows that $\widehat{v}(\xi) := \exp(-\pi\xi^2)$ for any $\xi \in \mathbf{R}^N$. [] By the Tonelli and Fubini theorems, we have

$$\begin{aligned}&\int_{\mathbf{R}^N} \widehat{u}(\xi)v(\xi)e^{2\pi i\xi\cdot x}\, d\xi = \iint_{\mathbf{R}^N \times \mathbf{R}^N} u(y)e^{-2\pi i \xi\cdot y} v(\xi) e^{2\pi i \xi\cdot x}\, dyd\xi \\ &= \int_{\mathbf{R}^N} u(y)\widehat{v}(y-x)\, dy = \int_{\mathbf{R}^N} u(x+z)\widehat{v}(z)\, dz.\end{aligned}$$

Let us now replace $v(\xi)$ by $\varepsilon_\varepsilon(\xi) := v(\varepsilon\xi)$, for any $\varepsilon > 0$. As $\widehat{(\varepsilon_\varepsilon)}(z) = \varepsilon^{-N}\widehat{v}(z/\varepsilon)$, by setting $s = z/\varepsilon$ we get

$$\int_{\mathbf{R}^N} \widehat{u}(\xi)v(\varepsilon\xi)e^{2\pi i\xi\cdot x}\, d\xi = \int_{\mathbf{R}^N} u(x+\varepsilon s)\widehat{v}(s)\, ds.$$

As u is continuous and bounded, by the dominated convergence theorem, as $\varepsilon \to 0$ we get

$$v(0)\int_{\mathbf{R}^N} \widehat{u}(\xi)e^{2\pi i\xi\cdot x}\, d\xi = u(x)\int_{\mathbf{R}^N} \widehat{v}(s)\, ds.$$

As $v(0) = 1$ and $\int_{\mathbf{R}^N} \widehat{v}(s)\, ds = 1$, we get (1.16). □

By Proposition 1.1, the regularity assumptions of Theorem 1.8 are actually needed, as $\bar{u} = \mathcal{F}(\overline{\widehat{u}})$. However, by a more refined argument one could show that (1.16) holds under the only hypotheses that $u, \widehat{u} \in L^1$. Of course, a posteriori one then gets $u, \widehat{u} \in C^0 \cap L^\infty$.

Under the assumptions of this theorem, we also have

$$\widehat{\widehat{u}}(x) = u(-x) \qquad \forall x \in \mathbf{R}^N. \tag{1.17}$$

By Theorem 1.8, $u = 0$ is the only element of $L^1 \cap C^0 \cap L^\infty$ such that $\mathcal{F}(u) = 0$. This yields the next statement.

Corollary 1.9 *The Fourier transform $L^1 \to C^0 \cap L^\infty$ is injective. [Ex]*

By (1.13) it is clear that $\widehat{u} \in \mathcal{E}$ iff u decreases at infinity more rapidly than any negative power of $|x|$. By the next classical theorem, $\widehat{u}$ is analytic iff u has compact support.

Theorem 1.10 *(Paley-Wiener) Any $u \in C^\infty(\mathbf{R}^N)$ has bounded support iff $\mathcal{F}(u)$ can be extended to an analitic function $\mathbf{C}^N \to \mathbf{C}$ [also denoted by $\mathcal{F}(u)$].* [(2)] *Moreover,* $\operatorname{supp} u \subset \overline{B(0,R)}$ *iff*

$$\forall m \in \mathbf{N}, \exists C_m \geq 0 : \forall z \in \mathbf{C}^N, \quad [\mathcal{F}(u)](z)| \leq C_m \frac{e^{2\pi R|Im(z)|}}{(1+|z|)^m}, \tag{1.18}$$

or equivalently,

$$\forall \varepsilon > 0, \exists C_\varepsilon \geq 0 : \forall z \in \mathbf{C}^N, \quad [\mathcal{F}(u)](z)| \leq C_\varepsilon e^{2\pi(R+\varepsilon)|Im(z)|}. \qquad []$$

This extended function $\mathcal{F}(u) : \mathbf{C}^N \to \mathbf{C}$ is named the *Fourier-Laplace transform* of u.

Overview of the Fourier Transform in L^1. We defined the classical Fourier transform $\mathcal{F} : L^1 \to C^0_b$, and derived its basic properties. In particular, we saw that:

(i) the operator $\mathcal{F}$ transforms partial derivatives to multiplication by powers of the independent variable (up to a multiplicative constant), and conversely. This is at the basis of the application of the Fourier transform to the study of linear partial differential equations with constant coefficients on the whole $\mathbf{R}^N$, that we shall outline ahead.

(ii) the operator $\mathcal{F}$ establishes a correspondence between the regularity of u and the order of infinitesimum of $\widehat{u}$ at ∞, and conversely between the order of infinitesimum of u at ∞ and the regularity of $\widehat{u}$. In the limit case of a compactly supported function (and only in this case), its Fourier transform can be extended to an entire analytic function $\mathbf{C}^N \to \mathbf{C}$; this is called the Fourier-Laplace transform of u.

(iii) the operator $\mathcal{F}$ maps the convolution of two functions to the product of their transforms.

(iv) Under suitable regularity restrictions, the inverse transform exists, and has an integral representation analogous to that of the direct transform. The properties of the two transforms are then similar; this accounts for the duality of the statements (i) and (ii). However the assumptions are not perfectly symmetric; in the next section we shall see a different functional framework where this is remedied.

The inversion formula (1.16) also provides an interpretation of the Fourier transform. (1.16) represents u as a weighted average of the *harmonic components* $x \mapsto e^{2\pi i \xi \cdot x}$. For any $\xi \in \mathbf{R}^N$, $\widehat{u}(\xi)$ is the *amplitude* of the component having *vector frequency* ξ (that is, frequency ξ_i in each direction

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(2) A function $\mathbf{C}^N \to \mathbf{C}$ is said analytic iff it is separately analytic with respect to each variable. For any $z \in \mathbf{C}^N$, we set $|z| := \left(\sum_{i=1}^N |z_i|^2\right)^{1/2}$ and $\mathrm{Im}(z) := (\mathrm{Im}(z_1), ..., \mathrm{Im}(z_N))$.

We recall the reader that we denote the closed ball of $\mathbf{R}^N$ with center the origin and radius R by $B(0,R)$.

x_i). Therefore any function which fulfills (1.16) can equivalently be represented by specifying either the value $u(x)$ at a.a. point $x \in \mathbf{R}^N$, or the amplitude $\widehat{u}(\xi)$ for a.a. frequency $\xi \in \mathbf{R}^N$. (2)

In passing note that the Fourier integral (1.1) converges for any ξ, although any function $f_\xi : y \mapsto e^{-2\pi i \xi \cdot x}$ is not integrable over $\mathbf{R}^N$, not even in the sense of the principal value.

Loosely speaking, the Paley-Wiener theorem entails that any non-identically vanishing $u \in \mathcal{D}$ has harmonic components of arbitrarly large frequencies.

The analogy between the Fourier transform and the Fourier series is obvious, and will be briefly discussed at the end of the next section.

Exercises. (i) For any N and any $u \in L^1$, (1.1) also reads

$$\widehat{u}(\xi) := \int_{\mathbf{R}^N} \cos(\xi \cdot x) u(x)\, dx + i \int_{\mathbf{R}^N} \sin(\xi \cdot x) u(x)\, dx \qquad \forall \xi \in \mathbf{R}^N. \tag{1.19}$$

The first integral is named the *cosine transform,* the second one the *sine transform.* They coincide with the even and the odd parts of the Fourier transform. Note that these transforms coincide with their inverses.

Show that:

$$\begin{aligned} &u \text{ is even} \quad \Leftrightarrow \quad \widehat{u}(\xi) = 2\textstyle\int_{\mathbf{R}^+} \cos(\xi \cdot x) u(x)\, dx \quad \forall \xi \in \mathbf{R}^N, \\ &u \text{ is odd} \quad \Leftrightarrow \quad \widehat{u}(\xi) = 2i\textstyle\int_{\mathbf{R}^+} \sin(\xi \cdot x) u(x)\, dx \quad \forall \xi \in \mathbf{R}^N. \end{aligned}$$

Therefore, for any $u \in L^1$, the cosine transform of any u coincides with the Fourier transform of the even part of u, and the sine transform of any u coincides with the Fourier transform of the even part of u multiplied by $-i$.

(ii) Under suitable assumptions on u and v, prove that $\widehat{(uv)} = \widehat{u} * \widehat{v}$.

(iii) Notice that the function $u : \mathbf{R} \to \mathbf{R} : x \mapsto \exp(-\pi x^2)$ solves the differential equation $Du + 2\pi x u = 0$. By means of Proposition 1.4, derive the corresponding equation in terms of the Fourier transform $\widehat{u}$, couple it with the initial condition $\widehat{u}(0) = \int_{\mathbf{R}^N} u(x)\, dx$, and derive the expression of $\widehat{u} : \mathbf{R} \to \mathbf{R} : \xi \mapsto \exp(-\pi \xi^2)$ by solving this Cauchy problem. Check that the result is consistent with (1.1).

(iv) Check that, for any $u \in L^1$,

$$u \text{ is real and even} \quad \Rightarrow \quad \widehat{u} \text{ is real and even,} \tag{1.20}$$

$$u \text{ is real and odd} \quad \Rightarrow \quad \widehat{u} \text{ is imaginary and odd,} \tag{1.21}$$

$$u \text{ is imaginary and even} \quad \Rightarrow \quad \widehat{u} \text{ is imaginary and even,} \tag{1.22}$$

$$u \text{ is imaginary and odd} \quad \Rightarrow \quad \widehat{u} \text{ is real and odd.} \tag{1.23}$$

2. Extensions of the Fourier Transform

In this section we extend the Fourier transform to the space of measures, to the space $\mathcal{S}$ of rapidly decreasing functions, to the space $\mathcal{S}'$ of temperate distributions, and to the space L^2 of square-integrable functions.

Fourier Transform of Measures. The Fourier transform can be extended to any finite complex Borel measure μ on $\mathbf{R}^N$, simply by replacing $u(x)\, dx$ with $d\mu(x)$ in (1.1):

$$\widehat{\mu}(\xi) := \int_{\mathbf{R}^N} e^{-2\pi i \xi \cdot x}\, d\mu(x) \qquad \forall \xi \in \mathbf{R}^N, \forall \text{ Borel measure } \mu \text{ on } \mathbf{R}^N. \tag{2.1}$$

(2) In *signal theory,* the transformation $u \mapsto \widehat{u}$ is referred to as the Fourier analysis, and its inverse as the Fourier synthesis; $\widehat{u}$ is usually named the *spectrum* of the signal u.

This is also called the *Fourier-Stieltjes transform.* These transformed functions are elements of C_b^0, just as for the functions of L^1. For instance, $\widehat{\delta}_a = e^{-2\pi i\xi\cdot a}$ for any $a \in \mathbf{R}^N$; in particular, $\widehat{\delta}_0 = 1$. However $\widehat{\delta}_a(\xi)$ does not vanish as $|\xi| \to +\infty$, at variance with Riemann-Lebesgue's Proposition 1.6.

Fourier Transform in $\mathcal{S}$. By Theorems 1.8 and 1.10, $u, \widehat{u} \in \mathcal{D}$ only if u is analytic, whence $u \equiv 0$. Thus $\mathcal{D}$ is not stable by Fourier transform. Loosely speaking, this means that the set of frequencies of the harmonic components of any non-identically vanishing $u \in \mathcal{D}$ is unbounded. This led L. Schwartz to introduce the space of rapidly decreasing functions, $\mathcal{S}$, then to restrict the Fourier transform to this space, and finally to extend this transform to the dual space of temperate distributions. We shall review the basic elements of that theory.

First we see that several of the formulas we saw in Sect. 1 also hold in $\mathcal{S}$ and $\mathcal{S}'$ without any restriction.

Proposition 2.1 *(The restriction of) $\mathcal{F}$ operates in $\mathcal{S}$ and is continuous (w.r.t. the topology of this space). Moreover, for any $u, v \in \mathcal{S}$,*

$$(2\pi i)^{|\alpha|}\xi^\alpha \widehat{u} = (D_x^\alpha u)\widehat{\ }, \tag{2.2}$$

$$D_\xi^\alpha \widehat{u} = (-2\pi i)^{|\alpha|}(x^\alpha u)\widehat{\ }, \tag{2.3}$$

$$\int_{\mathbf{R}^N} \widehat{u}\, v\, dx = \int_{\mathbf{R}^N} u\, \widehat{v}\, dx, \tag{2.4}$$

$$(uv)\widehat{\ } = \widehat{u} * \widehat{v}, \tag{2.5}$$

$$(u*v)\widehat{\ } = \widehat{u}\, \widehat{v}, \tag{2.6}$$

$$u(x) = \int_{\mathbf{R}^N} e^{2\pi i\xi\cdot x}\widehat{u}(\xi)d\xi \ \ (=: \tilde{\mathcal{F}}(\widehat{u})) \qquad \forall x \in \mathbf{R}^N. \quad [Ex] \tag{2.7}$$

Fourier Transform in $\mathcal{S}'$. We shall use the notations $\mathcal{F}$ and $\widehat{\ }$ also for several restrictions and extensions of the Fourier transform that we shall define. Next we extend the Fourier operator $\mathcal{F}$ to $\mathcal{S}'$ by transposition: we define the operator $\bar{\mathcal{F}} : \mathcal{S}' \to \mathcal{S}'$ by setting

$$\langle \bar{\mathcal{F}}(T), v\rangle := \langle T, \mathcal{F}(v)\rangle \qquad \forall v \in \mathcal{S}, \forall T \in \mathcal{S}'. \tag{2.8}$$

As $\mathcal{S}$ is dense in L^1, this is the unique continuous extension of the Fourier transform from L^1 to $\mathcal{S}'$. Henceforth we shall identify $\bar{\mathcal{F}}$ with $\mathcal{F}$.

Proposition 2.2 *The formulae (2.2)–(2.6) hold also in $\mathcal{S}'$. [Ex]*

The differentiation rules of distributions hold also in $\mathcal{S}'$, as this is a subspace of $\mathcal{D}'$. For any $T \in \mathcal{S}'$ and any $v \in \mathcal{D}$,

$$\begin{aligned} {}_{\mathcal{S}'}\langle (2\pi i)^{|\alpha|}\xi^\alpha \widehat{T}, v\rangle_{\mathcal{S}} &= {}_{\mathcal{S}'}\langle \widehat{T}, (2\pi i)^{|\alpha|}\xi^\alpha v\rangle_{\mathcal{S}} = {}_{\mathcal{S}'}\langle T, [(2\pi i)^{|\alpha|}\xi^\alpha v]\widehat{\ }\rangle_{\mathcal{S}} \\ &= {}_{\mathcal{S}'}\langle T, (-D)^\alpha \widehat{v}\rangle_{\mathcal{S}} = {}_{\mathcal{S}'}\langle D^\alpha T, \widehat{v}\rangle_{\mathcal{S}} = {}_{\mathcal{S}'}\langle (D^\alpha T)\widehat{\ }, v\rangle_{\mathcal{S}}. \end{aligned}$$

Hence

$$(2\pi i)^{|\alpha|}\xi^\alpha \widehat{T} = (D_x^\alpha T)\widehat{\ } \in \mathcal{S}' \qquad \forall T \in \mathcal{S}', \forall \alpha \in \mathbf{N}^N. \tag{2.9}$$

Lemma 2.3 *For any $T \in \mathcal{E}'$ and any $\xi \in \mathbf{R}^N$, $\widehat{T}(\xi) = {}_{\mathcal{E}'}\langle T, e^{-2\pi i x\cdot\xi}\rangle_{\mathcal{E}}$. This expression can be extended to any $\xi \in \mathbf{C}^N$, and is an analytic function. [This extends the Fourier-Laplace transform of Sect. 1.]*

Proof. For any $\varepsilon > 0$, let us define the mollifier ρ_ε as above, and set $(T * \rho_\varepsilon)(x) := \langle T_y, \rho_\varepsilon(x - y)\rangle$ for any $x \in \mathbf{R}^N$. (The index y indicates that T acts on the variable y; here x is just a parameter.)

As $\varepsilon \to 0$, $T * \rho_\varepsilon \to T$ weakly in $\mathcal{E}'$, hence also weakly in $\mathcal{S}'$, as $\mathcal{S}$ is dense in $\mathcal{S}'$. Therefore

$$\widehat{(T * \rho_\varepsilon)} \to \widehat{T} \qquad \text{in } \mathcal{S}'. \tag{2.10}$$

On the other hand, as $T * \rho_\varepsilon \in \mathcal{E}$ and $\int_{\mathbf{R}^N} \rho_\varepsilon(x)\,dx = 1$, we have

$$\begin{aligned}\widehat{(T * \rho_\varepsilon)}(\xi) &= \int_{\mathbf{R}^N \times \mathbf{R}^N} e^{-2\pi i \xi\cdot x} \langle T_y, \rho_\varepsilon(x-y)\rangle\,dxdy \\ &= \langle T_y, e^{-2\pi i\xi\cdot y} \int_{\mathbf{R}^N} e^{-2\pi i\xi\cdot(x-y)} \rho_\varepsilon(x-y)\,dx\rangle = \langle T_y, e^{-2\pi i\xi\cdot y}\rangle \widehat{\rho}_\varepsilon(\xi),\end{aligned}$$

and this is an analytic function of ξ. As $\varepsilon \to 0$, $\widehat{\rho}_\varepsilon(\xi) \to 1$ uniformly on any compact subset of $\mathbf{R}^N$. Therefore

$$\widehat{(T * \rho_\varepsilon)}(\xi) = \langle T_y, e^{-2\pi i\xi\cdot y}\rangle \widehat{\rho}_\varepsilon(\xi) \to \langle T_y, e^{-2\pi i\xi\cdot y}\rangle \qquad \text{in } \mathcal{S}'.$$

By (2.10) we then conclude that $\widehat{T}(\xi) = \langle T_y, e^{-2\pi i\xi\cdot y}\rangle$ for any $\xi \in \mathbf{R}^N$, and this function is analytic. □

Theorem 2.4 *(Paley-Wiener-Schwartz) Any temperate distribution T has bounded support (i.e., $T \in \mathcal{E}'$) iff $\mathcal{F}(T)$ can be extended to an analitic function $\mathbf{C}^N \to \mathbf{C}$ [which is also denoted by $\mathcal{F}(T)$].*
Moreover, $\operatorname{supp} T \subset \overline{B(0,R)}$*) iff*

$$\exists m \in \mathbf{N}_0, \exists C \geq 0 : \forall z \in \mathbf{C}^N, \quad |[\mathcal{F}(T)](z)| \leq C(1+|z|)^m e^{2\pi R|Im(z)|}. \qquad [] \tag{2.11}$$

Fourier Transform in $\mathcal{D}'$. The Fourier transform can also be extended to the whole $\mathcal{D}'$ by transposition, but in this case the transform maps $\mathcal{D}'$ to a proper subset of $\mathcal{D}'$ itself. Since $\mathcal{F}$ does not map $\mathcal{D}$ to $\mathcal{D}$, we define the space $\mathcal{Z} := \{v \in \mathcal{S} : \mathcal{F}(v) \in \mathcal{D}\}$, and equip it with the projective topology induced by $\mathcal{F}$, namely, the coarsest among the topologies such that (the restriction of) $\mathcal{F} : \mathcal{Z} \to \mathcal{S}$ is continuous.

$\mathcal{Z}$ is a proper and dense subspace of $\mathcal{S}$; hence $\mathcal{Z} \subset \mathcal{S} \subset \mathcal{S}' \subset \mathcal{Z}'$, with continuous injections. As we noticed, $\mathcal{Z} \cap \mathcal{D}$ is reduced to the null function. We can now define $\widehat{\mathcal{F}} : \mathcal{D}' \to \mathcal{Z}'$ as follows: [3]

$${}_{\mathcal{Z}'}\langle \widehat{\mathcal{F}}(T), v\rangle_{\mathcal{Z}} := {}_{\mathcal{D}'}\langle T, \mathcal{F}(v)\rangle_{\mathcal{D}} \qquad \forall v \in \mathcal{Z}, \forall T \in \mathcal{D}'. \tag{2.12}$$

This mapping is one-to-one, and extends $\mathcal{F}$.

Fourier Transform in L^2. As $L^2 \subset \mathcal{S}'$, any function of L^2 has a Fourier transform that belongs to $\mathcal{S}'$. The next statement is more precise.

• **Theorem 2.5** *(Plancherel) For any $u \in \mathcal{S}'$, we have $u \in L^2$ iff $\widehat{u} \in L^2$. The (restriction of the) Fourier transform is an isometry in this space, that is*

$$\|\widehat{u}\|_{L^2} = \|u\|_{L^2} \qquad \forall u \in L^2. \tag{2.13}$$

Moreover,

$$U_R(\xi) := \int_{]-R,R[^N} e^{-2\pi i\xi\cdot x} u(x)\,dx \to \widehat{u}(\xi) \qquad \textit{in } L^2, \textit{ as } R \to +\infty, \forall u \in L^2. \tag{2.14}$$

The functions U_R thus converge also in measure on any bounded subset of $\mathbf{R}^N$, but not necessarily a.e. in $\mathbf{R}^N$, a priori. Nevertheless, as R diverges along a suitable sequence (which may depend on u), these functions converge a.e. in $\mathbf{R}^N$.

(3) $\mathcal{F}(T)$ is sometimes called an *ultradistribution,* but (helas) the same term is also used for other notions in the literature.

Proof. For any $u \in \mathcal{S}$, we know that $\widehat{u} \in \mathcal{S}$. Moreover, by (1.14) and (1.17) we have

$$\int_{\mathbf{R}^N} |\widehat{u}|^2\,dx = \int_{\mathbf{R}^N} \widehat{u}\bar{\widehat{u}}\,dx = \int_{\mathbf{R}^N} u\widehat{\bar{\widehat{u}}}\,dx = \int_{\mathbf{R}^N} u(x)\overline{\widehat{\widehat{u}}(-x)}\,dx = \int_{\mathbf{R}^N} u\bar{u}\,dx = \int_{\mathbf{R}^N} |u|^2\,dx.$$

Therefore, as $\mathcal{S} \subset L^2$ with density, the restriction of $\mathcal{F}$ to L^2 is an isometry with respect to the L^2-metric. Hence $\mathcal{F}$ maps L^2 to itself.

In order to prove (2.14), for any $R > 0$ and any $x \in \mathbf{R}$, let us set $\chi_R(x) := 1$ if $|x_i| \leq R$ for $i = 1, ..., N$, and $\chi_R(x) := 0$ otherwise. Then $u\chi_R \in L^1 \cap L^2$ and $u\chi_R \to u$ in L^2. Hence, by (2.13),

$$\int_{]-R,R[^N} e^{-2\pi i\xi\cdot x}u(x)\,dx = \int_{\mathbf{R}^N} e^{-2\pi i\xi\cdot x}u(x)\chi_R(x)\,dx = \widehat{(u\chi_R)}(\xi) \to \widehat{u}(\xi) \qquad \text{in } L^2. \qquad \square$$

Remarks. (i) In any Hilbert space the scalar product is determined by the norm, as $2(u, v) = \|u + v\|^2 - \|u\|^2 - \|v\|^2$. (2.13) then entails that

$$\int_{\mathbf{R}^N} u(x)\bar{v}(x)\,dx = \int_{\mathbf{R}^N} \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,d\xi \qquad \forall u, v \in L^2. \tag{2.15}$$

(ii) The above argument allows one to extend the inversion Theorem 1.8 to L^2.

(iii) The representation (2.14) is more general:

$$\int_{]-R,R[^N} e^{-2\pi i\xi\cdot x}u(x)\,dx \to \widehat{u}(\xi) \qquad \text{in } \mathcal{S}', \text{ as } R \to +\infty, \forall u \in \mathcal{S}' \cap L^1_{\text{loc}}. \tag{2.16}$$ $\square$

The Lebesgue-integral representation (1.1) is meaningful only if $u \in L^1$. Anyway it may be useful to know of cases in which the (extended) Fourier transform maps functions to functions. For any $p \in [1, 2]$, any function of $u \in L^p$ can be written as the sum of a function of L^1 and one of L^2, i.e., $L^p \subset L^1 + L^2$. Indeed, setting $\chi := 1$ where $|u| \geq 1$ and $\chi := 0$ elsewhere, we have $u\chi \in L^1$, $u(1-\chi) \in L^2$ and $u = u\chi + u(1-\chi)$. Hence $\mathcal{F}(u) = \mathcal{F}(u\chi) + \mathcal{F}(u(1-\chi)) \in L^\infty + L^2$; in particular $\mathcal{F}(u)$ is an a.e.-defined function, although it may admit no integral representation.

This is made more precise by the next result, which is a direct consequence of the classical Riesz-Thorin Theorem on *interpolation* of L^p-spaces.

Theorem 2.6 *(Hausdorff-Young) Let $p \in [1, 2]$ and $p' := p/(p-1)$ if $p > 1$, $p' = \infty$ if $p = 1$. Then (the restriction of) $\mathcal{F}$ is a linear and continuous operator $L^p \to L^{p'}$. More precisely, for any $u \in L^p$, $\widehat{u} \in L^{p'}$ and $\|\widehat{u}\|_{L^{p'}} \leq \|u\|_{L^p}$.*

In this statement we regard $\mathcal{F}$ as a restriction of the operator defined in $\mathcal{S}'$. The results established for $\mathcal{F}$ in $\mathcal{S}$ and in $\mathcal{S}'$, in particular Theorems 2.1 and 2.5, hold also for the inverse Fourier transform $\mathcal{F}^{-1}$. This is clear, because of the analytic expression of the latter for functions of $\mathcal{S}$, see (1.16).

Fourier Transform vs. Fourier Series. Here we take $N = 1$, although the discussion might be extended to any N. For any $T > 0$, we say that

a function $f : \mathbf{R} \to \mathbf{C}$ is T-periodic iff $f(t + T) = f(t)$ for any $t \in \mathbf{R}$,

a distribution $u \in \mathcal{D}'(\mathbf{R})$ is T-periodic iff $\langle u, \varphi(\cdot + T)\rangle = \langle u, \varphi\rangle$ for any $\varphi \in \mathcal{D}(\mathbf{R})$.

We shall denote the T-periodic distributions (temperate distributions, resp.) by $\mathcal{D}'_T$ ($\mathcal{S}'_T$, resp.). At variance with the nonperiodic setting, the following result holds.

Proposition 2.7 *Any periodic distribution is temperate, i.e., $\mathcal{D}'_T = \mathcal{S}'_T$. For any $T > 0$ and any sequence $\{\varepsilon_n\}$ of $\mathcal{D}'_T$,*

$$\varepsilon_n \to v \text{ in } \mathcal{D}' \quad \Leftrightarrow \quad \varepsilon_n \to v \text{ in } \mathcal{S}'. \qquad [] \tag{2.17}$$

The relation between the Fourier transform and the Fourier series is illustrated by the next result, which, loosely speaking, states that the harmonic components of a periodic temperate distribution only have integer frequencies. Here by δ_a we denote the Dirac measure concentrated at a point $a \in \mathbf{R}$, and say that a series $\sum_{k\in\mathbf{Z}} c_k$ converges iff $\lim_{n\to\infty} \sum_{k=-n}^{n} c_k$ exists (this definition is reminiscent of the principal value, cf. Sect. VIII.2).

Theorem 2.8 *(Fourier Series in $\mathcal{D}'$) Let $u \in \mathcal{S}'$ and $T > 0$. The next three statements are then equivalent:*

$$u \in \mathcal{D}'_T, \tag{2.18}$$

$$\exists \{a_k\}_{k\in\mathbf{Z}} \subset \mathbf{C} : u(x) = \textstyle\sum_{k\in\mathbf{Z}} a_k e^{2k\pi i x/T} \qquad \text{in } \mathcal{D}', \tag{2.19}$$

$$\exists \{a_k\}_{k\in\mathbf{Z}} \subset \mathbf{C} : \widehat{u}(\xi) = \textstyle\sum_{k\in\mathbf{Z}} a_k \delta_{k/T}(\xi) \qquad \text{in } \mathcal{D}'. \tag{2.20}$$

The sequence $\{a_k\}$ is uniquely determined by u. Moreover, if $u \in L^1_{\rm loc}$, then

$$a_k = \frac{1}{T}\int_0^T e^{-2k\pi i x/T} u(x)\, dx \qquad \forall k \in \mathbf{Z}. \qquad [] \tag{2.21}$$

If either (2.19) or (2.20) hold, then by Proposition 2.7 both series also converge weakly star in $\mathcal{S}'$. This constrains the growth at $\pm\infty$ of the Fourier coefficients, a_k:

$$\exists C > 0, \exists M, m > 0 : \forall k \in \mathbf{Z}, \qquad |a_k| \le M(1 + |k|^m).$$

The series in (2.19) and the sequence of the a_ks are respectively named *Fourier series* and *Fourier coefficients* of u.

Theorem 2.9 *(Fourier Series in L^2) Let u be any T-periodic distribution. Then $u|_{]0,T[} \in L^2(0,T)$ iff, defining the a_ks as in (2.19), $\{a_k\} \in \ell^2$. Moreover, if the latter property holds, then*

$$\|u\|^2_{L^2(0,T)} = \textstyle\sum_{k\in\mathbf{Z}} |a_k|^2. \qquad [] \tag{2.22}$$

Overview of the Extensions of the Fourier Transform. The Fourier transform (1.1) has a natural extension to any complex Borel measure μ: formally, it suffices to replace $u(x)dx$ by $d\mu$. As (the restriction of) $\mathcal{F}$ maps the Schwartz space $\mathcal{S}$ to itself, we could extend $\mathcal{F}$ to $\mathcal{S}'$ by transposition. We also saw that $\mathcal{F}$ is an isometry in L^2, that in this space it admits an integral representation as a principal value, and that $\mathcal{F}$ is also linear and continuous from L^p to $L^{p/(p-1)}$, for any $p \in]1,2[$.

Finally, we pointed out that the Fourier series arise as Fourier transforms of periodic functions.

Exercises. (i) Show that $\mathcal{F}$ does not map L^p to L^q for any $q \ne p'$.

Hint: This may be proved by a dimensionality argument. Let $u \in L^p$ be such that $\mathcal{F}(u) \in L^q$. For any $\lambda > 0$, setting $u_\lambda(x) := u(\lambda x)$ for any x, (1.4) yields $\mathcal{F}(u_\lambda) = \lambda^{-N}\mathcal{F}(u)_{1/\lambda}$. Check that

$$\frac{\|\mathcal{F}(u_\lambda)\|_{L^q}}{\|u_\lambda\|_{L^p}} = \lambda^{N(-1+1/q+1/p)} \frac{\|\mathcal{F}(u)\|_{L^q}}{\|u\|_{L^p}}.$$

(ii) Show that the only possible eigenvalues of $\mathcal{F}$ in $\mathcal{S}'$ are $\pm i, \pm 1$.

Hint: Remind that the function $x \mapsto \exp(-\pi x^2)$ is an eigenfunction.

(iii) Evaluate $\mathcal{F}(\delta)$ and $\mathcal{F}(1)$.

Hint: For the latter, note that $\int \mathcal{F}^{-1}(v)\, dx = [\mathcal{F}\mathcal{F}^{-1}(v)](0) = v(0)$.

(iv) Check that the operators $\bar{\mathcal{F}} : u \mapsto \bar{\widehat{u}}$ and $\mathcal{F}^2$ are idempotent in $\mathcal{S}$; i.e. $\bar{\mathcal{F}}^2$ and $\mathcal{F}^4$ coincide with the identity.

(v) Show that $\widehat{(uv)} = \widehat{u} * \widehat{v}$ for any $u, v \in \mathcal{S}'$.

(vi) Set $T := \sum_{n\in\mathbf{Z}} \delta_n$, and check that $T \in \mathcal{S}'$ and $\widehat{T} = T$ in $\mathcal{S}'$.

3. The Laplace Transform: ... Omissis ...

4. Convolution: ... Omissis ...

5. Fourier Transform and P.D.E.s with Constant Coefficients

In this section we briefly illustrate the use of the Fourier transform in the analysis of P.D.E.s with constant coefficients set on the whole $\mathbf{R}^N$.

Any polynomial $P(\eta)$ of N complex variables of degree m is canonically associated to a linear differential operator $P(D)$ ($D := (\partial/\partial_1, ..., \partial/\partial_N)$) with constant (complex) coefficients of order m, and conversely:

$$P(\eta) := \textstyle\sum_{|\alpha|\leq m} c_\alpha \eta^\alpha \quad (\text{with } c_\alpha \in \mathbf{C},\ \forall\alpha) \quad \leftrightarrow \quad P(D) := \textstyle\sum_{|\alpha|\leq m} c_\alpha D^\alpha. \tag{5.1}$$

This establishes an isomorphism between the linear space of polynomials over $\mathbf{R}^N$ with complex coefficients and that of linear differential operators with constant complex coefficients. The Fourier transform exploits this isomorphism as follows. For any $u = u(x) \in \mathcal{S}$, by Proposition 2.1 we have [4]

$$\mathcal{F}[P(D_x)u] = P(2\pi i\xi)\mathcal{F}u, \qquad \mathcal{F}[P(-2\pi ix)u] = P(D_\xi)\mathcal{F}u \qquad \text{in } \mathbf{R}^N,$$

and the corresponding formulae for the inverse transform, for any function $v = v(\xi) \in \mathcal{S}$:

$$\mathcal{F}^{-1}[P(D_\xi)v] = P(-2\pi ix)\mathcal{F}^{-1}v, \qquad \mathcal{F}^{-1}[P(2\pi i\xi)v] = P(D_x)\mathcal{F}^{-1}v \qquad \text{in } \mathbf{R}^N.$$

The polynomial $P(2\pi i\xi)$ is called the *symbol* of $P(D)$, [5] and is an element of $\mathcal{S}'$. Hence

$$P(D)u = \mathcal{F}^{-1}[P(2\pi i\xi)\mathcal{F}u] \overset{(2.5)}{=} \big(\mathcal{F}^{-1}[P(2\pi i\xi)]\big)*u \in \mathcal{S} \qquad \forall u \in \mathcal{S}. \tag{5.2}$$

An analogous result holds for any $u \in \mathcal{E}'$ (i.e., any compactly supported distribution), by transposition, as $\mathcal{S}' * \mathcal{E}' \subset \mathcal{S}'$. In particular,

$$\mathcal{F}^{-1}[P(2\pi i\xi)] = \big(\mathcal{F}^{-1}[P(2\pi i\xi)]\big)*\delta = P(D)\delta\ (\in \mathcal{S}'). \tag{5.2'}$$

These properties may be applied to the study of P.D.E.s. This method is necessarily restricted to linear equations with constant coefficients that are set on the whole $\mathbf{R}^N$. Let us fix a non-identically-vanishing differential operator $P(D)$, a function $f \in \mathcal{S}$, and consider the equation

$$u \in \mathcal{S}, \qquad P(D)u = f \qquad \text{in } \mathcal{S}. \tag{5.3}$$

Let us assume that $u \in \mathcal{S}$ solves this equation. By applying the Fourier transform to both members of this differential equation, we get

$$\widehat{u} \in \mathcal{S}, \qquad P(2\pi i\xi)\widehat{u}(\xi) = \widehat{f}(\xi) \qquad \text{in } \mathcal{S}. \tag{5.4}$$

If $P(2\pi i\xi) \neq 0$ for any $\xi \in \mathbf{R}^N$, then $\widehat{u}(\xi) = P(2\pi i\xi)^{-1}\widehat{f}(\xi)$ $(\in \mathcal{S})$. [Ex] In this case the differential equation (5.3) has a unique solution (in $\mathcal{S}$): [6]

$$u = \mathcal{F}^{-1}[P(2\pi i\xi)^{-1}\widehat{f}(\xi)] = (\mathcal{F}^{-1}[P(2\pi i\xi)^{-1}])*f \in \mathcal{S}. \tag{5.5}$$

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[4] In passing notice that these polynomials are naturally restricted to imaginary variables, but their values are not restricted to either real or imaginary numbers.

[5] The occurrence of the factor 2π depends on the definition we gave of the Fourier transform.

[6] In alternative and for any P, by (5.2) we have $(\mathcal{F}^{-1}[P(2\pi i\xi)])*u = P(D)u = f$. But in general to invert a convolution is not an easy task.

(A priori this convolution makes sense only in $\mathcal{S}'$.)

Note that, whenever $P(2\pi i\xi) \neq 0$ for any $\xi \in \mathbf{R}^N$, $P(2\pi i\xi)^{-1} \in \mathcal{E} \cap \mathcal{S}'$ [Ex] Moreover, as $P(2\pi i\xi)^{-1}\mathcal{S} \subset \mathcal{S}$, by transposition we get

$$P(2\pi i\xi)^{-1}\widehat{f}(\xi) \in \mathcal{S}' \qquad \forall f \in \mathcal{S}'. \tag{5.5'}$$

The first equality of (5.5) then holds also if $u, f \in \mathcal{S}'$; however, the convolution makes sense only if $f \in \mathcal{E}'$. We have thus shown the next statement.

Proposition 5.1 *Let $P(D)$ $(\not\equiv 0)$ be a linear differential operator with constant coefficients. If $P(2\pi i\xi) \neq 0$ for any $\xi \in \mathbf{R}^N$, then*

$$P(D)^{-1} = (\mathcal{F}^{-1}[P(2\pi i\xi)^{-1}]\mathcal{F}) = (\mathcal{F}^{-1}[P(2\pi i\xi)^{-1}])*: \ \mathcal{S} \to \mathcal{S} \ \text{ and } \ \mathcal{E}' \to \mathcal{S}'. \tag{5.6}$$

For instance, [7]

$$P(\eta) := 1 - \sum_{j=1}^{N} \eta_j^2 \quad \leftrightarrow \quad P(D) = I - \sum_{j=1}^{N} D_j^2 =: I - \Delta \qquad \text{on } \mathcal{S} \text{ or } \mathcal{S}'. \tag{5.7}$$

As $P(2\pi i\xi) = 1 + 4\pi^2|\xi|^2 > 0$ for any $\xi \in \mathbf{R}^N$, we get

$$\begin{aligned} &\forall f \in \mathcal{S} \ \ (f \in \mathcal{E}', \text{ resp.}),\\ &u = \mathcal{F}^{-1}[(1 + 4\pi^2|\xi|^2)^{-1}\widehat{f}] = \mathcal{F}^{-1}[(1 + 4\pi^2|\xi|^2)^{-1}]*f\\ &\text{is the unique solution in } \mathcal{S} \text{ (in } \mathcal{S}', \text{ resp.) of the equation } u - \Delta u = f. \end{aligned} \tag{5.8}$$

A similar conclusion does not hold for $P(D) = -\Delta$, as $P(2\pi i\xi) = 0$ for $\xi = 0$.

Remark. So far we used the Fourier transform, and thus confined ourselves either to $\mathcal{S}$ or to $\mathcal{S}'$. The next formula shows that any differential operator may be represented as the convolution with a compactly supported distribution:

$$P(D)u = P(D)(\delta * u) = [P(D)\delta] * u \qquad \forall u \in \mathcal{D}'. \tag{5.8'}$$

Pseudo-Differential Operators. (5.2) also reads as an identity between operators:

$$P(D) = \mathcal{F}^{-1}[P(2\pi i\xi)\mathcal{F}] = (\mathcal{F}^{-1}[P(2\pi i\xi)]) * \qquad \text{either in } \ \mathcal{S} \to \mathcal{S} \ \text{ or } \ \mathcal{E}' \to \mathcal{S}'. \tag{5.9}$$

More generally, in this way one may define the *pseudo-differential operator* $P(D)$ in $\mathcal{S}$ (and then in $\mathcal{S}'$ by transposition) also for nonpolynomial functions $P \in \mathcal{S}'$, provided that $P(2\pi i\xi)\mathcal{S} \subset \mathcal{S}$. As (1.15) also applies to $\mathcal{S}'$, one may define $P(D)$ by means of (5.9); if $1/P \in \mathcal{S}'$, (5.6) then follows also in this case. For instance, one may define any positive power of the negative Laplace operator $-\Delta := -\sum_{j=1}^{N} D_j^2$:

$$(-\Delta)^{s/2}u := \mathcal{F}^{-1}\left[|2\pi\xi|^s\mathcal{F}(u)\right] = (2\pi)^s\mathcal{F}^{-1}(|\xi|^s)*u \in \mathcal{S} \qquad \forall u \in \mathcal{S}, \forall s > 0. \tag{5.10}$$

This definition may be extended to any $s \in \,]-N, 0[$, since in this case the function $\xi \mapsto |\xi|^s$ is an element of $L^1_{\rm loc} \subset \mathcal{S}'$. As

$$\forall s \in \,]-N, 0[, \exists c_s > 0 : \mathcal{F}^{-1}(|\xi|^s) = c_s|x|^{-(s+N)} \ (\in L^1_{loc} \subset \mathcal{S}'), \qquad [] \tag{5.11}$$

[7] In passing notice that $I - \Delta : v \mapsto v - \Delta v$ is a linear operator, but $1 - \Delta : v \mapsto 1 - \Delta v$ is not so.

we have [8]

$$(-\Delta)^{s/2}u := (2\pi)^s\mathcal{F}^{-1}(|\xi|^s)*u = (2\pi)^s c_s|x|^{-(s+N)}*u \in \mathcal{S} \qquad \forall u \in \mathcal{S}, \forall s \in\,]-N,0[; \tag{5.12}$$

by transposition, the same applies for any $u \in \mathcal{S}'$. In conclusion,

$$\forall s > -N,\ (-\Delta)^{s/2} : \mathcal{S} \to \mathcal{S} \text{ and } \mathcal{E}' \to \mathcal{S}' \text{ (linear and continuous in both cases).} \tag{5.13}$$

A similar statement applies to any power of the operator $I - \Delta$, as $(1 + |2\pi\xi|)^{s/2} \in \mathcal{S}'$ whence $\mathcal{F}^{-1}[(1 + |2\pi\xi|)^{s/2}] \in \mathcal{S}'$ for any $s \in \mathbf{R}$. [Ex] The order of a pseudo-differential operator $P(D)$ is defined in terms of the asymptotic behavior of the function $P(2\pi i\xi)$ as $|\xi| \to +\infty$. [9] For instance $(I - \Delta)^{s/2}$ has order s, for any $s \in \mathbf{R}$.

Fundamental Solutions. An operator $P(D)$ ($\not\equiv 0$) is said *elliptic* iff its principal part (i.e., the sum of the terms of leading order) is associated to a polynomial that vanishes only at the origin of $\mathbf{C}^N$. This class includes e.g. $(-\Delta)^m + \tilde{P}(D)$ for any $m \in \mathbf{N}$ and any polynomial $\tilde{P}$ of degree lower than m. All elliptic operators are of even order. [Ex]

Theorem 5.2 *(Ehrenpreis-Malgrange-Hörmander ....) Let $P(D)$ ($\not\equiv 0$) be a linear differential operator with constant coefficients, and Ω be a convex domain of $\mathbf{R}^N$. Then:*

$$P(D)\mathcal{E}(\Omega) = \mathcal{E}(\Omega), \tag{5.14}$$

$$P(D)\mathcal{D}'_F(\Omega) = \mathcal{D}'_F(\Omega), \tag{5.15}$$

$$P(D)\mathcal{D}'(\Omega) = \mathcal{D}'(\Omega) \qquad \textit{(Ehrenpreis-Malgrange)}, \tag{5.16}$$

$$P(D)\mathcal{S}' = \mathcal{S}' \qquad \textit{(Hörmander)}. \tag{5.17}$$

If $P(D)$ is elliptic, then (5.14) and (5.15) hold also for nonconvex domains. []

For any of these four equalities the inclusion "$\subset$" is trivial (and holds for any domain Ω); the element of interest stays in the inclusion "$\supset$", which corresponds to the existence of a solution of the P.D.E. (5.3). A famous example due to Levy shows that this theorem may fail if the coefficients of the operator are nonconstant, even if they are assumed to be C^∞ functions of x.

Remarks. (i) The equalities (5.14) – (5.17) are of the form $P(D)X = X$, for various selections of the space X. This entails that $X \subset P(D)^{-1}X$, but not $X = P(D)^{-1}X$. For instance:
— in general $P(D)^{-1}\mathcal{E} \not\subset \mathcal{E}$. E.g., if $P(D) = D_t^2 - D_x^2$, the wave equation $P(D)u = 0$ has solutions $u \notin \mathcal{E}$ (u may even be discontinuous!).
— in general $P(D)^{-1}\mathcal{S}' \not\subset \mathcal{S}'$. Actually, the equation $P(D)u = f \in \mathcal{S}'$ might have not only a solution in $\mathcal{S}'$ (by (5.17)), but also one or more in $\mathcal{D}' \setminus \mathcal{S}'$; an example is provided ahead.
(ii) It is easily seen that

$$P(D)\mathcal{E}'(\Omega) \subset \mathcal{E}'(\Omega) \qquad \forall \text{ domain } \Omega \subset \mathbf{R}^N. \tag{5.18}$$

But the opposite inclusion fails: for instance, for $N = 1$, $\Omega = \mathbf{R}$, $P(D) = D$ and $f = \chi_{]0,1[}$ (the characteristic function of the interval $]0,1[$), the solutions of the equation (5.3) are of the form

[8] This holds also for $s \le -N$, provided that the function $g(x) = |x|^{-(s+N)}$ ($\notin L^1_{\rm loc}$) is replaced by a *regularization*, namely by a distribution T whose restriction to $\mathbf{R}^N \setminus \{0\}$ coincides with g.

[9] In this connection we have the following result.
Let $P : \mathbf{R}^N \to \mathbf{C}$ be a smooth function such that (denoting by $[n/2]$ the integer part of $n/2$),

$$\forall \alpha \text{ with } |\alpha| \le [n/2] + 1, \qquad \xi \mapsto |\xi|^{|\alpha|}|D^\alpha P(\xi)| \text{ is uniformly bounded in } \mathbf{R}^N.$$

The operator $P(D)$ is then linear and continuous in $L^p(\mathbf{R}^N)$.

$u = x^+ - (x-1)^+ + C$. Consistently with (5.15) and (5.17), these are elements of $\mathcal{D}'_F \cap \mathcal{S}'$, but not of $\mathcal{E}'$. [Ex] Similarly one may see that

$$P(D)\mathcal{D}(\Omega) \subset \mathcal{D}(\Omega), \quad P(D)\mathcal{S}(\Omega) \subset \mathcal{S}(\Omega) \qquad \forall \text{ domain } \Omega \subset \mathbf{R}^N, \tag{5.19}$$

but these inclusions are strict.

Let $P(\eta)$ be a polynomial of N complex variables of degree m, as in (5.1). Any $E \in \mathcal{D}'$ is called a *fundamental solution* of $P(D)$ iff $P(D)E = \delta$. If $E \in \mathcal{S}'$ we may apply the Fourier transform, getting $P(2\pi i\xi)\widehat{E} = 1$; thus $\widehat{E}$ is a regularization of the function $P(2\pi i\xi)^{-1}$.

Proposition 5.3 *No fundamental solution E has compact support: $E \notin \mathcal{E}'$.*

Proof. By the Paley-Wiener-Schwartz Theorem 2.4, if E had compact support then $\widehat{E}$ would be an entire function. By applying the Fourier transform to the equation $P(D)E = \delta$, we would then get $P(2\pi i\xi)\widehat{E} = 1$. But $1/P(2\pi i\xi)$ can be an entire function only if P is constant, because any other polynomial has zeroes. E would then be proportional to δ, and this is not consistent with the equation $P(D)E = \delta$. □

By the next statement we see that any nontrivial operator $P(D)$ has a fundamental solution in any open set Ω, and that this allows one to construct a solution of the equation (5.3) for any $f \in \mathcal{E}'$. The next two results respectively deal with the existence and the uniqueness of the solution of (5.3), in particular of the fundamental solution of the operator $P(D)$.

Proposition 5.4 *Let $P(D)$ ($\not\equiv 0$) be a linear differential operator with constant coefficients. Then:*
(i) there exists a fundamental solution $E \in \mathcal{D}'_F \cap \mathcal{S}'$;
(ii) $E* : \mathcal{E}' \to \mathcal{S}'$, and

$$P(D)(E * f) = f \qquad \forall f \in \mathcal{E}'; \tag{5.20}$$

(iii) let Ω be a bounded domain of $\mathbf{R}^N$. For any $f \in L^1(\Omega)$, setting $\tilde{f} := f$ in Ω and $\tilde{f} := 0$ outside Ω, we have $P(D)(E * \tilde{f})|_\Omega = f$ in $\mathcal{D}'(\Omega)$.

Proof. (i) As $\delta \in \mathcal{D}'_F \cap \mathcal{S}'$, this follows from (5.15) and (5.17).
(ii) One can show that $E * f \in \mathcal{S}'$ for any $f \in \mathcal{E}'$. [] (As the support of E is not compact, f is assumed to be compactly supported in order to give a meaning to $E * f$.) Moreover, by the differential properties of the convolution,

$$P(D)(E * f) = [P(D)E] * f = \delta * f = f.$$

(iii) As $\tilde{f} \in \mathcal{E}'$, by (5.20) $P(D)(E * \tilde{f}) = \tilde{f}$ in $\mathcal{D}'(\mathbf{R}^N)$. □

Let us set

$$\begin{aligned} &\varepsilon_\eta(x) := e^{2\pi i\eta\cdot x} \qquad \forall \eta \in \mathbf{C}^N, \forall x \in \mathbf{R}^N, \\ &\mathcal{Z}_P := \{\eta \in \mathbf{C}^N : P(2\pi i\eta) = 0\}, \\ &\Sigma_P := \text{ span of } \{\varepsilon_\eta : \eta \in \mathcal{Z}_P\} \ (\subset \mathcal{D}'). \end{aligned} \tag{5.21}$$

By the fundamental theorem of algebra $\mathcal{Z}_P \neq \emptyset$. As $|\varepsilon_\eta(x)| = e^{-2\pi Im(\eta)\cdot x}$, the distribution ε_η has exponential growth iff $\eta \notin \mathbf{R}^N$. Thus $\varepsilon_\eta \in \mathcal{S}'$ iff $\eta \in \mathbf{R}^N$ (this actually corresponds to $\varepsilon_\eta \in L^\infty$).

Proposition 5.5 *Let $P(D)$ ($\not\equiv 0$) be a linear differential operator with constant coefficients. Then:*
(i) Σ_P coincides with the kernel of $P(D)$ in $\mathcal{S}'$;
(ii) the set of all fundamental solutions of $P(D)$ in $\mathcal{S}'$ coincides with the affine space $E + \Sigma_P$, where E is any fundamental solution. [Ex]

Although the fundamental solution is not unique in $\mathcal{D}'$, next we see that it may be unique in $\mathcal{S}'$.

Proposition 5.6 *Let $P(D)$ ($\not\equiv 0$) be a linear differential operator with constant coefficients. Then:*
(i) the restriction of the operator $P(D)$ to $\mathcal{S}'$ is injective (in particular, there is at most one fundamental solution in $\mathcal{S}'$) iff $P(2\pi i\xi) \neq 0$ for any $\xi \in \mathbf{R}^N$;
(ii) if $P(2\pi i\xi) \neq 0$ for any $\xi \in \mathbf{R}^N$, then $E := \mathcal{F}^{-1}([P(2\pi i\xi)]^{-1})$ ($\in \mathcal{S}' \cap L^1_{\rm loc}$) is the unique fundamental solution of $P(D)$ in $\mathcal{S}'$.

Proof. (i) Note that for any $v \in \mathcal{S}'$

$$P(D)v = 0 \quad \Leftrightarrow \quad P(2\pi i\xi)\widehat{v}(\xi) = 0 \quad \forall \xi \in \mathbf{R}^N \quad \Leftrightarrow \quad \widehat{v} \text{ is supported in } \mathcal{Z}_P \cap \mathbf{R}^N.$$

Therefore $v = 0$ is the only solution $v \in \mathcal{S}'$ of the equation $P(D)v = 0$ (namely, $P(D)$ is injective in $\mathcal{S}'$) iff $\mathcal{Z}_P \cap \mathbf{R}^N = \emptyset$, i.e., iff $P(2\pi i\xi) \neq 0$ for any $\xi \in \mathbf{R}^N$.
(ii) This follows from part (i) and (5.5). □

Remarks. (i) If $P(2\pi i\xi) \neq 0$ for any $\xi \in \mathbf{R}^N$, we saw that $E = \mathcal{F}^{-1}([P(2\pi i\xi)]^{-1}) \in \mathcal{S}' \cap L^1_{\rm loc}$ and the restriction of the operator $P(D)$ to $\mathcal{S}'$ is injective. By (5.20), $E*$ may then be regarded as the inverse of the operator $P(D)$ in $\mathcal{E}'$.
(ii) If $P(2\pi i\xi) = 0$ for some $\xi \in \mathbf{R}^N$, then $P(D)$ has either no locally integrable fundamental solution or more than one. This depends upon the multiplicity of these roots.
For instance, for any $m \in \mathbf{N}$, the polynomial

$$P(2\pi i\xi) = \left(\sum_{j=1}^{m}(2\pi\xi_j)^2\right)^m = |2\pi\xi|^{2m}$$

is the symbol of the elliptic operator $P(D) = (-\sum_{j=1}^m D_j^2)^m = (-\Delta)^m$. As

$$P(2\pi i\xi) = |2\pi\xi|^{2m} = 0 \quad \Leftrightarrow \quad \xi = 0 \quad \text{(with algebraic multiplicity } 2m\text{)},$$

we have

$$[P(2\pi i\xi)]^{-1} \in \mathcal{S}' \cap L^1_{loc} \quad \Leftrightarrow \quad 2m < N.$$

Therefore:
— if $2m < N$ then $(-\Delta)^m$ has a fundamental solution $E \in \mathcal{S}' \cap L^1_{loc}$; for any polynomial $Q(x)$ of degree $< 2m$, $\tilde{E} = E + Q(x)$ is also a fundamental solution in $\mathcal{S}' \cap L^1_{loc}$;
— if $2m \geq N$ then $(-\Delta)^m$ has no fundamental solution in L^1_{loc}.
Thus, for instance, the operator $(-\Delta)^m$ has a locally integrable fundamental solution in $\mathbf{R}^3$ iff $m = 1$.

Fundamental Solutions of Cauchy Problems. Let us consider a Cauchy problem of the form

$$\begin{cases} D_t u + P(D_x)u = 0 & \forall t > 0 \\ u(\cdot, 0) = u^0, \end{cases} \tag{5.22}$$

for any $u^0 \in \mathcal{E}'(\mathbf{R}^N)$. Any mapping $E : \mathbf{R}^+ \to \mathcal{D}'(\mathbf{R}^N)$, such that the function $\mathbf{R}^+ \mapsto \langle E(\cdot, t), \varphi\rangle$ is smooth for any $\varphi \in \mathcal{D}(\mathbf{R}^N)$, is said a fundamental solution of the Cauchy problem associated to the operator $D_t + P(D_x)$ iff

$$\begin{cases} D_t E + P(D_x)E = 0 & \text{in } \mathcal{D}'(\mathbf{R}^N), \forall t > 0 \\ E(\cdot, 0) = \delta_{x=0} & \text{in } \mathcal{D}'(\mathbf{R}^N). \end{cases} \tag{5.23}$$

Actually, setting $\tilde{E}(\cdot, t) = E(\cdot, t)$ for any $t \geq 0$ and $\tilde{E}(\cdot, t) = 0$ for any $t < 0$, (5.23) is equivalent to

$$D_t\tilde{E} + P(D_x)\tilde{E} = \delta_{x=0} \otimes \delta_{t=0} = \delta_{(x,t)=(0,0)} \qquad \text{in } \mathcal{D}'(\mathbf{R}^{N+1}), \tag{5.24}$$

which means that $\tilde{E}$ is a fundamental solution of the operator $D_t + P(D_x)$ in $\mathbf{R}^{N+1}$ (with support confined to $\mathbf{R}^N \times \mathbf{R}^+$).

Defining the extension $\tilde{u}$ similarly to $\tilde{E}$, the Cauchy problem (5.22) also reads

$$D_t\tilde{u} + P(D_x)\tilde{u} = u^0 \otimes \delta_{t=0}; \tag{5.25}$$

(5.24) then entails that $u(x,t) = \tilde{E} * (u^0 \otimes \delta_{t=0}) = \langle E(x-y,t), u^0(y)\rangle$ solves (5.22). This definition of fundamental solution of the Cauchy problem may also be extended to equations of higher order in time.

Examples of Fundamental Solutions. The construction of a fundamental solution may not be trivial. Here we just illustrate some simple examples.

(i) The Heaviside function (denoted by H) is a fundamental solution of the derivative D in $\mathbf{R}$. By Proposition 5.6 all fundamental solutions of D are of the form $H + c$, with $c \in \mathbf{R}$.

(ii) For any $C \in \mathbf{C}$, the function $E(x) = x(H(x) + C)$ is a fundamental solution of D^2 in $\mathbf{R}$. By Proposition 5.6 all fundamental solutions of D^2 are of the form $E(x) + ax + b$, for any $a, b \in ci$. [Ex]

(iii) The rotational invariance of the Laplace operator Δ suggests that the equation $\Delta E = \delta$ might have a radial solution $E(x) := \varphi(r)$ $(r := |x|)$. By representing the Laplace operator in radial coordinates, we have

$$\varphi''(r) + \frac{N-1}{r}\varphi'(r) = 0 \qquad \forall r > 0 \text{ (in } \mathbf{R}^N).$$

Prescribing the appropriate singular behaviour at the origin, we then get the fundamental solution

$$E(x) := \begin{cases} 2^{-1}|x| & \forall x \in \mathbf{R}^N \setminus \{0\}, \text{ if } N = 1, \\ (2\pi)^{-1}\log|x| & \forall x \in \mathbf{R}^N \setminus \{0\}, \text{ if } N = 2, \\ -\dfrac{|x|^{2-N}}{(N-2)\omega_N} & \forall x \in \mathbf{R}^N \setminus \{0\}, \text{ if } N > 2; \end{cases} \tag{5.26}$$

ω_N being the $(N-1)$-dimensional measure of the unit sphere of $\mathbf{R}^N$. By Proposition 5.6 the fundamental solution of Δ is unique up to the sum of polynomials of degree one.

(iv) The function

$$E(x,t) := (4\pi t)^{-N/2} \exp\left(-|x|^2/4t\right) \qquad \forall t > 0 \tag{5.27}$$

is a fundamental solution of the heat operator $D_t - \Delta_x$ in $\mathbf{R}^{N+1}$ in the above sense. By Proposition 5.6 the fundamental solution of this operator is unique up to additive constants.

(v) The function

$$E(x,t) := (2c)^{-1}[H(x+ct) - H(x-ct)] = \begin{cases} 0 & \text{if } |x| \geq ct \\ (2c)^{-1} & \text{if } |x| < ct \end{cases} \tag{5.28}$$

is a fundamental solution of the Cauchy problem for the wave operator $D_t^2 - c^2D_x^2$ in $\mathbf{R}^2$, in the sense that

$$L(D)E = 0 \quad \forall t > 0, \qquad E(x,0) = 0, \quad D_tE(x,0) = \delta \quad \text{ for } x \in \mathbf{R}. \tag{5.29}$$

By Proposition 5.6 the fundamental solution of this operator is not unique.

6. A Glance at System Theory

In *signal analysis* [(10)] and other branches of engineering, some of the above results are used in the framework of the *theory of systems* with a different language, alternative to that of functional analysis — thus without referring to function spaces, and with attenuated mathematical rigor.

(10) Here is classical reference: A. Papoulis: Signal analysis. McGraw-Hill, New York 1977

Filters. The action of a system (representing a technical device) acting on time-dependent *signals* (i.e., functions of time) defines a *filter* (i.e., a linear operator) $L : u \mapsto f$. Here we just deal with linear SISO (i.e., single input - single output) systems. This filter may be represented, e.g., by a (linear) ODE: $P(D)u = f$; other filters are represented by multiplication by a constant (if this number is real and > 1, the filter is named an amplifier; if it is < 1, the filter is named an attenuator), by time-differentiation, by time-integration, by time-translation, and so on. An important class consists of the filters L that are translation-invariant (or *time-invariant)*, i.e., setting $\rho_\tau u(t) := u(t-\tau)$ for any $t, \tau \in \mathbf{R}$,

$$L\rho_\tau u = \rho_\tau Lu \qquad \text{for any admissible input } u, \forall \tau \in \mathbf{R}. \tag{6.1}$$

For instance, if $L = P(D)$ this property is fulfilled iff the coefficients of P are constant.

Convolution Filters. Henceforth we shall assume that δ is an admissible input for the system, i.e., δ is in the domain of L. Setting

$$\bar{h}(t,\tau) := [L(\rho_\tau \delta)](t) = [L(\delta(\cdot - \tau))](t) \qquad \forall \tau \in \mathbf{R}, \tag{6.2}$$

for any admissible input u we have

$$(Lu)(t) = [L(\delta * u)](t) = [L\langle \rho_\tau \delta(\cdot), u(\tau)\rangle](t) = \langle [L(\rho_\tau \delta)](t), u(\tau)\rangle = \langle \bar{h}(t,\tau), u(\tau)\rangle; \tag{6.3}$$

in general the latter is not a convolution.

The filter L is time-invariant (if and) only if $\bar{h}$ is of the form $\bar{h}(t,\tau) = h(t-\tau)$ for any t, τ; [Ex] thus $h = L\delta$. In this case the outcome of (6.3) is a convolution:

$$L(u) = L(\delta * u) = \langle h(t - \cdot), u\rangle = (L\delta) * u \quad \text{for any admissible input } u, \tag{6.4}$$

i.e., $L = (L\delta)*$; L is accordingly said a *convolution filter.*

Transfer Functions. By (6.3), the response of the system is determined by the response $L\delta$ to the unit impulse δ. $L\delta$ and $\widehat{L\delta}$ are respectively called the *transfer function in time* and *transfer function in frequency* (or *spectrum)* of the system L.

Henceforth we shall restrict ourselves to time-invariant filters. In these systems signals may conveniently be represented either as functions of time or (via Fourier transform) as functions of the frequency. For any admissible input u, we have

$$Lu = L(\delta * u) = (L\delta) * u \quad \Leftrightarrow \quad \widehat{Lu} = \widehat{L\delta}\, \widehat{u}; \tag{6.5}$$

the first formula holds for a.e. time t, the second one for a.e. frequency ω (assuming that these are regular distributions). Hence

$$|\widehat{Lu}|^2 = |\widehat{L\delta}|^2\, |\widehat{u}|^2 \quad \text{ for a.e. frequency } \omega. \tag{6.6}$$

As $\int |\widehat{u}(\omega)|^2\, d\omega = \int |u(t)|^2\, dt$ is interpreted as the power of the signal u, $|\widehat{u}|^2$ is named the *power spectrum* of the signal, and $|\widehat{L\delta}|^2$ is named the *transfer function of the energy* (both are functions of the frequency). The formula (6.6) may then be interpreted as follows:

"the power spectrum of the response equals the product of the transfer function of the energy by the power spectrum of the input."

Filter Compositions. For instance, a filter L may be constructed by combining two filters L_1, L_2 in parallel (in series, resp.). This means that $L = L_1 + L_2$ ($L = L_1 \circ L_2$, resp.), and entails

$$L = [L_1\delta + L_2\delta] * \quad \big(L = (L_1\delta) * (L_2\delta)*, \text{ resp.}\big) \quad \text{ in time,} \tag{6.7}$$

$$\widehat{L\delta} = \widehat{L_1\delta} + \widehat{L_2\delta} \quad \big(\widehat{L\delta} = \widehat{L_1\delta} \cdot \widehat{L_2\delta}, \text{ resp.}\big) \quad \text{in frequency} \tag{6.8}$$

(the latter formula holds only if L_1, L_2 are convolution filters). There are further ways to construct filters. For instance, a *feedback system* is represented by a mapping $f \mapsto g$ that is implictly defined by the equation

$$g = L_1(f + L_2(g)) \qquad \text{in time,} \tag{6.9}$$

L_1 and L_2 being linear filters. Let us assume that L_1 and L_2 are convolution filters, set $H_i := \widehat{L_i(\delta)}$ ($i = 1, 2$), and also assume that $1 - H_1 H_2$ is invertible. By applying the Fourier transform to this equation we get

$$\widehat{g} = H_1\big(\widehat{f} + H_2\widehat{g}\big), \qquad \text{i.e.} \quad \widehat{g} = \frac{H_1}{1 - H_1 H_2}\widehat{f} \qquad \text{in frequency.} \tag{6.10}$$

We conclude that the feedback system defines a convolution filter, and that its response in time to the unit impulse is $\mathcal{F}^{-1}(H_1/(1 - H_1 H_2))$.

Fundamental Relation. The next statement establishes a fundamental relation between a filter, its transfer function, and sinusoidal signals (i.e., exponential functions with imaginary exponent). Let us set $\varepsilon_\omega(t) := e^{2\pi i \omega t}$ for any $\omega, t \in \mathbf{R}$.

Theorem 5.7 *Let Φ be a subspace of $\mathcal{S}'$ such that $\delta, \varepsilon_\omega \in \Phi$ for any ω, and $L : \Phi \to \Phi$ be a time-invariant linear system. Any ε_ω is then an eigenfunction of L, and the spectrum $\widehat{L\delta}(\omega)$ is the corresponding eigenvalue. That is, for any ω,*

$$\varepsilon_\omega \in \Phi \quad \Rightarrow \quad L(\varepsilon_\omega) = \widehat{L\delta}(\omega)\, \varepsilon_\omega \quad \textit{in time.} \tag{6.11}$$

Proof. We have

$$\begin{aligned} [L(\varepsilon_\omega)](t) = [L(\delta * \varepsilon_\omega)](t) = [(L\delta) * \varepsilon_\omega](t) &= \int_{\mathbf{R}} [L\delta](\tau) e^{2\pi i \omega (t-\tau)}\, d\tau \\ &= \int_{\mathbf{R}} [L\delta](\tau) e^{-2\pi i \omega \tau}\, d\tau\; e^{2\pi i \omega t} = \widehat{L\delta}(\omega)\, \varepsilon_\omega(t). \end{aligned} \tag{6.12}$$ □

In several cases one may take $\Phi = \mathcal{S}'$.
(The latter result may be extended to the Laplace transform...)

Differential Filters (ODEs). For these filters we retrieve some known results. After (5.4), if $L = P(D)$ then

$$\widehat{L\delta}(\omega) = P(2\pi i \omega)\, \widehat{\delta}(\omega) = P(2\pi i \omega). \tag{6.13}$$

Hence $L\delta = \mathcal{F}^{-1}(P(2\pi i \omega))$, so that we retrieve (5.20):

$$Lu = (L\delta) * u = \mathcal{F}^{-1}(P(2\pi i \omega)) * u \quad \Leftrightarrow \quad \widehat{Lu} = \widehat{L\delta}\, \widehat{u} = P(2\pi i \omega)\widehat{u}. \tag{6.14}$$

Moreover, the transfer function of the energy of $L = P(D)$ is $|\widehat{L\delta}|^2 = |P(2\pi i \omega)|^2$:

$$|\widehat{Lu}|^2 = |\widehat{L\delta}|^2 |\widehat{u}|^2 = |P(2\pi i \omega)|^2\, |\widehat{u}|^2 \qquad \text{in frequency.} \tag{6.15}$$

For any $\omega \in \mathbf{R}$, ε_ω is an eigenfunction of any differential filter $L = P(D) = \sum_{n=0}^m c_n D^n$, and corresponds to the eigenvalue $\widehat{L\delta} \overset{(6.13)}{=} P(2\pi i \omega) = \sum_{n=0}^m c_n (2\pi i \omega)^n$. In this case Theorem 5.7 is reduced to the obvious formula

$$\sum_{n=0}^m c_n D^n e^{2\pi i \omega t} = \sum_{n=0}^m c_n (2\pi i \omega)^n\, e^{2\pi i \omega t}. \tag{6.16}$$