Introduction

Hysteresis occurs in several phenomena. In physics we encounter it in plasticity, friction, ferromagnetism, ferroelectricity, superconductivity, adsorption and desorption, for instance. More generally, hysteresis arises in phase transitions, a typical example being undercooling effects prior to nucleation. *Shape memory* effects have been observed and exploited in some recently developed materials. Hysteresis also occurs in engineering; thermostats are a very usual example. Others are met in porous media filtration, granular motion, semiconductors, spin glasses, mechanical damage and fatigue, for instance. Hysteresis also appears in chemistry, biology, economics, even in experimental psychology, and so on.

This is not a treatise on hysteresis. Our aim is to formulate a small collection of mathematical models of hysteresis, and to study their properties. As far as applications are concerned, we shall mainly deal with mechanical hysteresis; however, some of the equations we shall study stem from ferromagnetism.

One can distinguish two parts in this book. Chaps. I – VI deal with *hysteresis operators,* and Chaps. VII – XI with partial differential equations (P.D.E.s) with hysteresis. Let us outline this content.

Some Introductory Material. One can distinguish between rate dependent and rate independent memory effects. Rate dependent memory is typically *fading,* hence scale dependent. Rate independent memory is persistent, and scale invariant: this we name *hysteresis.* In Chap. I we illustrate this concept, and review some typical mechanisms generating hysteresis. The so called *play* and the *Prandtl-Reuss model* (or *stop)* allow to represent several friction and plasticity phenomena, and are the most elementary examples of continuous hysteresis models.

Then we consider a system for which equilibrium can be represented by the minimization of a linearly perturbed *nonconvex* functional. Nonconvexity is a source of *bistability,* and the latter leads to hysteresis in evolution. The approach of *elementary catastrophe theory* is partly followed in this presentation. For instance, such a setting occurs in the Weiss theory of ferromagnetism, because of a *positive feedback* effect. A more general example is offered by the classical Landau theory of phase transitions, which is also briefly reviewed.

In Chap. II we deal with so called *rheological* and *circuital models;* namely, *lumped parameter models* representing mechanic and electromagnetic constitutive laws. In either case, we first define a small family of elementary models, corresponding to the main constitutive behaviours. Each of these elementary models is characterized by a law which relates state variables; composite models are then constructed by means of *parallel* and *serial combinations* of these elements, and the corresponding laws are obtained by means of simple algebraic rules. This procedure yields a collection of ideal bodies, which allow to describe the constitutive behaviour of materials exhibiting a superposition of basic behaviours. This method has just heuristic value; however, it is very general, and can be used to devise constitutive laws also for other sorts of phenomena. By arranging rate independent elementary models, we obtain composite models which fulfil the same property, and so are able to represent hysteresis effects. We also propose a model of *mechanical damage.*

Hysteresis Operators. The following three chapters are devoted to the study of some important classes of continuous hysteresis models.

In Chap. III, we first introduce the concept of *hysteresis operator,* which is essentially due to M.A. Krasnosel'skiĭ. Then we analyse the mathematical properties of models of elasto-plasticity. We start with *plays, stops* and their generalizations. Either finite or infinite (even continuous) families of such elements can be combined either in series or in parallel, yielding (generalized) *Prandtl-Ishlinskiĭ models.* The latter can be described either by systems of variational inequalities, or by rate independent *memory operators.* Some properties of these operators in function spaces are pointed out. An operator representing mechanical damage is also briefly considered.

In Chap. IV we deal with the *Preisach model.* Maybe this is the most powerful among the (not many) hysteresis models at our disposal. This consists in combining a continuous family of elementary discontinuous hysteresis models, named *delayed relays.* This model was originally proposed for (scalar) ferromagnetism, and offers a good qualitative description of several phenomena in that field. Later it was also applied to other hysteresis phenomena, for instance unsaturated flow through porous media. The Preisach model is still regarded by engineers and physicists as a fundamental tool to deal with ferromagnetic hysteresis; several generalizations have also been proposed. A detailed account, which includes physical applications and generalizations, can be found in the recent monograph of Mayergoyz [196].

The underlying idea of the Preisach model is quite simple, and can be conveniently outlined by comparing cases with and without hysteresis. Let us consider two relays, respectively without and with delay, see Figs. 1(a), 1(b). The graph of any monotone function can be approximated by graphs like that of Fig. 1(c), which is obtained by linear combination of a finite family of jump functions, cf. Fig. 1(a). Similarly, the linear combination of a finite family of *delayed relays,* cf. Fig. 1(b) yields a hysteresis loop like that of Fig. 1(d). This construction allows one to approximate a fairly large class of continuous hysteresis laws, and yields an operator which acts in the space of continuous time functions. Vectorial extensions are also presented.

Figure 1. Comparison between the approximation of monotone functions by means of a weighted average of jump functions (case without hysteresis), and the Preisach model (case with hysteresis). See text.

Chap. V is devoted to a hysteresis model, which was already studied by Duhem. It corresponds to the transformation $u \mapsto w$ defined by the following Cauchy problem:

$$\begin{cases} \dot{w} = g_1(u,w)(\dot{u})^+ - g_2(u,w)(\dot{u})^- & \text{in }]0,T[, \\ w(0) = w^0, \end{cases} \tag{1}$$

with g_1 and g_2 given continuous functions. Here the dot denotes the time derivative, and $v^+ := \frac{|v|+v}{2}$, $v^- := \frac{|v|-v}{2}$ for any $v \in \mathbf{R}$, as usual. This system defines an operator $\mathcal{M} : u \mapsto w$, whose properties are studied. Generalizations of this model are introduced, in particular for the vectorial case.

This concludes our (admittedly, only partial) analysis of continuous hysteresis operators.

In Chap. VI we study *discontinuous* hysteresis operators. We concentrate our attention on the delayed relay operator. We consider its closure in suitable function spaces, in view of the study of ordinary differential equations (O.D.E.s) and P.D.E.s containing an operator of this sort. This closure is represented by means of two coupled *variational inequalities.* These *hysteresis atoms* are then combined to form a Preisach-type model. We then construct models of vectorial hysteresis, by extending the formulation in terms of variational inequalities.

P.D.E.s with Hysteresis. The following chapters are mainly devoted to the analysis of some P.D.E.s with hysteresis terms. These equations are tackled by means of few fundamental methods:

(i) Formulation of the problem as a system of *variational inequalities;* see Chap. VII.

(ii) Formulation as a differential equation containing an *accretive* operator, and then application of the *theory of nonlinear semigroups of contractions;* see Chap. VIII.

(iii) Formulation as a fixed point, and use of the *contraction mapping principle*; see Chap. VII, Sects. X.1 and X.4.

(iv) Approximation, a priori estimates, and passage to the limit by means of *compactness* techniques; see Chaps. IX, X, XI.

In Chap. VII we deal with models of elasto-plasticity for *space-structured* systems (namely, with x-dependence). By coupling the dynamic equation with the constitutive laws corresponding to a Prandtl-Ishlinskiĭ model of either play- or stop-type (introduced in Chap. III), we get a system of (possibly infinite) variational inequalities. We prove well posedness and regularity results for dynamic and quasi-static problems, by using standard monotonicity techniques, in the style of Duvaut and Lions [342]. We also study the dynamic problem for an elasto-visco-plastic material.

In Chap. VIII we introduce a new formulation of a class of differential equations containing hysteresis operators. In particular, we consider the *quasilinear* equation

$$\frac{\partial}{\partial t}\,[u + \mathcal{F}(u)] - \Delta u = f \qquad \text{in } \Omega\times]0,T[; \tag{2}$$

here $\mathcal{F}$ denotes a (possibly discontinuous) *generalized play operator,* f is a given function, Ω is a domain of $\mathbf{R}^N$, and $\Delta := \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$, as usual. Obviously, suitable initial and boundary conditions must be specified. We also consider the following first order hyperbolic equation

$$\frac{\partial}{\partial t}\,[u + \mathcal{F}(u)] + \sum_{\ell=1}^{N} \frac{\partial}{\partial x_\ell}(b_\ell u) + cu = f \qquad \text{in } \Omega \times\,]0,T[\,, \tag{3}$$

with $\mathcal{F}$ as above, and $b_1, \cdots, b_N$ given functions $\Omega \to \mathbf{R}$. Setting $U := (u, w)$ and $F := (f, 0)$, by means of a suitable transformation we can write equations (2) and (3) in the form

$$\frac{\partial U}{\partial t} + \mathcal{A}(U) + \mathcal{L}U \ni F \qquad \text{in }]0,T[\,, \tag{4}$$

where $\mathcal{A}$ is an m-accretive operator in the Banach space $\mathbf{R}^2$, endowed with the norm $\|(u, w)\| := |u| + |w|$, and $\mathcal{L}$ is an elliptic (respectively, hyperbolic) operator. This allows us to apply results of the *theory of nonlinear semigroups of contractions.* In particular, existence, uniqueness and continuous dependence on the data of the integral solution (in the sense of Bénilan) are obtained for the associated Cauchy problem. This approach can be extended to a large class of hysteresis operators, including (a natural extension of) the Preisach model, without any continuity assumption on the operator. A dual model concerns *generalized stop operators.*

Chaps. IX and X are devoted to the analysis of some classes of P.D.E.s containing a continuous *memory operator* $\mathcal{F}$. Although we deal with such a general class, we are mainly interested in hysteresis operators. Our aim is to illustrate some simple techniques, without any ambition of completeness.

In Chap. IX we consider an initial and boundary value problem for the model equation (2). We assume that $\mathcal{F}$ is strongly continuous in $C^0([0, T])$, and fulfils a monotonicity-type property (named *piecewise monotonicity),* which extends to memory operators the standard monotonicity of superposition operators. These assumptions include important classes of hysteresis operators, such as the generalized Prandtl-Ishlinskiĭ and Preisach operators. By means of approximation, a priori estimates, and passage to the limit, we prove existence of a solution and several complementary properties. Following Hilpert [102], uniqueness of a solution is then proved for $\mathcal{F}$ equal to a generalized Prandtl-Ishlinskiĭ of play-type (a family which includes a large class of Preisach operators, too). We also define a notion of *generalized solution,* which coincides with the integral solution considered in Chap. VIII.

In Chap. X we study an initial boundary value problem for the semilinear parabolic equation

$$\frac{\partial u}{\partial t} - \Delta u + \mathcal{F}(u) = f \qquad \text{in } \Omega \times]0, T[. \tag{5}$$

We prove existence of a solution if the memory operator $\mathcal{F}$ is strongly continuous in $C^0([0, T])$, and uniqueness if $\mathcal{F}$ is also Lipschitz continuous. Similar results are obtained for the following first order semilinear hyperbolic equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \mathcal{F}(u) = f \qquad \text{in }]a, b[\times]0, T[, \tag{6}$$

under the assumption that $\mathcal{F}$ be piecewise monotone. We also briefly discuss a simple example of O.D.E. with hysteresis, which represents a travelling wave.

In Chap. XI we deal with P.D.E.s containing *discontinuous* hysteresis operators. The hysteresis relation defined by a delayed relay operator can be approximated by a sequence of differential inclusions, containing a nonmonotone function and a time relaxation term. The asymptotic behaviour as the relaxation time vanishes is discussed for systems obtained by coupling such a law with either an O.D.E. or a P.D.E..

We then prove existence (and uniqueness) of the solution of equation (2), with $\mathcal{F}$ replaced by a delayed relay operator. This includes a sort of *Stefan problem with hysteresis,* with two thresholds for phase transition (namely, with the delayed relay in place of the sign graph, in the weak formulation). The analogous problem for equation (5), this also a free boundary problem, has a solution, which can be nonunique. The corresponding problem for the quasilinear equation (3) is well posed, if a suitable generalization of the *entropy condition* is imposed.

Chap. XII is an appendix. There we recall the definitions of some function spaces, present some elementary properties of nonlinear operators acting in Banach spaces, review the main results of the theory of nonlinear semigroups of contractions, and recall the basic elements of nonconvex analysis. We also outline some simple results for equations with *order preserving operators*, and study the properties of a nonstandard convergence in $BV(0,T)$.

We conclude with some comments, and with two lists of references: one for hysteresis works, the other for the remainder of the literature quoted in this volume.

Notes. (i) In this volume physical equations are usually written without coefficients. By doing so, we do not assume that coefficients can be normalized by a suitable choice of measure units (which might not be always possible). We just do not display coefficients, and so simplify our formula layout, since they do not play any role in our developments. These would not be changed, if (positive) coefficients were included.

(ii) Chapters are labelled by Roman numbers. They are indicated in referring to theorems and formulae of other chapters, and to sections; in other cases they are omitted.