Partial Differential Equations – A. Visintin – 2014

The Program of the course is best represented by the classes and the posted notes. Here there is synthesis.

Prerequisites (in parentheses the corresponding courses in Trento):

Differential and Integral Calculus, with Fourier series and ODEs *(Analisi I, II e III).*

Measure theory and Lebesgue integration *(Analisi III).*

Linear algebra *(Geometria I).*

General topology *(Geometria II).*

Elementary notions on PDEs *(Fisica-Matematica).*

Banach and Hilbert spaces, functional spaces, linear and continuous operators *(Analisi Funzionale).*

Fourier and Laplace transforms *(Integral transforms,* at the first semester).

(The parallel course *Advanced Analysis* is strongly recommended.)

Review of the elementary theory of PDE:

Linear first order equations and basic linear second order equations: Laplace, heat and wave equations.

Basics of boundary- and initial-value problems for linear PDEs.

Resolution via variable separation. Characteristics. Classification of PDEs.

Main Issues:

1. Elementary Second-Order PDEs [ReRo; chap. 1]

Review of ordinary differential equations (ODEs). Existence and uniqueness for initial-value problems. Stability. Theorem of existence, uniqueness and continuous dependence on the data. Eigenvalue problem for the homogeneous boundary-value problem for the equation $y'' + \lambda y = 0$. Linear systems of ODEs. Fundamental solution and matrix function. Formula of variation of parameters. Gronwall's lemma.

Laplace/Poisson equation: boundary conditions. Solution by separation of variables. Exercises. Energy equality.

Green identities. Variational formulation of the Laplace/Poisson. Fundamental lemma of the calculus of variations. Maximum principle.

Heat equation: boundary and initial conditions. Solution by separation of variables. Backward heat equation. Energy inequality.

Duhamel principle for evolutionary PDEs. Variational formulation. Energy inequality. Maximum principle.

Wave equation: boundary and initial conditions. Solution by separation of variables. D'Alembert solution of the wave equation. Domain of dependence and domain of influence. Variational formulation. Energy conservation.

Comparison between the qualitative properties of the heat, wave and Schrödinger equations.

Linear elliptic operators in nondivergence and divergence forms.

Weak and strong maximum form of the maximum principle for linear elliptic and parabolic equations in nondivergence form in C^0. Monotone dependence of the solution on the data. Uniqueness of the solution. [ReRo; chap. 4]

2. Characteristics and First-Order PDEs [ReRo; chap. 2], [Lecture Notes]

Multi-indices. Principal part and symbol of a linear differential operator. Classification of nonlinear PDEs: quasilinear, semilinear, fully-nonlinear equations.

Classification of linear second-order PDEs: elliptic, parabolic, hyperbolic.

Characteristic (hyper-)surfaces and (bi)characteristic curves. (Bi)characteristics of semilinear and quasilinear equations. (Bi)characteristics of first-order systems. Projected and unprojected (bi)characteristic curves.

Integration of semilinear and quasilinear first-order equations via the method of characteristics. Lagrange method of first integrals.

Statement of the Cauchy-Kovalevskaya and Holmgren theorems (without proofs).

Diagonalization of semilinear systems of first-order equations: $u_t + A(x,t) \cdot u_x = h(u,x,t)$. Diagonalization and derivation of the characteristic system.

3. Conservation Laws [ReRo; chap. 3], [Lecture Notes]

Conservation Laws. Burgers equation and p-system.

Characteristic curves. Shocks, rarefaction waves and multiplicity of solutions.

Notion of weak solution. Rankine-Hugoniot condition. Lax shock condition.

Riemann problem: shock wave and rarefaction wave.

Entropy condition and viscosity solution.

4. Distributions and Fourier Transform [Lecture Notes]

Space of test functions, $\mathcal{D}$. Convergence in $\mathcal{D}$. Exercises.

Space of distributions, $\mathcal{D}'$. Regular and singular distributions. Characterization of distributions. Exercises. Convergence in $\mathcal{D}'$.

Differentiation of distributions. Comparison with the classical and the a.e. derivatives.

Order and support of distributions. Distributions of finite order.

Space of infinitely differentiable functions, $\mathcal{E}$, and space of compactly supported distributions, $\mathcal{E}'$.

Space of rapidly decreasing functions, $\mathcal{S}$, and space of tempered distributions, $\mathcal{S}'$.

Convolution and Young theorem.

Review of the Fourier transformation, $\mathcal{F}$, in L^1 e in L^2. Extension of $\mathcal{F}$ to L^2 and $\mathcal{S}'$.

P.D.E.s with Constant Coefficients. The Ehrenpreis-Malgrange-Hörmander theorem. Notion of fundamental solution for stationary and evolutionary problems. Green and Neumann functions.

5. Sobolev Spaces [Lecture Notes]

Euclidean domains of Hölder class. Cone property.

Sobolev spaces of positive order and basic properties.

Extension operators. Density results.

Sobolev spaces of real order.

Sobolev inequality and imbedding between Sobolev spaces. Morrey theorem. Sobolev and Morrey indices. Rellich compactness theorem.

L^p- and Sobolev spaces on manifolds. Trace theorems.

Friedrichs inequality.