# Notes on Sobolev Spaces — A. Visintin — a.a. 2017-18

**Contents:** 1. Hölder spaces. 2. Regularity of Euclidean domains. 3. Sobolev spaces of positive integer order. 4. Sobolev spaces of real integer order. 5. Sobolev and Morrey embeddings. 6. Traces. 7. On application to PDEs.

**Note.** The bullet • and the asterisk $*$ are respectively used to indicate the most relevant results and complements. The symbol [] follows statements the proof of which has been omitted, whereas [Ex] is used to propose the reader to fill in the argument as an exercise.

Here are some abbreviations that are used throughout:

a.a. = almost any; resp. = respectively; w.r.t. = with respect to.

$p'$: conjugate exponent of $p$, that is, $p' := p/(p-1)$ if $1 < p < +\infty$, $1' := \infty$, $\infty' := 1$.

$\mathbf{N}_0 := \mathbf{N} \setminus \{0\}$; $\mathbf{R}^N_+ := \mathbf{R}^{N-1} \times ]0, +\infty[$. $|A|$ := measure of the measurable set $A$.

## 1. Hölder spaces

First we state a result, that provides a procedure to construct normed spaces, and is easily extended from the product of two spaces to that of a finite family. This technique is very convenient, and we shall repeatedly use it.

**Proposition 1.1** *Let $A$ and $B$ be two normed spaces and $p \in [1, +\infty]$. Then:*

*(i) The vector space $A \times B$ is a normed space equipped with the* $p$-norm *of the product:*

$$\begin{aligned} &\|(v,w)\|_p := \left(\|v\|_A^p + \|w\|_B^p\right)^{1/p} \qquad \text{if } 1 \le p < +\infty, \\ &\|(v,w)\|_\infty := \max\{\|v\|_A, \|w\|_B\}. \end{aligned} \tag{1.1}$$

*Let us denote this space by $(A \times B)_p$. These norms are mutually equivalent.*

*(ii) If $A$ and $B$ are Banach spaces, then $(A \times B)_p$ is a Banach space.*

*(iii) If $A$ and $B$ are separable (reflexive, resp.), then $(A \times B)_p$ is also separable (reflexive, resp.).*

*(iv) If $A$ and $B$ are uniformly convex and $1 < p < +\infty$, then $(A \times B)_p$ is uniformly convex.*

*(v) If $A$ and $B$ are inner-product spaces (Hilbert spaces, resp.), equipped with the scalar product $(\cdot,\cdot)_A$ and $(\cdot,\cdot)_B$, resp., then $(A \times B)_2$ is an inner-product space (a Hilbert space, resp.) equipped with the scalar product*

$$\big((u_1,v_1),(u_2,v_2)\big)_2 := (u_1,u_2)_A + (v_1,v_2)_B \qquad \forall (u_1,v_1),(u_2,v_2) \in (A \times B)_2.$$

*$\|(\cdot,\cdot)\|_2$ is then the corresponding Hilbert norm.*

*(vi) $F \in (A \times B)'_p$ (the dual space of $(A \times B)_p$) iff there exists a (unique) pair $(g,h) \in A' \times B'$ such that*

$$\langle F, (u,v)\rangle = {}_{A'}\langle g, u\rangle_A + {}_{B'}\langle h, v\rangle_B \qquad \forall (u,v) \in (A \times B)_p. \tag{1.2}$$

*In this case*

$$\|F\|_{(A\times B)'_p} = \|(g,h)\|_{(A'\times B')_{p'}}. \tag{1.3}$$

*The mapping $(A \times B)'_p \to (A' \times B')_{p'} : F \mapsto (g,h)$ is indeed an isometric surjective isomorphism.*

(We omit the simple argument, that rests upon classical properties of Banach spaces.)

A variant of the above result consists in equipping Banach spaces with the graph norm, associated to a linear operator.

**Spaces of Continuous Functions.** Throughout this section, by $K$ we shall denote a compact subset of $\mathbf{R}^N$, and by $\Omega$ a (possibly unbounded) domain of $\mathbf{R}^N$.

The linear space of continuous functions $K \to \mathbf{C}$, denoted by $C^0(K)$, is a Banach space equipped with the sup-norm $p_K(v) := \sup_{x\in K} |v(x)|$ (this is actually a maximum). The corresponding topology induces the uniform convergence.

The linear space of continuous functions $\Omega \to \mathbf{C}$, denoted by $C^0(\Omega)$, is a locally convex Fréchet space equipped with a family of seminorms: $\{p_{K_n} : K \subset\subset \Omega\}$, where $\{K_n : n \in \mathbf{N}\}$ is a nondecreasing sequence of compact sets that invades $\Omega$, namely $\bigcup_{n\in\mathbf{N}} K_n = \Omega$. [(1)] This topology induces the **locally uniform convergence.**

The linear space of *bounded continuous* functions $\Omega \to \mathbf{C}$, denoted by $C^0_b(\Omega)$, is also a Banach space equipped with the sup-norm $p_\Omega(v) := \sup_{x\in\Omega} |v(x)|$, and is thus a subspace of $C^0(\Omega)$.

As $\Omega$ is a metric space, we can also deal with uniformly continuous functions. In the literature, the linear space of *bounded and uniformly continuous functions* $\Omega \to \mathbf{C}$ is often denoted by $BUC(\Omega)$ or $C^0(\bar\Omega)$, as these functions have a unique continuous extension to $\bar\Omega$. The latter notation is customary but slightly misleading: actually, $C^0\big(\overline{\mathbf{R}^N}\big) \neq C^0\big(\mathbf{R}^N\big)$, although obviously $\overline{\mathbf{R}^N} = \mathbf{R}^N$. If $\Omega$ is bounded then $K := \bar\Omega$ is compact, and $C^0(\bar\Omega)$ can be identified with the space $C^0(K)$ that we defined above. Notice that $C^0(\bar\Omega)$ $(= BUC(\Omega))$ is a closed subspace of $C^0_b(\Omega)$ for any domain $\Omega$ of $\mathbf{R}^N$, and that the inclusion is strict; for instance,

$$\{x \mapsto \sin(1/x)\} \in C^0_b(]0,1[) \setminus C^0(\overline{]0,1[}), \qquad \{x \mapsto \sin(x^2)\} \in C^0_b\big(\mathbf{R}\big) \setminus C^0\big(\bar{\mathbf{R}}\big). \tag{1.5}$$

In this section we shall see several other spaces *over* $\bar\Omega$ that are included into the corresponding space *over* $\Omega$.

**Spaces of Hölder-Continuous Functions.** After introducing the basic spaces of continuous functions, we define spaces of functions which have some (weak or strong, integer or fractional...) differentiability. Let us fix any $\lambda \in ]0,1]$. The bounded continuous functions $v : \Omega \to \mathbf{C}$ such that

$$p_{\Omega,\lambda}(v) := \sup_{x,y\in\Omega, x\neq y} \frac{|v(x) - v(y)|}{|x-y|^\lambda} < +\infty \tag{1.6}$$

are said **Hölder-continuous** of index (or exponent) $\lambda$, and form a linear space that we denote by $C^{0,\lambda}(\bar\Omega)$ and equip with the graph norm. If $\lambda = 1$ these functions are said to be **Lipschitz continuous.** Obviously Hölder functions are uniformly continuous, so $C^{0,\lambda}(\bar\Omega) \subset C^0(\bar\Omega)$. The functional $p_{\Omega,\lambda}$ is a seminorm on $C^0(\Omega)$. [Ex]

**Proposition 2.1** *For any $\lambda \in ]0,1]$, $C^{0,\lambda}(\bar\Omega)$ is a Banach space when equipped with the norm $p_\Omega + p_{\Omega,\lambda}$.*

The functions $v : \Omega \to \mathbf{C}$ that are Hölder-continuous of index $\lambda$ when restricted to any compact set $K \subset \Omega$ are called **locally Hölder-continuous.** They form a Fréchet space, denoted by $C^{0,\lambda}(\Omega)$, when equipped with the family of seminorms $\{p_K + p_{K,\lambda} : K \subset\subset \Omega\}$. Notice that

$$C^{0,\lambda}(\bar\Omega) \subset C^{0,\nu}(\bar\Omega) \qquad \forall \lambda, \nu \in ]0,1], \nu < \lambda, [Ex] \tag{1.7}$$

with continuous injections. [(2)] For instance for any $\lambda \in ]0,1]$, the function $x \mapsto |x|^\lambda$ is an element of $C^{0,\lambda}(\mathbf{R})$, but not of $C^{0,\nu}(\mathbf{R})$ for any $\nu > \lambda$, and not of $C^{0,\lambda}(\bar{\mathbf{R}})$ (here also the traditional notation is not very helpful).

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(1) We remind the reader that Fréchet spaces are linear spaces that are also complete metric spaces and such that the linear operations are continuous. Their topology can be generated by an at most countable family of seminorms.

(2) All the injections that we consider between function spaces will be continuous; so we shall not point it out any more.

Notice that $\bigcup_{\lambda\in]0,1]} C^{0,\lambda}([0,1]) \neq C^0([0,1])$; e.g., the function

$$u(x) := (\log x)^{-1} \qquad \forall x \in\, ]0,1/2], \qquad u(0) = 0 \tag{1.8}$$

is continuous, but is not Hölder-continuous for any index $\lambda$. Moreover,

$$\bigcap_{\lambda\in]0,1[} C^{0,\lambda}([0,1]) \neq C^{0,1}([0,1]).$$

For instance the function

$$u(x) := x \log|x| \qquad \forall x \in\, ]0,1/2], \qquad u(0) = 0$$

is element of $C^{0,\lambda}([0,1])$ for any $\lambda \in\, ]0,1[$, but is not Lipschitz continuous. [Ex]

**Spaces of Differentiable Functions.** Let us assume that $\Omega$ and $\lambda$ are as above and that $m \in \mathbf{N}$. Let us recall the multi-index notation, and set $D_i := \partial/\partial x_i$ for $i = 1, ..., N$.

We claim that the functions $\Omega \to \mathbf{C}$ that are $m$-times differentiable and are bounded and continuous jointly with their derivatives up to order $m$ form a Banach space, denoted by $C_b^m(\Omega)$, when equipped with the norm

$$p_{\Omega,m}(v) := \sum_{|\alpha|\leq m} \sup_{x\in\Omega} |D^\alpha v(x)| \qquad \forall m \in \mathbf{N}. \tag{1.9}$$

This is easily seen because, setting

$$k(m) := \frac{(N+m)!}{N!m!} = \text{ number of the multi-indices } \alpha \in \mathbf{N}^N \text{ such that } |\alpha| \leq m, \tag{1.10}$$

the mapping $C_b^m(\Omega) \to C_b^0(\Omega)^{k(m)} : v \mapsto \{D^\alpha v : |\alpha| \leq m\}$ is a (nonsurjective) isomorphism between $C_b^m(\Omega)$ and its range. Indeed, if $D^\alpha u_n \to u_\alpha$ uniformly in $\Omega$ for any $\alpha \in \mathbf{N}^N$ such that $|\alpha| \leq m$, then $u_\alpha = D^\alpha u_0$; thus $u_n \to u_0$ in $C_b^m(\Omega)$. For instance, $C_b^1(\mathbf{R}^2)$ is isomorphic to $\{(w, w_1, w_2) \in C_b^0(\mathbf{R}^2)^3 : w_i = \partial w/\partial x_i \text{ in } \mathbf{R}^2, \text{ for } i = 1,2\}$. Here one can define a norm via Proposition 1.1.

The functions $\Omega \to \mathbf{C}$ that are continuous with their derivatives up to order $m$ form a locally convex Fréchet space equipped with the family of seminorms $\{p_{K,m} : K \subset\subset \Omega\}$. This space is denoted by $C^m(\Omega)$ (or by $\mathcal{E}^m(\Omega)$).

The linear space of the functions $\Omega \to \mathbf{C}$ that are bounded with their derivatives up to order $m$, and whose derivatives of order $m$ are Hölder-continuous of index $\lambda$, can be equipped with the norm

$$p_{\Omega,m,\lambda}(v) := \sum_{|\alpha|\leq m} \sup_{x\in\Omega} |D^\alpha v(x)| + \sum_{|\alpha|=m} p_{\Omega,\lambda}(D^\alpha v), \tag{1.11}$$

with $p_{\Omega,\lambda}$ as above. By Proposition 1.1, this is a Banach space, that we denote by $C^{m,\lambda}(\bar{\Omega})$.

The linear space of the functions $\Omega \to \mathbf{C}$ whose derivatives up to order $m$ are Hölder-continuous of index $\lambda$ in any compact set $K \subset \Omega$ can be equipped with the family of seminorms $\{p_{K,m,\lambda} : K \subset\subset \Omega\}$. This is a locally convex Fréchet space, denoted by $C^{m,\lambda}(\Omega)$.

It is also convenient to set

$$\left.\begin{aligned} &C^{m,0}(\bar{\Omega}) = C^m(\bar{\Omega}) := \{v \in C^m(\Omega) : D^\alpha v \in C^0(\bar{\Omega}), \forall \alpha, |\alpha| \leq m\}, \\ &C^{m,0}(\Omega) = C^m(\Omega), \\ &C^\infty(\bar{\Omega}) = \bigcap_{m\in\mathbf{N}} C^m(\bar{\Omega}), \qquad C^\infty(\Omega) = \bigcap_{m\in\mathbf{N}} C^m(\Omega). \end{aligned}\right\} \quad \forall m \in \mathbf{N}. \tag{1.12}$$

In passing notice that $C^\infty(\bar{\Omega}) \cap L^p(\Omega)$ is a dense subset of $L^p(\Omega)$ for any $p \in [1, +\infty[$. This can be proved by convolution with a regularizing kernel.

**Some Embeddings.** We say that a topological space $A$ is **embedded** into another topological space $B$ whenever $A \subset B$ and the injection operator $A \to B$ (which is then called an *embedding)* is continuous.

For any $m \in \mathbf{N}$, some embeddings are obvious within the class of $C^m$-spaces,

$$m \geq \ell \quad \Rightarrow \quad C^m(\bar{\Omega}) \subset C^\ell(\bar{\Omega}), \tag{1.13}$$

as well within that of $C^{m,\lambda}$-spaces:

$$\nu \leq \lambda \quad \Rightarrow \quad C^{m,\lambda}(\bar{\Omega}) \subset C^{m,\nu}(\bar{\Omega}) \qquad \forall m. \tag{1.14}$$

Concerning inclusions between spaces of the two classes, apart from obvious ones like $C^{m,\lambda}(\bar{\Omega}) \subset C^m(\bar{\Omega})$, some regularity is needed for the domain. [(3)]

**Proposition 2.2** *Let either $\Omega = \mathbf{R}^N$, or $\Omega \in C^{0,1}$ and bounded. Then*

$$C^{m+1}(\bar{\Omega}) \subset C^{m,\lambda}(\bar{\Omega}) \qquad \forall m, \forall \lambda \in [0,1].[] \tag{1.15}$$

From the latter inclusion, it easily follows that

$$C^{m_2,\lambda_2}(\bar{\Omega}) \subset C^{m_1,\lambda_1}(\bar{\Omega}) \qquad \text{if } m_1 < m_2, \forall \lambda_1, \lambda_2 \in [0,1]. \tag{1.16}$$

**A Counterexample.** The next example shows that some regularity is actually needed for (1.15) to hold. Let us set

$$\Omega := \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 < 1,\ y < |x|^{1/2}\}. \tag{1.17}$$

Of course $\Omega \in C^{0,1/2} \setminus C^{0,\nu}$ for any $\nu > 1/2$. [(4)] For any $a \in ]1,2[$, the function $v : \Omega \to \mathbf{R} : (x,y) \mapsto (y^+)^a \operatorname{sign}(x)$ belongs to $C^1(\bar{\Omega}) \setminus C^{0,\nu}(\bar{\Omega})$ for any $\nu > a/2$. [Ex]

Example 3.2 ahead also shows that some regularity is needed for (1.15) to hold.

We just considered embeddings for Banach spaces "on $\bar{\Omega}$". It is easy to see that these results yield the analogous statements for the corresponding Fréchet spaces "on $\Omega$".

## 2. Regularity of Euclidean Domains

Open subsets of $\mathbf{R}^N$ may be very irregular; e.g., consider $\bigcup_{n \in \mathbf{N}} B(q_n, 2^{-n})$, where $\{q_n\}$ is an enumeration of $\mathbf{Q}^N$. This set is open and has finite measure, but it is obviously dense in $\mathbf{R}^N$.

Several notions can be used to define the regularity of a Euclidean open set $\Omega$, or rather that of its boundary $\Gamma$. Here we just introduce two of them.

**Open Sets of Class $C^{m,\lambda}$.** Let us denote by $B_N(x,R)$ the ball of $\mathbf{R}^N$ of center $x$ and radius $R$. For any $m \in \mathbf{N}$ and $0 \leq \lambda \leq 1$, we say that $\Omega$ is of class $C^{m,\lambda}$ (here $C^{m,0}$ stays for $C^m$), and write $\Omega \in C^{m,\lambda}$, iff for any $x \in \Gamma$ there exist:

(i) two positive constants $R = R_x$ and $\delta_x$,
(ii) a mapping $\varphi_x : B_{N-1}(x,R) \to \mathbf{R}$ of class $C^{m,\lambda}$,
(iii) a Cartesian system of coordinates $y_1, ..., y_N$,

(3) The regularity of domains is defined in the next section.
(4) According to the definition of the next section ...

such that the point $x$ is characterized by $y_1 = ... = y_N = 0$ in this Cartesian system, and, for any $y' := (y_1, ..., y_{N-1}) \in B_{N-1}(x, R)$,

$$\begin{array}{lll} y_N = \varphi(y') & \Rightarrow & (y', y_N) \in \Gamma, \\ \varphi(y') < y_N < \varphi(y') + \delta & \Rightarrow & (y', y_N) \in \Omega, \\ \varphi(y') - \delta < y_N < \varphi(y') & \Rightarrow & (y', y_N) \notin \bar{\Omega}. \end{array} \tag{1.16}$$

This means that $\Gamma$ is an $(N-1)$-dimensional manifold (without boundary) of class $C^{m,\lambda}$, and that $\Omega$ locally stays only on one side of $\Gamma$. We say that $\Omega$ is a continuous (Lipschitz, Hölder, resp.) open set whenever it is of class $C^0$ ($C^{0,1}$, $C^{0,\lambda}$ for some $\lambda \in ]0,1]$, resp.). [5]

For instance, the domain

$$\Omega_{a,b,\lambda} := \{(x,y) \in \mathbf{R}^2 : x > 0, ax^{1/\lambda} < y < bx^{1/\lambda}\} \qquad \forall \lambda \leq 1, \forall a, b \in \mathbf{R}, a < b \tag{1.17}$$

is of class $C^{0,\lambda}$ iff $a < 0 < b$. [Ex]

We say that $\Omega$ is **uniformly of class** $C^{m,\lambda}$ iff

$$\Omega \in C^{m,\lambda}, \quad \inf_{x \in \Gamma} R_x > 0, \quad \inf_{x \in \Gamma} \delta_x > 0, \quad \sup_{x \in \Gamma} \|\varphi_x\|_{C^{m,\lambda}(B_{N-1}(x,R))} < +\infty. \tag{1.18}$$

For instance, by compactness, this is fulfilled by any bounded domain $\Omega$ of class $C^{m,\lambda}$. For instance $\Omega = \{(x,y) \in \mathbf{R}^2 : |xy| < 1\}$ is nonuniformly of class $C^{m,\lambda}$ for any $m, \lambda$.

**Cone Property.** The above notion of regularity of open sets is not completely satisfactory, as it excludes sets like e.g. a ball with deleted center. We then introduce a further regularity notion.

We say that $\Omega$ has the cone property iff there exist $a, b > 0$ such that, defining the finite open cone

$$C_{a,b} := \left\{x := (x_1, ..., x_N) : x_1^2 + ... + x_{N-1}^2 \leq b x_N^2,\ 0 < x_N < a\right\},$$

any point of $\Omega$ is the vertex of a cone contained in $\Omega$ and congruent to $C_{a,b}$. For instance, any ball with deleted center and the plane sets

$$\begin{array}{ll} \Omega_1 := \{(\rho, \theta) : 1 < \rho < 2, 0 < \theta < 2\pi\} & (\rho, \theta : \text{polar coordinates}), \\ \Omega_2 := \{(x,y) \in \mathbf{R}^2 : |x|, |y| < 1, x \neq 0\} & \end{array} \tag{1.19}$$

have the cone property, but are not of class $C^0$. [Ex]

**Proposition 2.1** *Any bounded Lipschitz domain has the cone property. [Ex]*

For unbounded Lipschitz domains this may fail; $\Omega := \{(x,y) \in \mathbf{R}^2 : x > 1, 0 < y < 1/x\}$ is a counterexample. Note that a domain $\Omega$ is bounded whenever it has the cone property and $|\Omega| < +\infty$.

## 3. Sobolev Spaces of Positive Integer Order

In this section we introduce the Sobolev spaces of positive integer order, which consist of the complex-valued functions defined on a domain $\Omega \subset \mathbf{R}^N$ that fulfill certain integrability properties jointly with their distributional derivatives. We then see how these functions can be extended to $\mathbf{R}^N$ preserving their Sobolev regularity, and approximate them by smooth functions.

[5] This notation refers to Hölder spaces, that are defined half-a-page ahead ...

**Sobolev Spaces of Positive Integer Order.** Henceforth we shall denote by $D$ derivatives in the sense of distributions. For any domain $\Omega \subset \mathbf{R}^N$, any $m \in \mathbf{N}$ and any $p \in [1, +\infty]$, we set

$$W^{m,p}(\Omega) := \big\{ v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega),\ \forall \alpha \in \mathbf{N}^N, |\alpha| \le m \big\}. \tag{3.1}$$

(Thus $W^{0,p}(\Omega) := L^p(\Omega)$.) This is a vector space over $\mathbf{C}$, that we equip with the norm

$$\|v\|_{W^{m,p}(\Omega)} := \Bigg( \sum_{|\alpha| \le m} \|D^\alpha v\|^p_{L^p(\Omega)} \Bigg)^{1/p} \qquad \forall p \in [1, +\infty[, \tag{3.2}$$

$$\|v\|_{W^{m,\infty}(\Omega)} := \max_{|\alpha| \le m} \|D^\alpha v\|_{L^\infty(\Omega)}. \tag{3.3}$$

We shall also write $\|\cdot\|_{m,p}$ in place of $\|\cdot\|_{W^{m,p}(\Omega)}$. Equipped with the topology induced by this norm, $W^{m,p}(\Omega)$ is called a **Sobolev space of order** $m$ (and of integrability $p$).

By Proposition 1.1, in $W^{m,p}(\Omega)$ the $p$-norm is equivalent to any other $q$-norm:

$$\Bigg( \sum_{|\alpha| \le m} \|D^\alpha v\|^q_{L^p(\Omega)} \Bigg)^{1/q} \quad \text{if} \ \ 1 \le q < +\infty, \qquad \max_{|\alpha| \le m} \|D^\alpha v\|_{L^\infty(\Omega)} \quad \text{if} \ \ q = \infty,$$

The equivalent 1-norm $\sum_{|\alpha| \le m} \|D^\alpha v\|_{L^1(\Omega)}$ can also be used.

The next result follows from Proposition 1.1.

• **Proposition 3.1** *For any $m \in \mathbf{N}$ and any $p \in [1, +\infty]$ the following occurs:*
*(i) $W^{m,p}(\Omega)$ is a Banach space over $\mathbf{C}$.*
*(ii) If $1 \le p < +\infty$, $W^{m,p}(\Omega)$ is separable.*
*(iii) If $1 < p < +\infty$, $W^{m,p}(\Omega)$ is uniformly convex (hence reflexive).*
*(iv) $\|\cdot\|_{m,2}$ is a Hilbert norm. $W^{m,2}(\Omega)$ (which is usually denoted by $H^m(\Omega)$) is then a Hilbert space, equipped with the scalar product*

$$(u, v) := \sum_{|\alpha| \le m} \int_\Omega D^\alpha u \, \overline{D^\alpha v} \, dx \qquad \forall u, v \in W^{m,2}(\Omega). \tag{3.4}$$

*(v) If $p \ne \infty$, then for any $F \in W^{m,p}(\Omega)'$ there exists a family $\{f_\alpha\}_{|\alpha| \le m} \subset L^{p'}(\Omega)$ such that*

$$\langle F, v \rangle = \sum_{|\alpha| \le m} \int_\Omega f_\alpha D^\alpha v \, dx \qquad \forall v \in W^{m,p}(\Omega). \tag{3.5}$$

*This entails that*

$$\|F\|_{W^{m,p}(\Omega)'} = \Bigg( \sum_{|\alpha| \le m} \|f_\alpha\|^{p'}_{L^{p'}(\Omega)} \Bigg)^{1/p'} \qquad \textit{if} \ \ p \in \, ]1, +\infty[, \tag{3.6}$$

$$\|F\|_{W^{m,1}(\Omega)'} = \max_{|\alpha| \le m} \|f_\alpha\|_{L^\infty(\Omega)}. \tag{3.7}$$

*Conversely, for any family $\{f_\alpha\}_{|\alpha| \le m}$ as above, (3.5) defines a functional $F \in W^{m,p}(\Omega)'$.*

**Extension Operators.** We call a linear operator $E : L^1_{\rm loc}(\Omega) \to L^1_{\rm loc}(\mathbf{R}^N)$ a **(totally) regular extension operator** iff

(i) $Eu = u$ a.e. in $\Omega$ for any $u \in L^1_{\rm loc}(\mathbf{R}^N)$, and

(ii) for any $m \in \mathbf{N}$, $E$ is a *regular $m$-extension operator.* By this we mean that for any $p \in [1,+\infty]$, (the restriction of) $E$ is continuous from $W^{m,p}(\Omega)$ to $W^{m,p}(\mathbf{R}^N)$ for any $p \in [1,+\infty]$; that is, there exists a constant $C_{m,p}$ such that

$$\|Eu\|_{W^{m,p}(\mathbf{R}^N)} \leq C_{m,p}\|u\|_{W^{[s],p}(\Omega)} \qquad \forall u \in W^{m,p}(\Omega).$$

For instance the *trivial extension*

$$\tilde{u} := u \quad \text{in } \Omega, \qquad \tilde{u} := 0 \quad \text{in } \mathbf{R}^N \setminus \Omega, \tag{3.8}$$

is not a regular extension operator, whenever $\Omega$ is regular enough. For instance, if $\Omega$ is a ball then $u \equiv 1 \in W^{1,p}(\Omega)$, but obviously $\tilde{u} \notin W^{1,p}(\mathbf{R}^N)$. (Loosely speaking, the radial derivative of $\tilde{u}$ has a Dirac measure concentrated along $\partial\Omega$, so that $\nabla\tilde{u}$ is not even locally integrable.)

• **Theorem 3.2** *(Calderón-Stein) For any uniformly-Lipschitz domain of $\mathbf{R}^N$, there exists a regular extension operator. []*

We illustrate the necessity of assuming some regularity for the domain $\Omega$ by means of two counterexamples.

**Example 3.1.** Let us set $Q := ]0,1[^2$, fix any $\lambda \in ]0,1[$, and set

$$\Omega := \{(x,y) \in Q : y > x^\lambda\}, \qquad u_\gamma(x,y) := y^{-\gamma} \quad \forall (x,y) \in \Omega, \forall \gamma > 0. \tag{3.9}$$

For any $p \in [1,+\infty[$ a direct calculation shows that

$$u_\gamma \in W^{1,p}(\Omega) \quad \Leftrightarrow \quad p(\gamma+1) < 1 + \lambda^{-1}. \ [Ex] \tag{3.10}$$

Let us now assume that $(0 <)\gamma < (1+\lambda^{-1})/2-1$, namely $2(\gamma+1) < 1+\lambda^{-1}$; the inequality in (3.10) is then fulfilled by some $\tilde{p} > 2$. On the other hand $W^{1,\tilde{p}}(Q) \subset L^\infty(Q)$, by a result that we shall see in Sect. 3 (cf. Morrey's Theorem). Therefore the unbounded function $u_\gamma$ cannot be extended to any element of $W^{1,\tilde{p}}(Q)$.

This example shows that, even for bounded domains, in Theorem 3.2 the hypothesis of Lipschitz regularity of $\Omega$ cannot be replaced by the uniform $C^{0,\lambda}$-regularity for any $\lambda \in ]0,1[$. Note that for $\lambda = 1$ this construction fails, and actually in that case the Calderón-Stein Theorem 3.2 applies.

**Example 3.2.** Let us set (using polar coordinates $(\rho,\theta)$ besides the Cartesian coordinates $(x,y)$)

$$\begin{aligned} &\Omega = \{(x,y) \in \mathbf{R}^2 : 1 < \rho(x,y) < 2,\ 0 < \theta(x,y) < 2\pi\}, \\ &u : \Omega \to \mathbf{R} : (x,y) \mapsto \theta(x,y). \end{aligned} \tag{3.11}$$

Notice that $u \in W^{m,p}(\Omega)$ for any $m \in \mathbf{N}$ (actually, $u \in W^{m,p}(\Omega) \cap C^\infty(\Omega)$!), but it cannot be extended to any $w \in W^{m,p}(\mathbf{R}^2)$ for any $m \geq 1$. Actually $\Omega$ fulfills the cone property, but is not even of class $C^0$.

Extension results are often applied to generalize to $W^{m,p}(\Omega)$ properties that are known to hold for $W^{m,p}(\mathbf{R}^N)$. As the restriction operator is obviously continuous from $W^{m,p}(\mathbf{R}^N)$ to $W^{m,p}(\Omega)$, under the hypotheses of Theorem 3.2, $W^{m,p}(\Omega)$ consists exactly of the restrictions of the functions of $W^{m,p}(\mathbf{R}^N)$. The next statement then follows.

**Corollary 3.3'** *Let $\Omega$ be a uniformly-Lipschitz domain of $\mathbf{R}^N$. For any $m \in \mathbf{N}$ and any $p \in [1,+\infty]$, one can equip $W^{m,p}(\Omega)$ with the equivalent quotient norm*

$$\|v\| := \inf\{\|w\|_{W^{m,p}(\mathbf{R}^N)} : w \in W^{m,p}(\mathbf{R}^N), w|_\Omega = v\} \qquad \forall v \in W^{m,p}(\Omega). \quad [Ex] \tag{3.12}$$

**Density results.** Let us denote by $\mathcal{D}(\bar{\Omega})$ the space of restrictions to $\Omega$ of functions of $\mathcal{D}(\mathbf{R}^N)$. Equivalently, $\mathcal{D}(\bar{\Omega})$ is the space of functions $\Omega \to \mathbf{C}$ that can be extended to elements of $\mathcal{D}(\mathbf{R}^N)$.

• **Theorem 3.4** *Let $m \in \mathbf{N}$ and $p \in [1, +\infty[$.*
*(i) (Meyers and Serrin) For any domain $\Omega \subset \mathbf{R}^N$, $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$.*
*(ii) If $\Omega$ is uniformly-Lipschitz, then $\mathcal{D}(\bar{\Omega})$ is dense in $W^{m,p}(\Omega)$. []*

For $p = \infty$ both statements fail (even for $m = 0$).

Exercise: Discuss the validity of this theorem in Examples 3.1 and 3.2.

Let us set
$$W^{1,p}_{\mathrm{loc}}(\Omega) := \{v \in \mathcal{D}'(\Omega) : \varphi v \in W^{1,p}(\Omega), \forall \varphi \in \mathcal{D}(\Omega)\}.$$
Like $L^p_{\mathrm{loc}}(\Omega)$, this is not a normed space.

**Proposition 3.5** *(Calculus Rules) Let $\Omega$ be any domain of $\mathbf{R}^N$ and $p \in ]1, +\infty[$.*
*(i) For any $u, v \in W^{1,p}(\Omega) \cap L^{p'}(\Omega)$,*

$$uv \in W^{1,1}(\Omega), \qquad \nabla(uv) = (\nabla u)v + u\nabla v \qquad \text{a.e. in } \Omega. \tag{3.13}$$

*(ii) For any Lipschitz-continuous function $F : \mathbf{C} \to \mathbf{C}$ and any $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$,*

$$F(u) \in W^{1,p}_{\mathrm{loc}}(\Omega), \qquad \nabla F(u) = F'(u)\nabla u \qquad \text{a.e. in } \Omega. \tag{3.14}$$

By using Theorem 3.4(i), both statements can be proved via regularization. [Ex]
For any $h \in \mathbf{R}^N$ and any $\Omega \subset \mathbf{R}^N$, let us denote by $\tau_h$ the shift operator $v \mapsto v(\cdot + h)$.

**Theorem 3.6** *For any $p \in [1, +\infty]$,*

$$v \in W^{1,p}(\mathbf{R}^N) \quad \Rightarrow \quad \|\tau_h v - v\|_{L^p(\mathbf{R}^N)} \le |h| \|\nabla v\|_{L^p(\mathbf{R}^N)^N} \quad \forall h \in \mathbf{R}^N. \tag{3.15}$$

The converse holds if $p > 1$; that is, $v \in W^{1,p}(\mathbf{R}^N)$ whenever $v \in L^p(\mathbf{R}^N)$ and there exists a constant $C > 0$ such that for any $h \in \mathbf{R}^N$, $\|\tau_h v - v\|_{L^p(\mathbf{R}^N)} \le C|h|$. [] It is easily seen that this converse statement fails for $p = 1$ and $v = H$ (the Heaviside function).

* *Proof.* For $p = \infty$ the result is obvious; let us then assume that $p < +\infty$. By the Jensen inequality we have

$$|\tau_h v(x) - v(x)|^p = \Big| \int_0^1 h \cdot \nabla v(x + th)\, dt \Big|^p \le |h|^p \int_0^1 |\nabla v(x + th)|^p\, dt \qquad \text{for a.e. } x \in \mathbf{R}^N;$$

hence

$$\begin{aligned} &\|\tau_h v - v\|^p_{L^p(\mathbf{R}^N)} \le |h|^p \int_{\mathbf{R}^N} dx \int_0^1 |\nabla v(x + th)|^p\, dt \\ &= |h|^p \int_0^1 dt \int_{\mathbf{R}^N} |\nabla v(x + th)|^p\, dx = |h|^p \int_0^1 dt \int_{\mathbf{R}^N} |\nabla v(x)|^p\, dx = |h|^p \int_{\mathbf{R}^N} |\nabla v(x)|^p\, dx. \end{aligned} \qquad \square$$

**The Reflection Method.** We conclude this section by illustrating a technique that yields regular $m$-extension operators, for any integer $m \ge 1$. For any $x \in \mathbf{R}^N$, let us first set $x := (x', x_N)$ with $x' \in \mathbf{R}^{N-1}$ and $x_N \in \mathbf{R}$, and $\mathbf{R}^N_+ := \{(x', x_N) \in \mathbf{R}^N : x_N > 0\}$.

**Theorem 3.7** *For any $m \geq 1$, there exist $a_1, ..., a_m \in \mathbf{R}$ such that, defining*

$$Eu(x) := \begin{cases} u(x) & \textit{if } x_N > 0 \\ \displaystyle\sum_{j=1}^{m} a_j u(x', -jx_N) & \textit{if } x_N < 0 \end{cases} \qquad \forall u \in L^1_{\text{loc}}(\mathbf{R}^N_+), \tag{3.16}$$

*$E$ is a regular $m$-extension operator for $\Omega = \mathbf{R}^N_+$.*

* *Proof.* For any $p \in [1, +\infty[$ and any $u \in \mathcal{D}(\overline{\mathbf{R}^N_+})$, any derivative of $Eu \in L^p(\mathbf{R}^N)$ of order up to $m$ is uniformly bounded in $\mathbf{R}^N \setminus (\mathbf{R}^{N-1} \times \{0\})$. It is then clear that $Eu \in W^{m,p}(\mathbf{R}^N)$ iff all derivatives of $Eu$ of order up to $m-1$ match a.e. along the hyperplane $\mathbf{R}^{N-1} \times \{0\}$, that is,

$$\begin{gathered} \lim_{x_N \to 0^+} D^\ell_N D^\beta_{x'} Eu(x', x_N) = \lim_{x_N \to 0^-} D^\ell_N D^\beta_{x'} u(x', x_N) \\ \text{for a.e. } x' \in \mathbf{R}^{N-1}, \forall \ell \in \mathbf{N}, \forall \beta \in \mathbf{N}^{N-1} : \ell + |\beta| < m. \end{gathered} \tag{3.17}$$

As

$$D^\ell_N D^\beta_{x'} Eu(x', x_N) = \sum_{j=1}^{m} (-j)^\ell a_j D^\ell_N D^\beta_{x'} u(x', -jx_N) \qquad \forall x' \in \mathbf{R}^{N-1}, \forall x_N < 0,$$

(3.16) is tantamount to

$$\begin{gathered} D^\ell_N D^\beta_{x'} u(x', 0) = \sum_{j=1}^{m} (-j)^\ell a_j D^\ell_N D^\beta_{x'} u(x', 0) \\ \text{for a.e. } x' \in \mathbf{R}^{N-1}, \forall \ell \in \mathbf{N}, \forall \beta \in \mathbf{N}^{N-1} : \ell + |\beta| < m. \end{gathered}$$

By the arbitrariness of $u \in \mathcal{D}(\overline{\mathbf{R}^N_+})$, this holds iff

$$\sum_{j=1}^{m} (-j)^\ell a_j = 1 \qquad \text{for } \ell = 0, ..., m-1. \tag{3.18}$$

This is a linear system of $m$ equations with matrix $M = \{(-j)^{i-1}\}_{i,j=1,...,m}$ for the unknowns $a_1, ..., a_m$. The matrix $M$ is of the Vandermonde class, hence it is nonsingular. Therefore this system has exactly one solution.

By Theorem 3.4 the space $\mathcal{D}(\overline{\mathbf{R}^N_+})$ is dense in $W^{m,p}(\mathbf{R}^N_+)$. $E$ thus maps $\mathcal{D}(\overline{\mathbf{R}^N_+})$ to $W^{m,p}(\mathbf{R}^N)$. Finally, $E$ is continuous, since

$$\|Eu\|_{W^{m,p}(\mathbf{R}^N)} \leq \big(1 + m \max_{1 \leq j \leq m} \max_{0 \leq \ell \leq m-1} j^\ell |a_j|\big) \|u\|_{W^{m,p}(\mathbf{R}^N_+)} \qquad \forall u \in \mathcal{D}(\overline{\mathbf{R}^N_+}).$$

Therefore $E$ can be extended to a (unique) continuous operator $W^{m,p}(\mathbf{R}^N_+) \to W^{m,p}(\mathbf{R}^N)$. □

The latter result can also be generalized to domains of class $C^m$, by partition of the unity and local charts. (We shall not display this argument.)

## 4. Sobolev Spaces of Real Order

By part (ii) of Theorem 3.4, $\mathcal{D}(\mathbf{R}^N)$ is dense in $W^{m,p}(\mathbf{R}^N)$ for any $p \in [1,+\infty[$ and any $m \geq 1$. This holds for no other domain of class $C^0$; we just illustrate this issue via a simple example.

Let $\Omega$ be an open ball of $\mathbf{R}^N$, and set $u \equiv 1$ in $\Omega$; obviously $u \in W^{m,p}(\Omega)$ for any $m \geq 1$ and any $p \in [1,+\infty[$. By contradiction, let us assume that it is possible to approximate $u$ in the topology of $W^{m,p}(\Omega)$ by means of a sequence $\{u_n\} \subset \mathcal{D}(\Omega)$. The trivial extension operator $v \mapsto \tilde{v}$ (cf. (3.8)) is continuous from $\mathcal{D}(\Omega)$ to $\mathcal{D}(\mathbf{R}^N)$ w.r.t. the $W^{m,p}$-topologies, for it obviously maps Cauchy sequences to Cauchy sequences; hence $\tilde{u}_n \to \tilde{u}$ in $W^{m,p}(\mathbf{R}^N)$. But it is clear that $\widetilde{u} \notin W^{m,p}(\mathbf{R}^N)$. Thus $\mathcal{D}(\Omega)$ is not dense in $W^{m,p}(\Omega)$.

On account of this negative result, we set

$$W_0^{m,p}(\Omega) := \text{ closure of } \mathcal{D}(\Omega) \text{ in } W^{m,p}(\Omega) \qquad \forall m \in \mathbf{N}, \forall p \in [1,+\infty[, \tag{4.1}$$

for any domain $\Omega \subset \mathbf{R}^N$, and equip this space with the same norm as $W^{m,p}(\Omega)$. The properties of Proposition 3.1 also hold for $W_0^{m,p}(\Omega)$, which indeed is a closed subspace of $W^{m,p}(\Omega)$. From this discussion we infer that $\Omega = \mathbf{R}^N$ is the only domain of class $C^0$ such that $W_0^{m,p}(\Omega) = W^{m,p}(\Omega)$ for any $m > 0$.

By the next statement, for any $m > 1$ the functions of $W_0^{m,p}(\Omega)$ may be regarded as vanishing on $\partial\Omega$ jointly with their derivatives up to order $m-1$. (Under suitable regularity assumptions for $\Omega$, this property might be restated in terms of *traces* — a notion that we introduce ahead, where the regularity condition "$u \in C^{m-1}(\bar{\Omega})$" will be dropped.)

**Proposition 4.1** *Let the domain $\Omega$ be of class $C^m$, $m \geq 1$ be an integer, and $1 \leq p < +\infty$. Then*

$$(D^\alpha u)\big|_{\partial\Omega} = 0 \qquad \forall u \in W_0^{m,p}(\Omega) \cap C^{m-1}(\bar{\Omega}), \forall \alpha \in \mathbf{N}^N, |\alpha| \leq m-1. \tag{4.1$'$}$$

*Partial Proof.* We shall prove this statement just for $m = 1$, via a procedure that however can easily be extended to any $m > 1$. We shall also confine ourselves to the case of $\Omega = \mathbf{R}^N_+$ ($:= \{(x', x_N) \in \mathbf{R}^N : x_N > 0\}$). The result can then be extended to more general sets via partition of unity (by a method that we shall illustrate ahead).

Let $u \in W_0^{1,p}(\mathbf{R}^N_+) \cap C^0(\overline{\mathbf{R}^N_+})$, and $\{u_n\}$ be a sequence in $\mathcal{D}(\mathbf{R}^N_+)$ such that $u_n \to u$ in $W_0^{1,p}(\mathbf{R}^N_+)$. Thus

$$u_n(x', x_N) = \int_0^{x_N} D_N u_n(x', t)\, dt \qquad \forall (x', x_N) \in \mathbf{R}^N_+, \forall n. \tag{4.1$'$}$$

As $D_N u_n \to D_N u$ in $L^p(\mathbf{R}^N_+)$, this equality is preserved in the limit. Hence $u_n(x', 0) = 0$ for any $x' \in \mathbf{R}^{N-1}$. □

**Sobolev Spaces of Negative Order.** Next we set

$$W^{-m,p'}(\Omega) := W_0^{m,p}(\Omega)' \;\; (\subset \mathcal{D}'(\Omega)) \qquad \forall m \in \mathbf{N}, \forall p \in [1,+\infty[, \tag{4.2}$$

and equip it with the dual norm

$$\|u\|_{W^{-m,p'}(\Omega)} := \sup\left\{\langle u, v\rangle : v \in W_0^{m,p}(\Omega), \|v\|_{W^{m,p}(\Omega)} = 1\right\}$$

(here by $\langle \cdot, \cdot \rangle$ we denote the pairing between $W^{-m,p'}(\Omega)$ and $W_0^{m,p}(\Omega)$). [6]

The Sobolev spaces of negative order inherit several properties from their preduals.

[6] Notice that we have thus defined $W^{-m,q}(\Omega)$ only for $1 < q \leq +\infty$, and that for $m = 0$ we retrieve $W^{0,p'}(\Omega) = L^{p'}(\Omega)$.

**Proposition 4.2** *For any $m \in \mathbf{N}$ and any $p \in [1, +\infty[$, $W^{-m,p'}(\Omega)$ is a Banach space.*

*(i) If $1 < p < +\infty$, $W^{-m,p'}(\Omega)$ is separable and reflexive.*

*(ii) $\| \cdot \|_{-m,2}$ is a Hilbert norm, and $W^{-m,2}(\Omega)$ is a Hilbert space (that is usually denoted by $H^{-m}(\Omega)$).*

**Proposition 4.3** *(Characterization of Sobolev Spaces of Negative Integer Order) For any $m \in \mathbf{N}$ and any $p \in [1, +\infty[$,*

$$F \in W^{-m,p'}(\Omega) \quad \Leftrightarrow \quad \exists \{f_\alpha\}_{|\alpha| \leq m} \subset L^{p'}(\Omega) : \; F = \textstyle\sum_{|\alpha| \leq m} D^\alpha f_\alpha \quad \text{in } \mathcal{D}'(\Omega). \tag{4.3}$$

*[This representation of $F$ need not be unique.]*

*Proof.* By the Hahn-Banach theorem any $F \in W^{-m,p'}(\Omega)$ can be extended to a functional $\tilde{F} \in W^{m,p}(\Omega)'$. By part (v) of Proposition 3.1 then there exists a family $\{f_\alpha\}_{|\alpha| \leq m}$ in $L^{p'}(\Omega)$ such that

$$\langle \tilde{F}, v \rangle = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \int_\Omega f_\alpha D^\alpha v \, dx \qquad \forall v \in W^{m,p}(\Omega),$$

Restricting this equality to $v \in \mathcal{D}(\Omega)$, we then get $F = \sum_{|\alpha| \leq m} D^\alpha f_\alpha$ in $\mathcal{D}'(\Omega)$.

Conversely, any distribution of this form is obviously a functional of $W^{-m,p'}(\Omega)$. $\square$

**Sobolev Spaces of Positive Noninteger Order.** Let us fix any $p \in [1, +\infty[$, any $\lambda \in ]0, 1[$, set

$$[a_{\lambda,p}(v)](x, y) := \frac{v(x) - v(y)}{|x - y|^{\frac{N}{p} + \lambda}} \qquad \forall x, y \in \Omega \, (x \neq y), \forall v \in L^1_{\rm loc}(\Omega), \tag{4.4}$$

$$W^{\lambda,p}(\Omega) := \big\{ v \in L^p(\Omega) : a_{\lambda,p}(v) \in L^p(\Omega^2) \big\}, \tag{4.5}$$

and equip this space with the norm of the graph

$$\|v\|_{\lambda,p} := \big( \|v\|^p_{L^p(\Omega)} + \|a_{\lambda,p}(v)\|^p_{L^p(\Omega^2)} \big)^{1/p}, \tag{4.6}$$

or with any other equivalent $q$-norm.

In order to complete this picture we also set

$$W^{\lambda,\infty}(\Omega) := C^{0,\lambda}(\bar{\Omega}) \qquad \forall \lambda \in ]0, 1[. \tag{4.7}$$

For $\lambda = 1$ this equality holds [as a result, not as a definition!], only if the domain $\Omega$ is regular enough. (See (1.15) and the related counterexample; see also ahead.) For $\lambda = 0$ the equality obviously fails.

Let us next fix any positive $m \in \mathbf{N}$, and, still for any $p \in [1, +\infty[$, set

$$W^{m+\lambda,p}(\Omega) := \big\{ v \in W^{m,p}(\Omega) : D^\alpha v \in W^{\lambda,p}(\Omega), \; \forall \alpha \in \mathbf{N}^N, |\alpha| = m \big\}; \tag{4.8}$$

this is a normed space over $\mathbf{C}$ equipped with the norm of the graph

$$\begin{aligned} \|v\|_{m+\lambda,p} &:= \Bigg( \|v\|^p_{m,p} + \sum_{|\alpha| = m} \|D^\alpha v\|^p_{\lambda,p} \Bigg)^{1/p} \\ &= \Bigg( \sum_{|\alpha| \leq m} \int_\Omega |D^\alpha v|^p \, dx + \sum_{|\alpha| = m} \iint_{\Omega^2} |[a_{\lambda,p}(D^\alpha v)](x, y)|^p \, dx dy \Bigg)^{1/p}, \end{aligned} \tag{4.9}$$

or with any other equivalent $q$-norm. Let us also set

$$W^{m+\lambda,\infty}(\Omega) := C^{m,\lambda}(\bar{\Omega}) \qquad \forall m \in \mathbf{N}, \forall \lambda \in ]0, 1[. \tag{4.10}$$

The spaces $W^{m+\lambda,p}(\Omega)$ with $\lambda \in ]0,1[$ are called *Sobolev spaces of fractional order* (sometimes just *fractional Sobolev spaces),* or also *Slobodeckiĭ spaces.*

**Proposition 4.4** *For any $s \in \mathbf{R}^+$, the following occurs:*
*(i) If any $p \in [1,+\infty[$, $W^{s,p}(\Omega)$ is a Banach space over $\mathbf{C}$. equipped with the norm of the graph.*
*(ii) If $p < +\infty$, $W^{s,p}(\Omega)$ is separable.*
*(iii) If $1 < p < +\infty$, $W^{s,p}(\Omega)$ is uniformly convex (hence reflexive).*
*(iv) $\|\cdot\|_{s,2}$ is a Hilbert norm. $W^{s,2}(\Omega)$ (that will be denoted by $H^s(\Omega)$) is a Hilbert space, equipped with the scalar product (here by $m$ we denote the integer part of $s$)*

$$(u,v) := \sum_{|\alpha|\le m} \int_\Omega D^\alpha u(x)\, \overline{D^\alpha v}(x)\, dx + \sum_{|\alpha|=m} \iint_{\Omega^2} [a_{\lambda,2}(D^\alpha u)](x,y)\, [a_{\lambda,2}(D^\alpha v)](x,y)\, dxdy \qquad \forall u,v \in W^{s,2}(\Omega). \tag{4.11}$$

*Outline of the Proof.* If $p = +\infty$ we already know that $W^{m+\lambda,\infty}(\Omega) := C^{m,\lambda}(\bar{\Omega})$ is a Banach space. If $p < +\infty$, we set

$$L_1(v) := \{D^\alpha v : |\alpha| \le m\}, \quad L_2(v) := \{a_{\lambda,p}(D^\alpha v) : |\alpha| = m\} \qquad \forall v \in L^p(\Omega);$$

the thesis then follows by applying Proposition 1.1. □

**Proposition 4.5** *Let $\Omega$ be any nonempty domain of $\mathbf{R}^N$, and set $\Omega_n := \big\{x \in \Omega : d\big(x, \mathbf{R}^N \setminus \Omega\big) > 1/n\big\}$ for any $n \in \mathbf{N}$. Then*

$$\|u\|_{W^{s,p}(\Omega_n)} \to \|u\|_{W^{s,p}(\Omega)} \qquad \forall u \in W^{s,p}(\Omega), \forall s \ge 0, \forall p \in [1,+\infty]. \tag{4.12}$$

*Outline of the Proof.* With no loss of generality one may assume that $\Omega$ is bounded. For $p \ne \infty$, the statement then follows from the absolute continuity of the integral. For $p = \infty$ the proof is even simpler. [Ex] □

**Sobolev Spaces of Negative Noninteger Order.** This construction mimics that of Sobolev spaces of negative integer order. First we set

$$W_0^{s,p}(\Omega) := \text{ closure of } \mathcal{D}(\Omega) \text{ in } W^{s,p}(\Omega) \qquad \forall s \ge 0, \forall p \in [1,+\infty[, \tag{4.13}$$

and equip it with the topology induced by $W^{s,p}(\Omega)$. The properties stated in Proposition 3.1 hold also for $W_0^{s,p}(\Omega)$. (7) This is a normal space of distributions, hence its dual is also a space of distributions. We then set

$$W^{-s,p'}(\Omega) := W_0^{s,p}(\Omega)' \ \ (\subset \mathcal{D}'(\Omega)) \qquad \forall s \ge 0, \forall p \in [1,+\infty[, \tag{4.14}$$

and equip it with the dual norm

$$\|u\|_{-s,p'} := \sup\big\{\langle u,v\rangle : v \in W_0^{s,p}(\Omega), \|v\|_{s,p} = 1\big\}.$$

A result analogous to Proposition 4.2 holds for $W^{-s,p'}(\Omega)$.

We have thus completed the definition of the *scale* of Sobolev spaces. In the next statement we gather their main properties.

**Proposition 4.7** *Let $s \in \mathbf{R}$ and $p \in ]1,+\infty]$ (with $p = 1$ included if $s \ge 0$). Then:*
*(i) $W^{s,p}(\Omega)$ is a Banach space over $\mathbf{C}$.*

(7) Theorems 4.2—4.4 hold for fractional indices, too. []

*(ii) If* $p < +\infty$, $W^{s,p}(\Omega)$ *is separable.*
*(iii) If* $1 < p < +\infty$, $W^{s,p}(\Omega)$ *is reflexive.*
*(iv)* $\|\cdot\|_{s,2}$ *is a Hilbert norm, and* $W^{s,2}(\Omega)$ *(=:* $H^s(\Omega)$*) is a Hilbert space.*
*(v) If* $s \geq 0$, *the same properties hold for* $W_0^{s,p}(\Omega)$, *the closure of* $\mathcal{D}(\Omega)$ *in* $W^{s,p}(\Omega)$.

Let us set

$$W_{\rm loc}^{s,p}(\Omega) := \big\{v \in \mathcal{D}'(\Omega) : \varphi v \in W^{s,p}(\Omega), \forall \varphi \in \mathcal{D}(\Omega)\big\} \qquad \forall s \in \mathbf{R}, \forall p \in [1,+\infty]. \tag{4.15}$$

This is a Fréchet space, equipped with the family of seminorms $\{v \mapsto \|\varphi v\|_{s,p} : \varphi \in \mathcal{D}(\Omega)\}$; indeed this topology can be generated by a countable family of these seminorms.

*** Other Classes of Sobolev-Type Spaces.** There are also other Sobolev-type spaces of noninteger order. For instance, one may interpolate the Sobolev spaces of integer order, or use the Fourier transformation. By the latter method one sets [8]

$$\begin{aligned} &\tilde{H}^{s,p} := \big\{v \in \mathcal{S}' : \mathcal{F}^{-1}[(1+|\xi|^2)^{s/2}\mathcal{F}(v)] \in L^p\big\} \qquad \forall s \in \mathbf{R}, \forall p \in [1,+\infty],\\ &\|v\|_{\tilde{H}^{s,p}} = \|\mathcal{F}^{-1}[(1+|\xi|^2)^{s/2}\mathcal{F}(v)]\|_{L^p} \qquad \forall v \in \tilde{H}^{s,p}. \end{aligned} \tag{4.16}$$

These are known as *spaces of Bessel potentials* (or just *Bessel potentials),* or *Lebesgue spaces,* or *Liouville spaces,* or *Lizorkin spaces,* and so on... [9]

These are Banach spaces. If $p \in [1,+\infty[$ this space is separable, if $p \in ]1,+\infty[$ it is reflexive. $\tilde{H}^{s,2}$ is a Hilbert space and is denoted by $\tilde{H}^s$. In the definition of the latter space, the inverse transformation $\mathcal{F}^{-1}$ can be dropped, since $\mathcal{F}$ is an isometry in $L^2$.

For $p = 2$ the Plancherel theorem yields

$$\begin{aligned} \int_{\mathbf{R}^N} uv\,dx = \int_{\mathbf{R}^N} \hat{u}\hat{v}\,d\xi &= \int_{\mathbf{R}^N} [(1+|\xi|^2)^{s/2}\hat{u}]\;[(1+|\xi|^2)^{-s/2}\hat{v}]\,d\xi\\ &\leq \|u\|_{\tilde{H}^s}\;\|v\|_{\tilde{H}^{-s}} \qquad \forall u,v \in \mathcal{S}, \forall s \in \mathbf{R}; \end{aligned} \tag{4.17}$$

Hence $\tilde{H}^{-s} \subset (\tilde{H}^s)'$ with continuous injection. The opposite inclusion can also be proved. []

For any sufficiently smooth domain $\Omega \subset \mathbf{R}^N$ (e.g. uniformly of Lipschitz class), the spaces $\tilde{H}^{s,p}(\Omega)$ are defined as follows, in analogy with (3.12):

$$\begin{aligned} &\tilde{H}^{s,p}(\Omega) = \big\{w\big|_\Omega : w \in \tilde{H}^{s,p}(\mathbf{R}^N)\big\} \qquad \forall s \in \mathbf{R}, \forall p \in ]1,+\infty[,\\ &\|v\|_{\tilde{H}^{s,p}(\Omega)} = \inf\big\{\|w\|_{\tilde{H}^{s,p}(\mathbf{R}^N)} : w\big|_\Omega = v\big\} \qquad \forall v \in \tilde{H}^{s,p}. \end{aligned} \tag{4.18}$$

On the basis of the next statement, these spaces may be regarded as an alternative to Sobolev spaces of real order.

**Theorem 4.6** *For any domain* $\Omega$ *uniformly of Lipschitz class, the following holds:*
*(i) For any* $m \in \mathbf{Z}$ *and any* $p \in ]1,+\infty[$, $\tilde{H}^{m,p}(\Omega) = W^{m,p}(\Omega)$.
*(ii) For any* $s \in \mathbf{R}$, $\tilde{H}^s(\Omega) = H^s(\Omega)$.
*(iii) The classes of the spaces* $\tilde{H}^{s,p}(\Omega)$ *and* $W^{s,p}(\Omega)$ *are contiguous (in the sense of Gagliardo), that is,*

$$H^{s+\varepsilon,p}(\Omega) \subset W^{s,p}(\Omega) \subset H^{s-\varepsilon,p}(\Omega) \qquad \forall s \in \mathbf{R}, \forall p \in ]1,+\infty[, \forall \varepsilon > 0. \tag{4.19}$$

---

(8) We still write $L^p$ instead of $L^p(\mathbf{R}^N)$ and similarly, and denote the *Fourier transform* of any $v \in \mathcal{S}'$ by $\mathcal{F}(v)$ or $\hat{v}$.

(9) This class of spaces is so natural, that one may expect that they have been discovered over and over.

*However,* $H^{s,p}(\Omega) \neq W^{s,p}(\Omega)$ *whenever* $s \notin \mathbf{Z}$ *and* $p \neq 2$*.*

*Partial Proof.* It suffices to prove these results for $\Omega = \mathbf{R}^N$. The proof of the statement (ii) may be found e.g. in [Baiocchi-Capelo, p. 76-79]. Here we just show that

$$\tilde{H}^m = H^m \qquad \forall m \in \mathbf{Z}. \tag{4.20}$$

The equivalence between the norms of $\tilde{H}^m$ and $H^m$ is easily checked, since for any $\alpha \in \mathbf{N}^N$ $\mathcal{F}(D^\alpha u) = (i\xi)^\alpha \hat{u}$, whence by the Plancherel theorem

$$\|D^\alpha u\|_{L^2} = \|\mathcal{F}(D^\alpha u)\|_{L^2} = \|\xi^\alpha \hat{u}\|_{L^2}.$$

Moreover

$$\exists C_1, C_2 > 0 : \forall \alpha \in \mathbf{N}^N, \forall \xi \in \mathbf{R}^N, \qquad C_1(1+|\xi|^2)^{|\alpha|/2} \leq 1 + |\xi|^{|\alpha|} \leq C_2(1+|\xi|^2)^{|\alpha|/2}. \qquad [Ex]$$

By the definition of the norm of $\tilde{H}^m(\Omega)$, it follows that $\tilde{H}^m(\Omega) = H^s(\Omega)$. □

## 5. Sobolev and Morrey Embeddings

**Basic Embeddings.** Obviously

$$|\Omega| < +\infty \quad \Rightarrow \quad C^m(\bar{\Omega}) \subset W^{m,p}(\Omega) \qquad \forall m \in \mathbf{N}, \forall p \in [1, +\infty], \tag{5.1}$$

with strict inclusion, and $C^{m,1}(\bar{\Omega}) \subset W^{m+1,\infty}(\Omega)$ for any domain $\Omega$. Moreover

$$\Omega \in C^0 \quad \Rightarrow \quad C^{m,1}(\bar{\Omega}) = W^{m+1,\infty}(\Omega) \qquad \forall m \in \mathbf{N}. \quad [] \tag{5.2}$$

The following simple counterexample shows that the latter equality fails if $\Omega$ just fulfills the cone property. Let $\Omega_1$ be as in (2.4), and set $u(\rho, \theta) = \theta$ for any $(\rho, \theta) \in \Omega_1$. Then $u \in W^{m,p}(\Omega_1)$ for any $m \in \mathbf{N}$ and any $p \in [1, +\infty]$, but $u \notin C^0(\bar{\Omega}_1)$.

In (4.10) we already defined

$$W^{m+\lambda,\infty}(\Omega) := C^{m,\lambda}(\bar{\Omega}) \qquad \forall m \in \mathbf{N}, \forall \lambda \in ]0, 1[. \tag{5.3}$$

Next we compare Sobolev spaces having either different differentiability indices, $m$, and/or different integrability indices, $p$. Here we shall confine ourselves to the case of integer differentiability indices, although most of these results take over to real indices.

**Proposition 5.1** *For any domain* $\Omega \subset \mathbf{R}^N$*, any* $m \in \mathbf{N}$ *and any* $p_1, p_2 \in [1, +\infty]$*,*

$$|\Omega| < +\infty, \ p_1 < p_2 \quad \Rightarrow \quad W^{m,p_2}(\Omega) \subset W^{m,p_1}(\Omega) \quad \textit{(with density)}. \tag{5.4}$$

*For any* $\Omega$*, the same inclusion holds for the corresponding* $W_0$*- and* $W_{\text{loc}}$*-spaces.*

*Proof.* (5.4) directly follows from the analogous inclusions between $L^p$-spaces. □

**Proposition 5.2** *If* $\Omega$ *is uniformly-Lipschitz, then, for any* $m_1, m_2 \in \mathbf{N}$ *and for any* $p \in [1, +\infty]$*,*

$$m_1 \leq m_2 \quad \Rightarrow \quad W^{m_2,p}(\Omega) \subset W^{m_1,p}(\Omega) \quad \textit{(with density)}. \tag{5.5}$$

*For any $\Omega$, the same inclusion holds for the corresponding $W_0$- and $W_{\rm loc}$-spaces.*

*Proof.* These inclusions are obvious. As by Theorem 3.4 $\mathcal{D}(\bar\Omega)$ is dense in both spaces, the density follows. □

**The Sobolev Theorem.** Two further classes of embeddings are of paramount importance in the theory of Sobolev spaces; these are embeddings between Sobolev spaces and from Sobolev to Hölder spaces:

$$W^{r,p}(\Omega) \subset W^{s,q}(\Omega) \quad \text{and} \quad W^{r,p}(\Omega) \subset C^{\ell,\lambda}(\bar\Omega) \quad \text{(for suitable indices)}. \tag{5.6}$$

These results are first proved for $\Omega = \mathbf{R}^N$ and then generalized to any uniformly-Lipschitz domain via Calderón-Stein's Theorem 3.2.

In Propositions 5.1 and 5.2 we already considered the case in which the indices $m$ and $p$ vary in the same direction. What happens as one of these two indices increases and the other one decreases? We shall see that, under appropriate restrictions on the integrability indices, the larger is $m$ the smaller is the space. The converse always fails, independently of $p$ and $q$: for any domain $\Omega$,

$$\forall m_1, m_2 \in \mathbf{N}, \forall p, q \in [1, +\infty], \quad m_1 < m_2 \quad \Rightarrow \quad W^{m_1,p}(\Omega) \not\subset W^{m_2,q}(\Omega).[Ex] \tag{5.7}$$

The same applies if both $W$-type spaces are replaced by the corresponding $W_0$- or $W_{\rm loc}$-spaces.

Nontrivial embeddings between Sobolev spaces rest on the following fundamental inequality due to Sobolev.

• **Theorem 5.3** *(Sobolev Inequality) For any $N > 1$ and any $p \in [1, N[$, there exists a constant $C = C_{N,p} > 0$ such that, setting $p^* := Np/(N-p)$,*

$$\|u\|_{L^{p^*}(\mathbf{R}^N)} \le C \|\nabla u\|_{L^p(\mathbf{R}^N)^N} \qquad \forall u \in \mathcal{D}(\mathbf{R}^N). \quad [] \tag{5.8}$$

Although this inequality only applies to functions with bounded support ($u \equiv 1$ is an obvious counterexample), the constant $C$ does not depend on the support.

*Proof for $p = 1$ and $N = 2$.* In this case the argument is much simpler than in the general setting. For any $u \in \mathcal{D}(\mathbf{R}^2)$,

$$|u(x,y)| = \Big| \int_{-\infty}^{x} \frac{\partial u}{\partial \tilde x}(\tilde x, y)\, d\tilde x \Big| \le \int_{\mathbf{R}} |\nabla u(\tilde x, y)|\, d\tilde x \qquad \forall (x,y) \in \mathbf{R}^2,$$

and similarly $|u(x,y)| \le \int_{\mathbf{R}} |\nabla u(x, \tilde y)|\, d\tilde y$. Therefore

$$\begin{aligned}
\iint_{\mathbf{R}^2} |u(x,y)|^2\, dxdy &\le \iint_{\mathbf{R}^2} \Big( \int_{\mathbf{R}} |\nabla u(\tilde x, y)|\, d\tilde x \Big) \Big( \int_{\mathbf{R}} |\nabla u(x, \tilde y)|\, d\tilde y \Big)\, dxdy \\
&= \iint_{\mathbf{R}^2} |\nabla u(\tilde x, y)|\, d\tilde x dy \iint_{\mathbf{R}^2} |\nabla u(x, \tilde y)|\, dx d\tilde y \\
&= \Big( \iint_{\mathbf{R}^2} |\nabla u(x,y)|\, dxdy \Big)^2,
\end{aligned}$$

that is, $\|u\|_{L^2(\mathbf{R}^2)} \le \|\nabla u\|_{L^1(\mathbf{R}^2)^2}$. Of course $1^* = 2$ for $N = 2$. □

**Remark.** If we assume that an inequality of the form (5.8) is fulfilled for some pair $p, p^*$, then we can establish the relation between $p^*$ and $p$ via the following simple scaling argument. Let us fix any $u \in \mathcal{D}(\mathbf{R}^N)$ and set $v_t(x) := u(tx)$ for any $x \in \mathbf{R}^N$ and any $t > 0$. Writing (5.8) for $v_t$ we get

$$t^{-N/p^*} \|u\|_{L^{p^*}(\mathbf{R}^N)} \le C t^{1-N/p} \|\nabla u\|_{L^p(\mathbf{R}^N)^N} \qquad \forall u \in \mathcal{D}(\mathbf{R}^N), \forall t > 0.[Ex]$$

This inequality can hold for all $t > 0$ only if $-N/p^* = 1 - N/p$, that is, $p^* := Np/(N-p)$.

**Sobolev Embeddings.** As obviously $\|\nabla u\|_{L^p(\mathbf{R}^N)^N} \le \|u\|_{W^{1,p}(\mathbf{R}^N)^N}$ and $\mathcal{D}(\mathbf{R}^N)$ is dense in $W^{1,p}(\mathbf{R}^N)$, the Sobolev inequality (5.8) entails that

$$\|u\|_{L^{p^*}(\mathbf{R}^N)} \le C\|\nabla u\|_{W^{1,p}(\mathbf{R}^N)} \qquad \forall u \in W^{1,p}(\mathbf{R}^N).$$

This yields the basic Sobolev imbedding

$$W^{1,p}(\mathbf{R}^N) \subset L^{p^*}(\mathbf{R}^N) \ \big(=: W^{0,p^*}(\mathbf{R}^N)\big) \qquad \forall p \in [1,N[, \forall N > 1. \tag{5.9}$$

On this basis one can prove the following more general result.

• **Theorem 5.4** *(Sobolev Embeddings) Let $\Omega$ be a uniformly-Lipschitz domain of $\mathbf{R}^N$. For any $\ell, m \in \mathbf{N}$ and any $p, q \in [1,+\infty]$,*

$$p \le q, \quad \ell - \frac{N}{q} \le m - \frac{N}{p} \quad \Rightarrow \quad W^{m,p}(\Omega) \subset W^{\ell,q}(\Omega) \tag{5.10}$$

*with continuous injection, and also with density if $q \ne +\infty$.*

*These statements hold for any domain $\Omega$ of $\mathbf{R}^N$ if both $W$-spaces are replaced either by the corresponding $W_0$-spaces, or by the corresponding $W_{\rm loc}$-spaces.*

*Proof.* On account of the regularity of $\Omega$, by the Calderón-Stein's Theorem 3.2 it suffices to prove the inclusion for $\Omega = \mathbf{R}^N$. It also suffices to deal with $m = 1$ and $\ell = 0$, since by applying this result iteratively one can then get it in general.

Notice that

$$p \le q \le p^* \quad \Rightarrow \quad W^{1,p}(\mathbf{R}^N) \subset L^p(\mathbf{R}^N) \cap L^{p^*}(\mathbf{R}^N) \subset L^q(\mathbf{R}^N) \tag{5.11}$$

The first inclusion follows from the trivial embedding $W^{1,p}(\mathbf{R}^N) \subset L^p(\mathbf{R}^N)$ and the Sobolev embedding (5.9); the second inclusion is easily checked. [Ex] We conclude that $W^{1,p}(\mathbf{R}^N) \subset L^q(\mathbf{R}^N)$ whenever $p \le q \le p^*$.

We claim that the injection operator $j : W^{m,p}(\Omega) \to W^{\ell,q}(\Omega)$ is continuous. By the Closed Graph Theorem, it suffices to show that the set $G := \{(v, jv) : v \in W^{m,p}(\Omega)\}$ is closed in $W^{m,p}(\Omega) \times W^{\ell,q}(\Omega)$. If $(v_n, jv_n) \to (v, w)$ in this space, then, up to extracting a subsequence, $(v_n, jv_n) \to (v, w)$ a.e. in $\Omega$; hence $w = jv$ a.e.. □

**Remarks.** (i) We have $p \le q$ and $\ell - N/q \le m - N/p$ only if $\ell \le m$, consistently with (5.7).

(ii) If $|\Omega| < +\infty$, then in (5.10) the hypothesis $p \le q$ may be replaced by $\ell \le m$. [Ex]

**Morrey Embeddings.** Next we come to our second important class of embeddings, that read $W^{m,p}(\Omega) \subset C^{\ell,\lambda}(\bar{\Omega})$ under suitable hypotheses on $m, p, \ell, \lambda$. By an inclusion like this we mean that for any $v \in W^{m,p}(\Omega)$ there exists a (necessarily unique) $\hat{v} \in C^{\ell,\lambda}(\bar{\Omega})$ such that $\hat{v} = v$ a.e. in $\Omega$. That is, the equivalence class associated to any element of $W^{m,p}(\Omega)$ contains one (and only one) function of $C^{\ell,\lambda}(\bar{\Omega})$. Henceforth we shall systematically assume this convention, and select a continuous representative whenever it exists.

The next result only applies to the case of $(m-\ell)p > N$.

Similar to the Sobolev embeddings, these further embeddings also rest on a fundamental inequality.

• **Theorem 5.5** *(Morrey Inequality) For any $N \geq 1$ and any $p \in ]N, +\infty[$, there exists a constant $C = C_{N,p} > 0$ such that*

$$\sup_{x,y \in \mathbf{R}^N, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^{1-N/p}} \leq C \|\nabla u\|_{L^p(\mathbf{R}^N)^N} \qquad \forall u \in \mathcal{D}(\mathbf{R}^N). \tag{5.12}$$

The Morrey inequality entails the following result.

• **Theorem 5.6** *(Morrey Embeddings) Let $\Omega$ be a uniformly-Lipschitz domain of $\mathbf{R}^N$, $\ell, m \in \mathbf{N}$, $1 \leq p < +\infty$ and $0 < \lambda < 1$. Then*

$$\ell + \lambda \leq m - \frac{N}{p} \quad \Rightarrow \quad W^{m,p}(\Omega) \subset C^{\ell,\lambda}(\bar{\Omega}). \tag{5.13}$$

*Moreover,* [10]

$$W^{m+N,1}(\Omega) \subset C_b^m(\Omega). \tag{5.14}$$

*In both cases the corresponding injection is continuous. []*

*Proof of (5.14).* It suffices to show this statement for $\Omega = \mathbf{R}^N$ and for $m = 0$. We have

$$\begin{aligned} |u(x_1, \dots, x_N)| &= \left| \int_{-\infty}^{x_1} dy_1 \cdots \int_{-\infty}^{x_N} dy_N \, \frac{\partial^N u}{\partial y_1 \cdots \partial y_N}(y_1, \dots, y_N) \right| \\ &\leq \left\| \frac{\partial^N u}{\partial y_1 \cdots \partial y_N} \right\|_{L^1(\mathbf{R}^N)} \leq \|u\|_{W^{N,1}(\mathbf{R}^N)} \qquad \forall u \in \mathcal{D}(\mathbf{R}^N). \end{aligned}$$

As $\mathcal{D}(\mathbf{R}^N)$ is dense in $C_b^0(\mathbf{R}^N)$, we then get $\|u\|_{C_b^0(\mathbf{R}^N)} \leq \|u\|_{W^{N,1}(\mathbf{R}^N)}$ for any $u \in W^{N,1}(\mathbf{R}^N)$. [Ex] $\square$

The next result also follows from the Morrey inequality (5.12).

**Theorem 5.7** *(a.e. Fréchet-differentiability) Let $m \in \mathbf{N}$, $p \in ]N, +\infty]$ and $\alpha \in \mathbf{N}^N$ with $|\alpha| < m$. For any $u \in W^{m,p}$, any continuous representative of $D^\alpha u$ is a.e. Fréchet-differentiable.*

This yields an extension a classical result of Rademacher.

**Corollary 5.8**

$$W^{m+1,\infty}(\mathbf{R}^N) \subset C_b^{m,1}(\mathbf{R}^N) \qquad \forall m \in \mathbf{N}. \tag{5.15}$$

(The converse inclusion is obvious.)

**Remarks.** (i) Although for $N = 1$ (5.14) entails that $W^{1,1}(\Omega) \subset L^\infty(\Omega)$, we have

$$W^{1,N}(\Omega) \not\subset L^\infty(\Omega) \qquad \forall N > 1. \tag{5.16}$$

For instance, setting $\Omega := B(0, 1/2)$ (the ball of center the origin and radius 2) and for $N \geq 2$

$$v_\alpha(x) := (-\log|x|)^\alpha \qquad \forall x \in \Omega, \forall \alpha \in ]0, 1 - 1/N[, \tag{5.17}$$

it is easy to check that $v_\alpha \in W^{1,N}(\Omega)$, although of course $v_\alpha \notin L^\infty(\Omega)$.

(ii) The above results are extended to fractional Sobolev spaces. After (5.3), for any domain $\Omega$, $C^{m,\lambda}(\bar{\Omega}) = W^{m+\lambda,\infty}(\Omega)$. Setting $N/\infty := 0$, the Morrey embedding (5.13) might then be regarded

[10] By $C_b^m(\Omega)$ we denote the space of functions $\Omega \to \mathbf{C}$ that are continuous and bounded with their derivatives up to order $m$, possibly without being uniformly continuous.

as a limit case of the Sobolev embedding (5.10) for $q = \infty$. In this case however the Sobolev theorem does not apply, as $(m-\ell)p > N$. $\square$

**Regularity Indices.** Defining [10]

$$\text{the } \textbf{Sobolev index} \ \ I_S(m,p) := m - N/p, \tag{5.18}$$

$$\text{the } \textbf{Hölder index} \ \ I_H(m,\lambda) := m + \lambda, \tag{5.19}$$

under the assumptions of the respective theorems, the Sobolev and Morrey embeddings (5.10) and (5.13) respectively also read

$$p \le q, \quad \mathcal{I}_S(\ell,q,N) \le \mathcal{I}_S(m,p,N) \quad \Rightarrow \quad W^{m,p}(\Omega) \subset W^{\ell,q}(\Omega), \tag{5.20}$$

$$I_H(\ell,\lambda) \le I_S(m,p) \quad \Rightarrow \quad W^{m,p}(\Omega) \subset C^{\ell,\lambda}(\bar{\Omega}). \tag{5.21}$$

Next we see that if $\Omega$ is bounded and the inequality between the indices is strict, then these injections are compact.

• **Theorem 5.9** *(Compactness) Let $\Omega$ be a bounded Lipschitz domain of $\mathbf{R}^N$, $\ell, m \in \mathbf{N}_0$, $1 \le p < +\infty$ and $0 < \lambda < 1$. Then:*

$$p \le q, \quad m - N/p > \ell - N/q \quad \Rightarrow \quad W^{m,p}(\Omega) \subset\subset W^{\ell,q}(\Omega), \tag{5.22}$$

$$m - N/p > \ell + \lambda \quad \Rightarrow \quad W^{m,p}(\Omega) \subset\subset C^{\ell,\lambda}(\bar{\Omega}), \tag{5.23}$$

$$m_2 + \nu_2 > m_1 + \nu_1 \quad \Rightarrow \quad C^{m_2,\nu_2}(\bar{\Omega}) \subset\subset C^{m_1,\nu_1}(\bar{\Omega}). \tag{5.24}$$

*These $W$-spaces can be replaced by the corresponding either $W_0$- or $W_{\rm loc}$-spaces; in either case $\Omega$ may be any domain of $\mathbf{R}^N$.*

**Exercises.**

— * Let $\Omega$ be a uniformly-Lipschitz domain of $\mathbf{R}^N$ and $1 \le p \le +\infty$. For any $s \in \mathbf{R}$, let us denote by $W^{s,p}_c(\Omega)$ the subspace of compactly supported distributions of $W^{s,p}(\Omega)$. Prove the following equalities:

$$\bigcap_{s\in\mathbf{R}} W^{s,p}_c(\Omega) = \mathcal{D}(\Omega), \qquad \bigcup_{s\in\mathbf{R}} W^{s,p}_c(\Omega) = \mathcal{E}'(\Omega), \qquad \bigcap_{s\in\mathbf{R}} W^{s,p}_{loc}(\Omega) = \mathcal{E}(\Omega), \qquad \bigcup_{s\in\mathbf{R}} W^{s,p}_{loc}(\Omega) = \mathcal{D}'_F(\Omega)$$

(the latter is the space of distributions of finite order).

— Check that the bounded and uniformly continuous functions $\Omega \to \mathbf{C}$ have a unique continuous extension to $\bar{\Omega}$, even if the domain $\Omega$ is irregular.

— Why are not the Hölder spaces $C^{0,\lambda}(\Omega)$ defined for any $\lambda > 1$?

— Check that $f(x) = 1/\log|x/2| \in C^0([-1,1])$ but it belongs to no Hölder space.

— Find a domain of $\mathbf{R}^2$ that has the cone property but is not of class $C^{0,\lambda}$ for any $\lambda \in \,]0,1]$.

— Let $a, b, r, s \in \mathbf{R}$ be such that $a < b$ and $1 < r < s$. Discuss the regularity of the domain $\{(x,y) \in \mathbf{R}^2 : x^2 + y^2 < 1, x > 0, ax^s < y < bx^r\}$ for different choices of the parameters $a, b, r, s$.

— Give an example of a domain with boundary not of class $C^0$.

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[10] These definition can be extended to fractional spaces. As $C^{\ell,\lambda}(\bar{\Omega}) = W^{m+\lambda,\infty}(\Omega)$ (if $\Omega$ is a Lipschitz domain), by setting $N/\infty = 0$ we see that Hölder index may be reduced to the Sobolev index.

## 6. Traces

Dealing with PDEs it is of paramount importance to prescribe boundary- and initial-values. However, for functions of Sobolev spaces the restriction to a lower dimensional manifold $\mathcal{M} \subset \bar{\Omega}$ is meaningless, since $\mathcal{M}$ has vanishing Lebesgue measure and these functions are only defined a.e. in $\Omega$. Nevertheless by means of functional methods one can generalize the concept of *restriction* by introducing the notion of *trace.*

For instance, let $x_0 \in \Omega = ]0,1[$ and $\mathcal{M} = \{x_0\}$. For any $v \in C^1([0,1])$ and any $x \in ]0,1[$, we have $v(x_0) = v(x) + \int_x^{x_0} v'(\xi)\, d\xi$; hence

$$|v(x_0)| = \int_0^1 |v(x_0)|\, dx \leq \int_0^1 \Big(|v(x)| + \int_x^{x_0} |v'(\xi)|\, d\xi\Big)\, dx \leq \|v\|_{W^{1,1}(0,1)}.$$

The restriction $v \mapsto v(x_0)$ can thus be extended to a uniquely-defined continuous operator $W^{1,1}(0,1) \to \mathbf{R}$. Let us now set $\Omega = ]0,1[^2$. By a similar argument, one can easily check that $v(0,\cdot) \in L^p(0,1)$ whenever $v, D_{x_1}v \in L^p(\Omega)$, and moreover, for a suitable constant $C > 0$,

$$\|v(0,\cdot)\|_{L^p(0,1)} \leq C\big(\|v\|_{L^p(\Omega)} + \|D_{x_1}v\|_{L^p(\Omega)}\big) \qquad \text{if } v, D_{x_1}v \in L^p(\Omega). \tag{6.1}$$

**Sobolev Spaces on a Manifold.** Let $\mathcal{M} \subset \Omega$ be a *nonflat* $(M-1)$-dimensional manifold $\mathcal{M} \subset \Omega$. For any $s \geq 0$ and any $p \in [1,+\infty]$, if $\mathcal{M} \in C^{s,1}$ ($[s] :=$ integral part of $s$) and is compact, then the Sobolev space $W^{s,p}(\mathcal{M})$ can be defined via a local Cartesian representation of $\mathcal{M}$ as follows.

Let $\{\Omega_i\}_{i=1,\dots,m}$ be a finite open covering of $\mathcal{M}$, such that, for any $i$, $\mathcal{M} \cap \Omega_i$ is the graph of a function $B_i \to \mathbf{C}$ of class $C^{[s],1}$, the $B_i$'s being balls of $\mathbf{R}^{M-1}$. That is, there exist

(i) a mapping $\varphi_i : B_i \to \mathbf{R}$ of class $C^{m,\lambda}$, and

(ii) a Cartesian system of coordinates $y = A \cdot x$, $A$ being an orthogonal matrix, such that

$$\mathcal{M} \cap \Omega_i = \{(y', \varphi_i(y')) : y' := (y_1, ..., y_{N-1}) \in B_i\}. \tag{6.2}$$

Let $\{\psi_i\}$ be a partition of unity of class $C^\infty$ subordinate to the covering $\{\Omega_i\}$, and, for any function $u : \mathcal{M} \to \mathbf{C}$, let us set

$$u_i(y) := (\psi_i u)(y, \varphi_i(y)) \qquad \forall y \in B_i, \tag{6.3}$$

$$W^{s,p}(\mathcal{M}) := \{u : \mathcal{M} \to \mathbf{C} \text{ measurable: } u_i \in W^{s,p}(B_i), \forall i\}. \tag{6.4}$$

This is a Banach space equipped with the norm

$$\begin{aligned} \|u\|_{W^{s,p}(\mathcal{M})} &:= \Big(\sum_{i=1}^m \|u_i\|^p_{W^{s,p}(B_i)}\Big)^{1/p} \qquad \text{if } \ p < +\infty, \\ \|u\|_{W^{s,\infty}(\mathcal{M})} &:= \max_{i=1,\dots,m} \|u_i\|_{W^{s,\infty}(B_i)}. \end{aligned} \tag{6.5}$$

Although this norm depends on $\{(\Omega_i, \varphi_i, f_i)\}_{i=1,\dots,m}$, different choices of these families correspond to equivalent norms for the same space. []

Other function spaces can also be constructed on $\mathcal{M}$ via a similar local Cartesian representation. The class of regularity of these functions cannot be higher than that of $\mathcal{M}$: e.g., if $\mathcal{M} \in C^m$ then one can define $C^\ell(\mathcal{M})$ only for $\ell \leq m$. If $\mathcal{M} \in C^\infty$ then one can also define test functions and distributions on $\mathcal{M}$. One can also define fractional Sobolev spaces on the manifold: $W^{s,p}(\Gamma)$ for any $s \in \mathbf{R}$ and $1 < p < +\infty$. (This is obvious for $\Omega = \mathbf{R}^N_+$, and can be extended to sufficiently regular domains).

Spaces over manifolds share several properties with spaces over *(flat)* Euclidean domains, and most of the results of the previous sections can be extended to this setting.

**Traces.** Next we state two basic trace results. First notice that $\varGamma = \partial\varOmega$ can be equipped with the $(N-1)$-dimensional Hausdorff measure whenever $\varOmega$ is sufficiently regular. []

• **Theorem 6.1** *(Traces) Let* $1 < p < +\infty$, $s > 1/p$, *and* $\varOmega$ *be a bounded domain of* $\mathbf{R}^N$ *of class* $C^{[s],1}$ *([s] = integer part of s). Then*

$$\begin{aligned}&\exists\gamma_0 : W^{s,p}(\varOmega) \to W^{s-1/p,p}(\varGamma) \textit{ linear and continuous,}\\ &\textit{such that } \ \gamma_0 v = v|_\varGamma \quad \forall v \in \mathcal{D}(\bar{\varOmega});\end{aligned} \tag{6.6}$$

$$\begin{aligned}&\exists\mathcal{R} : W^{s-1/p,p}(\varGamma) \to W^{s,p}(\varOmega) \textit{ linear and continuous, such that}\\ &\gamma_0\mathcal{R}v = v \quad \forall v \in W^{1,p}(\varGamma).[]\end{aligned} \tag{6.6$'$}$$

If $\varOmega$ is uniformly of class $C^{[s],1}$, then the trace of order 0, $\gamma_0$, determines the first-order tangential derivatives (i.e., the tangential components of the gradient on the boundary). Jointly with the first-order normal derivative (i.e., the normal component of the gradient), $\gamma_0$ thus determines the boundary behaviour of all first-order derivatives. By applying this procedure to the derivatives, one can also deal with the trace of higher-order derivatives.

Theorem 6.1 entails the next result.

• **Theorem 6.2** *(Normal Traces – I) Let* $1 < p < +\infty$, $s > 1 + 1/p$, *and* $\varOmega$ *be a bounded domain of* $\mathbf{R}^N$ *of class* $C^{[s],1}$. *Then* [11]

$$\begin{aligned}&\exists\gamma_1 : W^{s,p}(\varOmega) \to W^{s-1/p,p}(\varGamma) \textit{ linear and continuous,}\\ &\textit{such that } \ \gamma_1 v = \partial v/\partial\vec{\nu} \ (= \vec{\nu}\cdot\nabla v) \textit{ on } \varGamma, \forall v \in \mathcal{D}(\bar{\varOmega}).[]\end{aligned} \tag{6.7}$$

**Use of the Green Formula.** Next we confine ourselves to the Hilbert setup, for the sake of simplicity. We assume that $\varOmega$ is a bounded domain of $\mathbf{R}^N$ of class $C^{0,1}$, set

$$L^2_{\rm div}(\varOmega)^N := \{\vec{v} \in L^2(\varOmega)^N : \nabla\cdot\vec{v} \in L^2(\varOmega)\}, \tag{6.8}$$

and equip it with the graph norm

$$\|\vec{v}\|_{L^2_{\rm div}(\varOmega)^N} := \left(\|\vec{v}\|^2_{L^2(\varOmega)^N} + \|\nabla\cdot\vec{v}\|^2_{L^2(\varOmega)}\right)^{1/2}. \tag{6.9}$$

By means of Proposition 1.1, it is easily checked that this is a Banach space, actually a subspace of $H^1(\varOmega)^N$.

**Theorem 6.3** *(Normal Traces – II) Let* $\varOmega$ *be a bounded domain of* $\mathbf{R}^N$ *of class* $C^{0,1}$. *Then*

$$\begin{aligned}&\exists\gamma_\nu : L^2_{\rm div}(\varOmega)^N \to H^{-1/2}(\varGamma) \ (= H^{1/2}(\varGamma)') \textit{ linear and continuous,}\\ &\textit{such that } \ \gamma_\nu\vec{v} = \vec{v}\cdot\vec{\nu} \quad \forall v \in \mathcal{D}(\bar{\varOmega});\end{aligned} \tag{6.9$'$}$$

*Moreover the following* generalized formula of partial integration *holds:*

$$-\int_\varOmega (\nabla\cdot\vec{u})v\,dx = \int_\varOmega \vec{u}\cdot\nabla v\,dx - {}_{H^{-1/2}(\varGamma)}\langle\gamma_\nu\vec{u}, v\rangle_{H^{1/2}(\varGamma)} \qquad \forall\vec{u} \in L^2_{\rm div}(\varOmega)^N, \forall v \in \mathcal{D}(\bar{\varOmega}). \tag{6.10}$$

[11] $\gamma_1$ is often denoted by $\gamma_\nu$.

*Outline of the Proof.* Let us write the classical formula of partial integration (essentially the Gauss-Green theorem) for a sequence $\{\vec{u}_n\} \subset \mathcal{D}(\bar{\Omega})^N$ that approximates $\vec{u}$ in $L^2_{\rm div}(\Omega)^N$:

$$-\int_\Omega (\nabla \cdot \vec{u}_n) v \, dx = \int_\Omega \vec{u}_n \cdot \nabla v \, dx - \int_\Gamma \vec{u}_n \cdot \vec{\nu} \, \gamma_0 v \, dS \qquad \forall v \in H^1(\Omega) \tag{6.10$'$}$$

(by $dS$ we denote the $(N-1)$-dimensional area element of $\Gamma$). For any $z \in H^{1/2}(\Gamma)$, let us select $v = \mathcal{R}z$ (so that $\gamma_0 v = z$), and notice that by (6.6') $\|\mathcal{R}z\|_{H^1(\Omega)} \le C\|z\|_{H^{1/2}(\Gamma)}$. By (6.10)' then

$$\Big| \int_\Gamma \vec{u}_n \cdot \vec{\nu} \, z \, dS \Big| \le \big( \|\vec{u}_n\|_{L^2(\Omega)^N} + \|\nabla \cdot \vec{u}_n\|_{L^2(\Omega)} \big) \|\mathcal{R}z\|_{H^1(\Omega)} \le C \|\vec{u}_n\|_{L^2_{\rm div}(\Omega)^N} \|\mathcal{R}z\|_{H^{1/2}(\Gamma)}. \tag{6.10$''$}$$

Hence

$$\|\vec{u}_n \cdot \vec{\nu}\|_{H^{1/2}(\Gamma)'} \le C \|\vec{u}_n\|_{L^2_{\rm div}(\Omega)^N} \qquad \forall n.$$

By passing to the limit in this inequality, we get (6.9'). (6.10) follows by passing to the limit in (6.10)'. □

Let $\Omega$ still be a bounded domain of $\mathbf{R}^N$ of class $C^{0,1}$, set

$$H^1_\Delta(\Omega) := \{ v \in H^1(\Omega) : \Delta v \in L^2(\Omega) \}, \tag{6.11}$$

and equip it with the graph norm

$$\|v\|_{H^1_\Delta(\Omega)} := \big( \|v\|^2_{H^1(\Omega)} + \|\Delta v\|^2_{L^2(\Omega)} \big)^{1/2}. \tag{6.12}$$

By means of Proposition 1.1, it is easily checked that this is a Banach space, with

$$H^2(\Omega) \subset H^1_\Delta(\Omega) \subset H^1(\Omega).$$

By applying Theorem 6.3 to the gradient of $u$, one easily gets the next statement.

**Corollary 6.4** *(Normal Traces – III) Let $\Omega$ be a bounded domain of $\mathbf{R}^N$ of class $C^{1,1}$. Then*

$$\begin{aligned} &\exists \widetilde{\gamma}_\nu : H^1_\Delta(\Omega) \to H^{-1/2}(\Gamma) \ \text{linear and continuous,} \\ &\text{such that } \ \widetilde{\gamma}_\nu v = \partial v / \partial \vec{\nu} \ (= \vec{\nu} \cdot \nabla v) \quad \forall v \in \mathcal{D}(\bar{\Omega}). \end{aligned} \tag{6.12$'$}$$

*Moreover the following* generalized formula of partial integration *holds:*

$$-\int_\Omega \Delta u \, v \, dx = \int_\Omega \nabla u \cdot \nabla v \, dx - {}_{H^{-1/2}(\Gamma)}\langle \widetilde{\gamma}_\nu u, v \rangle_{H^{1/2}(\Gamma)} \qquad \forall \vec{u} \in H^1_\Delta(\Omega)^N, \forall v \in \mathcal{D}(\bar{\Omega}). \tag{6.13}$$

**Two Characterizations.** Next we characterize the spaces $W^{1,p}_0$ and $W^{2,p}_0$ in terms of traces (cf. Proposition 4.1):

• **Proposition 6.5** *Let $\Omega$ be a bounded domain of $\mathbf{R}^N$ of class $C^{1,1}$. For any $p \in [1, +\infty]$,*

$$W^{1,p}_0(\Omega) = \{ v \in W^{1,p}(\Omega) : \gamma_0 v = 0 \ \ \text{a.e. on } \Gamma \}, \tag{6.14}$$

$$W^{2,p}_0(\Omega) = \{ v \in W^{2,p}(\Omega) : \gamma_1 v = \gamma_0 v = 0 \ \ \text{a.e. on } \Gamma \}. \tag{6.15}$$

More generally, for any integer $k \ge 1$, $W^{k,p}_0(\Omega)$ is the space of all functions of $W^{k,p}(\Omega)$ such that all the traces that make sense in $W^{k,p}(\Omega)$ vanish a.e on $\Gamma$. [] Thus for instance

$$W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) = \{ v \in W^{2,p}(\Omega) : \gamma_0 v = 0 \ \text{ a.e. on } \Gamma \} \ne W^{2,p}_0(\Omega). \tag{6.16}$$

**The Friedrichs Inequality.** The next result is often applied in the study of PDEs with Dirichlet boundary conditions.

**Theorem 6.6** *(Friedrichs Inequality) Assume that $\Omega$ is a bounded domain of $\mathbf{R}^N$ of class $C^{0,1}$, let $\Gamma_1 \subset \Gamma$ have positive $(N-1)$-dimensional measure, and $p \in ]1,+\infty[$. Then* (11)

$$v \mapsto \|v\| := \big(\|\nabla v\|^p_{L^p(\Omega)^N} + \|\gamma_0 v\|^p_{L^p(\Gamma_1)}\big)^{1/p} \tag{6.17}$$

*is an equivalent norm in $W^{1,p}(\Omega)$.*

* *Proof.* By the continuity of the trace operator $W^{1,p}(\Omega) \to L^p(\Gamma_1)$, there exists $C > 0$ such that $\|v\| \leq C\|v\|_{1,p}$ for any $v \in W^{1,p}(\Omega)$. The converse inequality holds if we show that there exists $\hat{C} > 0$ such that

$$\|v\|_{L^p(\Omega)} \leq \hat{C}\big(\|\nabla v\|^p_{L^p(\Omega)^N} + \|\gamma_0 v\|^p_{L^p(\Gamma_1)}\big)^{1/p} \qquad \forall v \in W^{1,p}(\Omega).$$

By contradiction, let us assume that for any $n \in \mathbf{N}$ there exists $v_n \in W^{1,p}(\Omega)$ such that

$$\|v_n\|_{L^p(\Omega)} > n\big(\|\nabla v_n\|^p_{L^p(\Omega)^N} + \|\gamma_0 v_n\|^p_{L^p(\Gamma_1)}\big)^{1/p}. \tag{6.18}$$

Possibly dividing this inequality by $\|v_n\|_{L^p(\Omega)}$, we can assume that $\|v_n\|_{L^p(\Omega)} = 1$ for any $n$. Thus

$$\big(\|\nabla v_n\|^p_{L^p(\Omega)^N} + \|\gamma_0 v_n\|^p_{L^p(\Gamma_1)}\big)^{1/p} < 1/n \qquad \forall n. \tag{6.19}$$

Therefore there exists $v \in W^{1,p}(\Omega)$ such that, possibly extracting a subsequence, $v_n \to v$ weakly in $W^{1,p}(\Omega)$. By (6.19), $\nabla v_n \to 0$ strongly in $L^p(\Omega)^N$ and $\gamma_0 v_n \to 0$ strongly in $L^p(\Gamma_1)$. Hence $\nabla v = 0$ a.e. in $\Omega$ and $\gamma_0 v = 0$ a.e. on $\Gamma_1$. As $\Omega$ is connected, this entails that $v = 0$ a.e. in $\Omega$. (11) On the other hand, as the injection $W^{1,p}(\Omega) \to L^p(\Omega)$ is compact, $\|v\|_{L^p(\Omega)} = \lim_{n\to+\infty} \|v_n\|_{L^p(\Omega)} = 1$, and this is a contradiction. □

* **Exercises.** (i) Characterize the closure of $\{v \in \mathcal{D}(\Omega)^N : \nabla\cdot v = 0\}$ in the topology of $L^2(\Omega)$.
(ii) Characterize the closure of $\{v \in \mathcal{D}(\Omega)^N : \nabla\cdot v \in L^2(\Omega)\}$ in the topology of $L^2(\Omega)$.
(iii) Characterize the closure of $\{v \in \mathcal{D}(\Omega) : \Delta v = 0\}$ in the topology of $L^2(\Omega)$.
(iv) Characterize the closure of $\{v \in \mathcal{D}(\Omega) : \Delta v \in L^2(\Omega)\}$ in the topology of $L^2(\Omega)$.

## 7. On Application to PDEs

Different formulations may be attached to the same problem, corresponding to different regularity hypotheses on data and solution. We outline this issue on the Dirichlet problem for the equation $-\Delta u + \lambda u = f$, for any $\lambda \geq 0$.

*Classical Formulation.* This setting refers to spaces of either continuous or Hölder-continuous functions. Here $f$ and $g$ are assumed to be (at least) continuous, $u$ is required to belong to $C^2(\Omega)\cap C^0(\bar{\Omega})$; the equation and the boundary condition are then assumed to hold at all points.

(11) $\Gamma_1$ is a manifold with boundary, and above we just defined Sobolev spaces on manifold without boundary. Anyway, we may define $\|\gamma_0 v\|_{L^p(\Gamma_1)} := \|\chi_{\Gamma_1}\gamma_0 v\|_{L^p(\Gamma)}$, where by $\chi_{\Gamma_1} : \Gamma \to \mathbf{R}$ we denote the characteristic function of $\Gamma_1$.

(11) Domain = connected open set...

*Strong Formulation.* Here we move to Sobolev spaces. We fix any $p \in [1, +\infty[$, and assume that $\Omega$ is at least of class $C^{0,1}$, so that $\gamma_0 : W^{1,p}(\Omega) \to W^{1-1/p,p}(\Gamma)$. For any $f \in L^p(\Omega)$ and any $g \in W^{1-1/p,p}(\Gamma)$, we search for $u \in W^{1,p}(\Omega)$ such that $\Delta u \in L^p(\Omega)$ and

$$\begin{cases} -\Delta u + \lambda u = f & \text{a.e. in } \Omega, \\ \gamma_0 u = g & \text{a.e. on } \Gamma. \end{cases} \tag{7.15}$$

*Weak Formulation.* The restriction "$\Delta u \in L^p(\Omega)$" is here removed by interpreting the equation in the sense of distributions. We assume that $f \in W^{-1,p}(\Omega)$, $g \in W^{1-1/p,p}(\Gamma)$, and search for $u \in W^{1,p}(\Omega)$ such that

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } \mathcal{D}'(\Omega), \\ \gamma_0 u = g & \text{a.e. on } \Gamma. \end{cases} \tag{7.16}$$

In the analysis of these problems, usually one first deals with the weak formulation. Proving existence of a solution is the first task; one then tries to derive its uniqueness and qualitative properties. Under stronger assumptions on the data, one also tries to establish regularity properties of the weak solution, aiming to show that this is a strong solution, or even a classical one.

The following classical result is often used in order to prove existence of a weak solution of linear elliptic equations in divergence form. (12)

**Theorem 7.1** *(Lax-Milgram) Let $H$ be a Hilbert space, and $A : H \to H$ be a linear and bounded operator such that, for some $\alpha > 0$,*

$$(Av, v) \geq \alpha \|v\|^2 \qquad \forall v \in H \text{ (coerciveness)}. \tag{7.}$$

*Then $A$ is bijective, and $\|A^{-1} f\| \leq \alpha^{-1} \|f\|$ for any $f \in H$.*

* *Proof.* The coerciveness yields $\alpha \|v\|^2 \leq (Av, v) \leq \|Av\| \|v\|$ for any $v \in H$, whence $\alpha \|v\| \leq \|Av\|$. This entails that $A$ is injective, and, for any sequence $\{v_n\}$ in $H$, that $\{v_n\}$ is a Cauchy sequence only if the same holds for $\{Av_n\}$. By the continuity of $A$, $A(H)$ is then a closed vector subspace of $H$. For any $v \in A(H)^\perp$, we have $\alpha \|v\|^2 \leq (Av, v) = 0$, whence $v = 0$. Therefore $A(H) = H$. The boundedness of $A^{-1}$ then follows from the stated inequality $\alpha \|v\| \leq \|Av\|$ for any $v \in H$. □

(12) This theorem generalizes to nonsymmetric operators the Riesz-Fréchet representation of the theory of Hilbert spaces.