CHAPTER 8

Introduction to Stefan-Type Problems

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Contents
0. Introduction .......................................................... 381
1. The Stefan model .................................................... 383
   1.1. Weak formulation .............................................. 383
       The energy balance ............................................ 384
       The temperature-phase rule ................................. 385
   1.2. Classical formulation ......................................... 386
       The Stefan condition ........................................ 387
       Metastability ................................................. 387
       The one-dimensional Stefan problem ......................... 388
   1.3. Comparison between the weak and the classical formulation
       An ideal experiment ......................................... 389
       Equivalence between the CSP and the WSP .................. 390
       Comparison of analytical properties ....................... 391
   1.4. A Stefan-type problem arising in ferromagnetism
       A vector free boundary problem ................................ 394
   1.5. Other Stefan-type problems .................................. 394
       The quasi-steady Stefan problem and the Hele–Shaw problem.
       The hyperbolic Stefan problem ............................. 395
   1.6. Historical note .................................................. 397
       Free boundary problems ...................................... 398
2. More general models of phase transitions ........................ 399
   2.1. The Gibbs–Thomson law ...................................... 399
       Undercooling and superheating ................................ 399
       The Gibbs–Thomson law ...................................... 400
       Contact angle condition .................................... 400
       Free energy .................................................... 401
2.2. Kinetic undercooling and phase relaxation

First mode: Directional solidification
Second mode: Equiaxed solidification (or phase relaxation).
Comparison of the two modes.
Glass formation.
The entropy balance.
Linearization.

2.3. Phase transitions in heterogeneous systems

Mass diffusion.
Phase separation.
A transformation of variable.
Linearized constitutive laws.
A nonparabolic system of equations.

2.4. Approach via nonequilibrium thermodynamics

Balance laws and Gibbs formula.
Entropy balance.
Phenomenological laws.
A transformation of the state variables.

2.5. Diffuse-interface models and length-scales

Double wells.
Two relaxation dynamics.
The Penrose–Fife and phase-field models.
A limit procedure.
Macroscopic-mesoscopic, and microscopic length-scales.
Discussion.

3. Analysis of the weak formulation of the Stefan model

3.1. $L^2$-techniques

Interpretation.

3.2. $L^1$- and $L^\infty$-techniques

An $L^1$-result.
$L^\infty$-results.

3.3. Two integral transformations

Time-integral transformation.
Inversion of the Laplace operator.

3.4. Semigroup techniques

Change of pivot space.
$L^2$-semigroups.
$L^1$-semigroups.

4. Phase relaxation with nonlinear heat diffusion

4.1. Weak formulation

4.2. Existence of a weak solution

Modelling remarks.

5. Convexity and other analytical tools

5.1. Convex and lower semicontinuous functions

5.2. Legendre–Fenchel transformation and subdifferential

5.3. Saddle points

5.4. Compactness by strict convexity

5.5. Maximal monotone operators

5.6. M-accretive operators and semigroups

Cauchy problem.
T-accretiveness.

5.7. Perimeter and curvature
Abstract

The classical Stefan model is a free boundary problem that represents thermal processes in phase transitions just by accounting for heat-diffusion and exchange of latent heat. The weak and the classical formulations of the basic Stefan system, in one and in several dimensions of space, are here reviewed. The basic model is then improved by accounting for surface tension, for nonequilibrium, and dealing with phase transitions in binary composites, where both heat and mass diffuse. The existence of a weak solution is proved for the initial- and boundary-value problem associated to the basic Stefan model, and also for a problem with phase relaxation and nonlinear heat-diffusion. Some basic analytical notions are also briefly illustrated: convex calculus, maximal monotonicity, accretiveness, and others.
0. Introduction

Il n’y a pas des problèmes résolus, il n’y a que des problèmes plus ou moins résolus.
(There are no solved problems, there are just problems that are more or less solved.)
Henri Poincaré

The model of solid-liquid transitions that Josef Stefan formulated in 1889 provides a
good example for this aphorism of Poincaré. Existence of a solution for that problem was
proved by Lev Rubinstein in 1947. Tenths of thousands of papers and a number of meetings
have then been devoted to this model and its extensions, and research on this topic is still
in full development.¹

The present work has a twofold purpose: to introduce the basic Stefan problem and some
more refined models of (first-order) phase transitions, and also to illustrate some methods
for the analysis of associated nonlinear initial- and boundary-value problems. These two
aspects might hardly be separated, for the interplay between modelling and analysis is the
blood and life of research on Stefan-type problems.

Phase transitions occur in many relevant processes in physics, natural sciences, and en-
gineering: almost every industrial product involves solidification at some stage. Examples
include metal casting, steel annealing, crystal growth, thermal welding, freezing of soil,
freezing and melting of the earth surface water, food conservation, and others. All of these
processes are characterized by two basic phenomena: heat-diffusion and exchange of la-
tent heat of phase transition. A model accounts for this basic behaviour in terms of partial
differential equations (shortly, PDEs): this is known as the Stefan problem, and was exten-
sively studied in the last half century.

Because of the size of the existing literature, it has been mandatory to operate a dras-
tic selection: important topics like numerical approximation, solid–solid phase transitions,
shape-memory alloys, and others will be omitted. Here we shall confine throughout to mod-
elling and analysis of solid-liquid transitions.² It is natural to recognize this phenomenon
as an example of free boundary problem (shortly, FBP), for the evolution of the domains
occupied by the phases is not known a priori. This is also labelled as a moving boundary
problem, for the interface between the two phases evolves in time. Many mathematicians
addressed the Stefan problem from this point of view, especially for univariate systems and
in the framework of classical function spaces (i.e., \( C^k \)).

Phase transitions may also be regarded from a different perspective. Heat diffusion and
exchange of latent heat may also be formulated in weak form, since they are accounted
for by the energy balance equation, provided that this is meant in the sense of distribu-
tions. This leads to the formulation of an initial- and boundary-value problem in a fixed
space–time domain for a nonlinear parabolic equation. This nonlinearity is expressed via a
maximal monotone graph, and the problem may thus be reduced to a variational inequality.
The natural framework is here provided by the Sobolev spaces.

¹ This provides evidence of the richness of Stefan-type problems and of the phenomena they represent, and also
confirms the fertility of the academic mind in finding new results to offer to the attention of the community.
² In recent years the term phase transition has come in use among applied analysts with reference to stationary
two-phase systems, too. Here we remain with the more traditional use: for us phase transitions are transitions,
namely processes.
The two approaches above are known as the classical and the weak formulation of the Stefan problem. However, rather than being two formulations of the same problem, these represent two alternative models of phase transitions, that turn out to be equivalent only in special cases. The classical model is a genuine FBP, for it is based on the assumption that the phases are separated by an (unknown) smooth interface that also evolves smoothly; this approach allows for the onset of metastable states at the interior of the phases. On the other hand the weak formulation makes no direct reference to any phase interface: this may or may not exist, anyway it does not explicitly occur in the statement of the model. Solid and liquid phases may actually be separated by a set having nonempty interior, a so-called mushy region, that represents a fine-length-scale mixture of the two phases. In this respect the weak formulation is more general than the classical one, but it excludes metastable states. These issues are illustrated in Section 1.

The Stefan model is simple to be stated, combines analytical and geometrical aspects, has a suggestive physical substrate, is relevant for a number of applications, and is the prototype of a large class of evolutive FBPs. However it provides an oversimplified picture of (first-order) phase transitions, and is far from accounting for the richness of the physics of this large class of phenomena. In Section 2 we then improve the basic Stefan model by accounting for surface tension and for nonequilibrium at the phase interface. We also consider composite materials, in which the heat equation must be coupled with the equation of mass-diffusion, and the transition temperature depends on the chemical composition. Dealing with coupled diffusion, it seems especially convenient to use an approach based on the entropy balance and on the second principle of thermodynamics. This leads to the formulation of an initial- and boundary-value problem for a parabolic system of equations with two nonlinearities.

In Section 3 we then deal with some methods for the analysis of the weak formulation of the basic Stefan problem in several space dimensions. Several analytical procedures may be applied for that purpose: $L^p$-techniques, transformation by either space- or time-integration, and semigroup methods provide well-posedness and regularity properties in the framework of Sobolev spaces.

In Section 4 we study a multi-nonlinear extension of the Stefan problem, that accounts for nonlinear heat conduction and phase relaxation. We provide the weak formulation of an initial- and boundary-value problem in the framework of Sobolev spaces, and prove existence of a solution in any time interval, via a procedure that rests upon the notion of saddle point.

Convex calculus is often applied in the analysis of FBPs. In Section 5 we review basic results of that theory, and also illustrate some other analytical tools that are used in this work: maximal monotonicity, accretiveness, De Giorgi’s notion of $I^*$-convergence, and so on. We conclude with a bibliographical note and with a collection of few hundred references on Stefan-type problems – just a small sample from an overwhelming literature. An effort has been done to quote some references also for the most investigated issues, where making a wise selection is hardly possible.

This paper is just meant as an introduction to Stefan-type models of phase transitions. Sections 3 and 4 are devoted to the analysis of nonlinear PDEs, and may be read independently of Sections 1 and 2 that deal with modelling – however the reader should be aware that divorcing analysis from modelling somehow spoils this theory. In the spirit of
this Handbook, it seemed appropriate to devote special attention to the analysis of the weak formulation of the basic Stefan problem. For the benefit of the less experienced reader, we provide detailed arguments in Sections 3 and 4, devote some room to illustrate the basic analytical tools in Section 5, and also quote a number of fundamental monographs. A somehow didactical attitude has been maintained throughout, although this author cannot forget the Italian saying “chi sa fa, chi non sa insegnà”.³

A part of this paper is based on Chapters II, IV, V of this author’s monograph [453], the reader is referred to for a more detailed account. On the other hand, the analysis of Section 4 provides new results.

1. The Stefan model

This first part is mainly devoted to the construction of two alternative formulations of the basic Stefan model of phase transitions, and to illustrate some related problems.⁴ First we introduce the main physical assumptions and the weak formulation in several space dimensions, and then state the associated classical formulation. We illustrate how these models are based on partially different physical hypotheses, and compare some of their properties. In this part we deal with a homogeneous material, neglect surface tension, and assume either local stability or local metastability; these restrictions will be dropped in Section 2.

We shall also outline a vector extension of the Stefan model that accounts for processes in ferromagnetic materials having negligible hysteresis, the quasi-steady Stefan problem, the Hele-Shaw model, and the hyperbolic Stefan problem. Finally, a brief historical note updates Section IV.9 of Visintin [453].

1.1. Weak formulation

We shall always deal with solid–liquid (and liquid–solid) phase transitions, although our developments also apply to other first-order phase transitions. Here we shall represent phase transitions in an especially simplified way, focusing upon the thermal aspects, that is, heat-diffusion and exchange of latent heat. We shall neglect stress and deformation in the solid, convection in the liquid, change of density.⁵

We shall assume that the process occurs at constant volume, although in experiments usually it is the pressure that is maintained constant. Our developments however take over to systems that are maintained at constant pressure, at the only expense of some minor changes in the terminology: for instance, the term internal energy should then be replaced by enthalpy.⁶

³ “Who knows, makes. Who does not know, he teaches.”
⁴ Throughout this work, we shall just refer to first-order phase transitions. These exhibit a latent heat of phase transition, at variance with second-order phase transitions.
⁵ These simplifications might hardly be assumed in the case of vapour–liquid systems. This is the main reason for our preference for solid–liquid systems.
⁶ See e.g. the discussion of Section 7 of Penrose and Fife [373].
DEFINITION. Let us denote by $\Omega$ a bounded domain of the Euclidean space $\mathbb{R}^3$, which is occupied by a homogeneous material capable of attaining two phases, liquid and solid. Let us fix a constant $T > 0$, set $Q := \Omega \times [0, T]$, and use the following notation:

$u$: density of internal energy – namely, internal energy per unit volume,

$\theta$: relative temperature – namely, difference between the actual absolute (Kelvin) temperature $\tau$ and the value $\tau_E$ at which a planar solid–liquid interface is at thermodynamic equilibrium,

$\chi(\in [-1, 1])$: rescaled liquid fraction: $\chi = -1$ in the solid, $\chi = 1$ in the liquid (the actual liquid fraction is $\frac{1}{2}(\chi + 1)$),

$\vec{q}$: heat flux per unit surface,

$k(\theta, \chi)$: thermal conductivity – a positive-definite $3 \times 3$-tensor,

$f$: intensity of a space-distributed heat source (or sink) – namely, heat either produced or absorbed per unit volume.

One may assume that at a mesoscopic length-scale (namely, an intermediate scale between that of macroscopic observations and that of molecular phenomena) just liquid and solid phases may be present, that is, $\chi = \pm 1$ at each point. At the macroscopic length-scale however a so-called mushy region (or mushy zone), namely a fine solid–liquid mixture, may appear. This is characterized by $-1 < \chi < 1$, which corresponds to a liquid concentration $0 < (\chi + 1)/2 < 1$.

The energy balance. Let us assume that the density of internal energy $u$ is a known function of the state variables $\theta$ and $\chi$:

$$u = \hat{u}(\theta, \chi) \quad \text{in} \quad Q.$$ (1.1.1)

This functional dependence is characteristic of the specific material. Under the above physical restrictions, the global energy balance reads

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{q} = f \quad \text{in} \quad D'(Q) \quad (\nabla \cdot := \text{div}).$$ (1.1.2)

This equation may just be expected to hold in the sense of distributions, for in general $u$ and $\vec{q}$ will be discontinuous at phase interfaces, as we shall see ahead. We couple this balance with the Fourier conduction law

$$\vec{q} = -k(\theta, \chi) \cdot \nabla \theta \quad \text{in} \quad Q,$$ (1.1.3)

the thermal conductivity $k$ being a prescribed positive-definite tensor function. We thus get the global heat equation

$$\frac{\partial u}{\partial t} - \nabla \cdot [k(\theta, \chi) \cdot \nabla \theta] = f \quad \text{in} \quad D'(Q).$$ (1.1.4)

A phase-temperature relation is then needed to close the system, besides of course appropriate boundary- and initial-conditions.
The temperature-phase rule. Although Eq. (1.1.2) describes processes that are outside (and possibly far from) equilibrium, here we assume local thermodynamic equilibrium. By this we mean that in a neighbourhood of each point the system is governed by the same constitutive relations as at equilibrium.

This hypothesis excludes the occurrence of undercooled and superheated states, that we shall illustrate ahead. This thus yields the following temperature-phase rule:

\[ \theta \geq 0 \text{ in } Q_1, \quad \theta \leq 0 \text{ in } Q_2, \]

(1.1.5)

where by \( Q_1 \) and \( Q_2 \) we denote the subsets of \( Q \) that correspond to the liquid and solid phases, respectively. Defining the multi-valued sign function,

\[ \text{sign}(\xi) := \begin{cases} -1 & \text{if } \xi < 0, \\ \{ \pm 1 \} & \text{if } \xi > 0, \end{cases} \]

(1.1.6)

the conditions (1.1.5) also read

\[ \chi \in \text{sign}(\theta) \text{ in } Q. \]

(1.1.7)

By eliminating \( \chi \) from (1.1.1) and (1.1.7), for the density of internal energy and the temperature we get a relation of the form

\[ u \in \alpha(\theta) \text{ in } Q, \]

(1.1.8)

where \( \alpha : \mathbb{R} \to 2^\mathbb{R} \) is a multi-valued maximal monotone function (cf. Section 5.5); see Figure 1.7

Notice that under the hypothesis of local thermodynamic equilibrium \( k(\theta, \chi) \cdot \nabla \theta \) is independent of \( \chi \); indeed \( k \) is determined by \( \chi \) where \( \theta \neq 0 \), and \( \nabla \theta = \vec{0} \) in the interior of the set where \( \theta \neq 0 \). Setting \( \tilde{k}(\theta) := k(\theta, \alpha(\theta)) \), we may then write \( \vec{q} = -\tilde{k}(\theta) \cdot \nabla \theta \text{ in } Q. \)

The system (1.1.5), (1.1.8) must be coupled with an initial condition for \( u \) and with boundary conditions either for \( \theta \) or for the normal component of the heat flux. This constitutes the weak formulation of the two-phase Stefan problem in several space dimensions.

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Footnote: By \( 2^A \) we denote the power set of any set \( A \).
(Traditionally, one speaks of a two-phase problem, for the temperature evolution is unknown in both phases.)

Let us denote by $\vec{\nu}$ the outward-oriented unit normal vector on the boundary $\Gamma$ of $\Omega$. For instance, one may choose a partition $\{\Gamma_D, \Gamma_N\}$ of $\Gamma$ and assume that

$$\theta = \theta_D \quad \text{on } \Sigma_D := \Gamma_D \times ]0, T[, \quad \tilde{k}(\theta) \cdot \frac{\partial \theta}{\partial \nu} = h \quad \text{on } \Sigma_N := \Gamma_N \times ]0, T[, \quad (1.1.9)$$

for a prescribed boundary temperature $\theta_D$ and a prescribed incoming heat flux $h$. We may now formulate a problem in the framework of Sobolev spaces.

**PROBLEM 1.1.1** (Weak formulation of the multi-dimensional two-phase Stefan problem). Find $\theta \in L^2(0, T; H^1(\Omega))$ and $u \in L^2(Q)$ such that (1.1.8), (1.1.9) are fulfilled and

$$\int_0^T \int_{\Omega} \left( u^0 - u \right) \frac{\partial v}{\partial t} + (\tilde{k}(\theta) \cdot \nabla \theta) \cdot \nabla v \, dx \, dt = \int_0^T \int_{\Sigma_N} hv \, dS \, dt$$

$$\forall v \in H^1(Q), \quad v = 0 \text{ on } (\Omega \times \{T\}) \cup \Sigma_D. \quad (1.1.11)$$

Here by $dS$ we denote the bidimensional Hausdorff measure. Note that by integrating (1.1.11) by parts in space and time we retrieve (1.1.4), (1.1.10) and an initial condition for $u$. This problem will be studied in Section 3.

### 1.2. Classical formulation

The classical formulation of the Stefan problem is based on two main hypotheses, which are at variance with those that underlie the weak formulation:

(i) no mushy region is either initially present or is formed during the process,

(ii) the liquid and solid phases are separated by a regular surface that also evolves regularly.

On the other hand here the condition of local thermodynamic equilibrium is restricted to the phase interface. Consistently with the hypotheses (i), we assume that all the solid (either initially present or formed during the process) is in the crystalline state, so that it is free of latent heat; this is at variance with the behaviour of amorphous solids like glasses and polymers, see Section 2.2. Let us label quantities relative to the liquid and solid phases by 1 and 2, respectively, assume that the constitutive function $\hat{u}$ is differentiable, and use the following further notation:

- $Q_i$: open subset of $Q$ corresponding to the $i$th phase, for $i = 1, 2$,
- $S := \partial Q_1 \cap \partial Q_2$: (possibly disconnected) manifold of $\mathbb{R}^4$ representing the space–time points at solid–liquid interfaces,
- $S_t := S \cap (\Omega \times \{t\})$: configuration of the solid–liquid interface at the instant $t \in [0, T]$,
- $C_V := \partial u/\partial \theta$ (cf. (1.1.1)): heat capacity (at constant volume) per unit volume – namely, heat needed to increase the temperature of a unit volume by one degree; this equals the product between the mass density and the specific heat,
- $L := \partial u/\partial \chi$: density of latent heat of phase transition – namely, heat needed to melt a unit volume of solid.
We thus get
\[
\frac{\partial u}{\partial t} = C_V(\theta, \chi) \frac{\partial \theta}{\partial t} + \frac{L(\theta, \chi)}{2} \frac{\partial \chi}{\partial t} \text{ in } D'(Q). \tag{1.2.1}
\]
(Note that $\chi/2$ has a unit jump across the interface.) This relation is set in the sense of distributions, for the phase function $\chi$ is discontinuous across phase interfaces. In each phase the energy balance reads $\frac{\partial u}{\partial t} = -\nabla \cdot \vec{q} + f$. Denoting the heat capacity and the thermal conductivity in the phase $i$ by $C_{V_i}(\theta)$ and $k_i(\theta)$, we then retrieve the heat equation in each phase:
\[
C_{V_i}(\theta) \frac{\partial \theta}{\partial t} - \nabla \cdot \left[ k_i(\theta) \nabla \theta \right] = f \text{ in } Q_i \ (i = 1, 2). \tag{1.2.2}
\]

The Stefan condition. Let us assume that $S$ is sufficiently regular and that the temperature $\theta$ is continuous across the solid–liquid interface $S$. At any instant $t$ let us denote by $\vec{n} \in \mathbb{R}^3$ a unit vector field normal to $S_t$ oriented from the liquid to the solid, by $\vec{q}_i$ the heat flux per unit surface that is either contributed or absorbed by the $i$th phase through $S$. For instance, let us assume that in a small time interval $dt$ an element $dS$ of the phase interface advances with normal velocity $\vec{v} \cdot \vec{n}$ through the solid phase. Excluding any tangential contribution along the interface, the net heat flux absorbed by $dS$ in $dt$ then equals $dQ = (\vec{q}_1 \cdot \vec{n} - \vec{q}_2 \cdot \vec{n}) dS$. The melting process transforms this heat into an amount of latent heat that is proportional to the volume spanned by $dS$ in $dt$. Thus
\[
\vec{q}_1 \cdot \vec{n} - \vec{q}_2 \cdot \vec{n} = L(\theta) \vec{v} \cdot \vec{n} \text{ on } S. \tag{1.2.3}
\]
This equality also holds in case of freezing; in that case both members are negative, and represent the heat released at the solid–liquid interface. In either case, by the Fourier law, (1.2.3) yields the Stefan condition
\[
k_1(\theta) \frac{\partial \theta_1}{\partial n} - k_2(\theta) \frac{\partial \theta_2}{\partial n} = -L(\theta) \vec{v} \cdot \vec{n} \text{ on } S, \tag{1.2.4}
\]
where we denote by $\frac{\partial \theta_i}{\partial n}$ the normal derivative of $\theta$ on $S$ relative to the $i$th phase. If $g \in C^1(Q)$ is such that $S = \{(x, t) \in Q: g(x, t) = 0\}$ and $\nabla g \neq 0$, then $\nabla g \cdot \vec{v} + \frac{\partial g}{\partial t} = 0$ and $\vec{n} |\nabla g| = \nabla g$ on $S$ (possibly after inverting the sign of $g$). The condition (1.2.4) is then equivalent to
\[
\left[ k_1(\theta) \nabla \theta_1 - k_2(\theta) \nabla \theta_2 \right] \cdot \nabla g = L(\theta) \frac{\partial g}{\partial t} \text{ on } S. \tag{1.2.5}
\]

Metastability. In the framework of the classical formulation of the Stefan problem, we allow for the occurrence of metastable states at the interior of the phases, namely, undercooling (also called supercooling), i.e., $\theta < 0$ in the liquid, and superheating, i.e., $\theta > 0$ in the solid.

Nevertheless we assume local thermodynamic equilibrium at the phase interface. For a homogeneous material, neglecting surface tension effects this corresponds to
\[
\theta = 0 \text{ on } S. \tag{1.2.6}
\]
\[8\] The moving interface is not a material surface, and only the normal component of its velocity has a physical meaning.
In the Stefan condition (1.2.4) we may then replace $L(\theta)$ by $L(0)$ and $k_i(\theta)$ by $k_i(0)$ ($i = 1, 2$).

The evolution of the solid–liquid interface is unknown. In principle, this lack of information is compensated by setting two quantitative conditions at the free boundary $S$, namely, (1.2.4) and (1.2.6). Appropriate conditions on the initial value of $\theta$ and on the initial phase configuration must also be provided, as well as boundary conditions like (1.1.9) and (1.1.10). As an example we consider the following model problem, under natural regularity assumptions on the data $f, \theta_0, \theta_D, h$.

**PROBLEM 1.2.1 (Classical formulation of the multi-dimensional two-phase Stefan problem).** Find $\theta \in C^0(\bar{Q})$ and a partition $\{Q_1, Q_2, S\}$ of $Q$ such that:

(i) $Q_1$ and $Q_2$ are open sets;

(ii) $S = \partial Q_1 \cap \partial Q_2$ is a regular 3-dimensional manifold, and $S_t := S \cap (\Omega \times \{t\})$ is a regular surface, for any $t \in [0, T]$;

(iii) $\partial \theta / \partial t, \partial^2 \theta / \partial x_i \partial x_j$ (for $i, j \in \{1, 2, 3\}$) exist and are continuous in $Q_1$ and in $Q_2$;

(iv) the normal derivative $\partial \theta / \partial n$ exists on the respective sides of $S$;

(v) $\bar{S} \cap (\Omega \times \{0\})$ is prescribed, and

\[
C_{Vi}(\theta) \frac{\partial \theta}{\partial t} - \nabla \cdot [k_i(\theta) \cdot \nabla \theta] = f \quad \text{in } Q_i \ (i = 1, 2),
\]

\[
k_1 \frac{\partial \theta_1}{\partial n} - k_2 \frac{\partial \theta_2}{\partial n} = -L \vec{v} \cdot \vec{n} \quad \text{on } S,
\]

\[
\theta = 0 \quad \text{on } S;
\]

\[
\theta = \theta_D \quad \text{on } \Sigma_D, \quad k(\theta) \frac{\partial \theta}{\partial n} = h \quad \text{on } \Sigma_N,
\]

\[
\theta = \theta^0 \quad \text{in } \Omega \times \{0\}.
\]

(Here and elsewhere, $S$ and $S_t$ might be disconnected.) If the temperature vanishes identically in one of the phases, one often speaks of a one-phase Stefan problem.\(^9\) Besides phase transitions, this problem may represent a number of physical phenomena.\(^10\) If the source term $f$ vanishes identically, the occurrence of undercooled and superheated states may be excluded by assuming natural sign conditions on the initial and boundary data, because of the maximum and minimum principles.

**The one-dimensional Stefan problem.** Next we deal with a univariate system, e.g. an infinite slab, that we represent by a finite interval $\Omega = [a, b]$. We assume that $a < s^0 < b$, and that for instance the interval $[a, s^0]$ ($[s^0, b]$, resp.) represents the solid (liquid, resp.) phase at $t = 0$. If we exclude the formation of new phases, the solid–liquid interface $S$ then coincides with the graph of a function $s : [0, T] \rightarrow [a, b]$ such that $s(0) = s^0$. Let us

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9 This traditional terminology is slightly misleading, for there are two-phases, although the temperature is unknown just in one of them.

10 For instance, dissolution of a bubble gas in liquid, see e.g. Friedman [229, part III]; diffusion with chemical reaction, see e.g. Boley [70]; Darcy’s filtration through porous media, see e.g. Bear [52,53]; swelling of polymers, see e.g. Astarita and Sarti [32]. These and other examples are illustrated in the rich account of Primicerio [378].
assume that
\[ CV_i \in C^0(\mathbb{R}), \quad k_i \in C^1(\mathbb{R}), \quad C V_i, k_i > 0 \quad (i = 1, 2), \]
\[ f \in C^0(\bar{Q}), \quad \theta_a, \theta_b \in C^0([0, T]), \quad \theta_a < 0, \quad \theta_b > 0, \]
\[ \theta^0 \in C^0([a, b]), \quad \theta^0 < 0 \text{ in } ]a, s^0[, \quad \theta^0 > 0 \text{ in } ]s^0, b[. \]

The previous equations coupled with natural initial and boundary conditions yield the following problem; cf. Figure 2.

**PROBLEM 1.2.2 (Classical formulation of the one-dimensional two-phase Stefan problem).** Find
\[ s \in C^0([0, T]) \cap C^1([0, T]) \text{ and } \theta \in C^0(\bar{Q}) \]
such that, setting
\[ Q_i := \{ (x, t) \in Q : x > s(t) \} \quad (i = 1, 2), \]
\[ \frac{\partial \theta}{\partial t}, \frac{\partial^2 \theta}{\partial x^2} \in C^0(Q_i) \quad (i = 1, 2), \]
the limits \([k_i(\theta) \frac{\partial \theta}{\partial x}](s(t) \pm 0, t)\) exist for any \( t \in ]0, T[ \), and
\[ CV_i(\theta) \frac{\partial \theta}{\partial t} - \frac{\partial}{\partial x} \left( k_i(\theta) \frac{\partial \theta}{\partial x} \right) = f \quad \text{in } Q_i \quad (i = 1, 2), \]
\[ \left( k_1(\theta) \frac{\partial \theta}{\partial x} \right)(s(t) + 0, t) - \left( k_2(\theta) \frac{\partial \theta}{\partial x} \right)(s(t) - 0, t) = -L(\theta) \frac{ds}{dt}(t) \quad \text{for } 0 < t < T, \]
\[ \theta(s(t), t) = 0 \quad \text{for } 0 < t < T, \]
\[ \theta(a, t) = \theta_a(t), \quad \theta(b, t) = \theta_b(t) \quad \text{for } 0 < t < T, \]
\[ s(0) = s^0, \quad \theta(x, 0) = \theta^0(x) \quad \text{for } a < x < b. \]

1.3. **Comparison between the weak and the classical formulation**

Despite of the terminology, in general the classical and the weak formulations of the Stefan problem (CSP and WSP, resp.) are not different formulations of the same physical model, and rest upon different physical assumptions.

*An ideal experiment.* The next example looks especially enlightening of the difference between the CSP and the WSP. Let a solid system be initially at a uniform temperature,
\[ \theta(\cdot, 0) = \theta^0 < 0, \text{ and be exposed to a constant and uniform heat source of intensity} \]
\[ \hat{f} = 1, \text{ e.g. by infrared radiation. If the system is thermally insulated, according to the} \]
\[ \text{WSP the temperature remains uniform in} \ \Omega, \text{ and the energy balance (1.1.4) is reduced to} \]
\[ \text{the ordinary differential equation} \ \frac{d\theta}{dt} = 1 \text{, namely,} \]
\[ (1.3.1) \]
\[ CV(\theta, \chi) \frac{d\theta}{dt} + \frac{L(\theta) d\chi}{2} = 1 \quad \text{in} \ [0, T]. \]

Initially the temperature thus increases linearly. As \( \theta \) vanishes it stops, melting starts without superheating, and \( \chi \) increases linearly in time from \(-1\) to \(1\), uniformly throughout \( \Omega \).

For a time-interval the temperature remains null, and the whole body consists of a mush with increasing liquid content. As \( \chi \) reaches the value \(1\), this mush has completed the liquid transformation, and the temperature increases again. Thus, according to the WSP, no phase interface is formed, and phase transition occurs throughout by the transformation process “solid \(\rightarrow\) mush \(\rightarrow\) liquid.”

The CSP provides a completely different picture: no front of phase transition is formed, and \( \theta \) just grows linearly in time, so that the solid is indefinitely superheated. Physically this is obviously unrealistic; it would be more reasonable to expect that a liquid phase is nucleated as a certain threshold is attained. But this behaviour is not accounted for by the CSP.

**Equivalence between the CSP and the WSP.** Let us now exclude the occurrence of mushy regions and of metastable states, and assume that the solid–liquid interface is regular and evolves regularly, so that both the weak and the classical formulation apply. The next statement bridges the two models.

**Proposition 1.3.1.** Let the pair \((\theta, S)\) fulfill the regularity conditions of Problem 1.2.1, and let \( \chi \) fulfill (1.1.7). The system (1.2.2), (1.2.4) is then equivalent to the distributional equation
\[ (1.3.2) \]
\[ CV(\theta, \chi) \frac{\partial \theta}{\partial t} + \frac{L(\theta) \partial \chi}{2} - \nabla \cdot [k(\theta, \chi) \cdot \nabla \theta] = f \quad \text{in} \ D'(Q). \]

**Proof.** Let us denote by \( \vec{v} := (\vec{v}_x, v_t) \in \mathbb{R}^4 \) the unit vector field normal to \( S \), oriented towards \( Q_1 \), say. As the vector \((\nabla g, \partial g/\partial t)\) is parallel to \( \vec{v} \), the Stefan condition (1.2.5) also reads
\[ (1.3.3) \]
\[ [k_1(\theta) \cdot \nabla \theta_1 - k_2(\theta) \cdot \nabla \theta_2] \cdot \vec{v}_x = L(\theta) v_t \quad \text{on} \ S. \]

Denoting the duality pairing between \( D'(Q) \) and \( D(Q) \) by \( \langle \cdot, \cdot \rangle \), a simple calculation yields
\[ (1.3.4) \]
\[ \int_Q \left\{ CV(\theta, \chi) \frac{\partial \theta}{\partial t} + \frac{L(\theta) \partial \chi}{2} - \nabla \cdot [k(\theta, \chi) \cdot \nabla \theta] \right\} \varphi \ dx \ dt \]
\[ = \int_Q \int_S \left\{ CV(\theta, \chi) \frac{\partial \theta}{\partial t} \varphi - \chi \frac{\partial L(\theta) \varphi}{\partial t} + [k(\theta, \chi) \cdot \nabla \theta] \cdot \nabla \varphi \right\} \ dx \ dt \]
\[ \pm \int_S \left\{ L(\theta) v_t - \vec{v}_x \cdot [k_1(\theta) \cdot \nabla \theta_1 - k_2(\theta) \cdot \nabla \theta_2] \right\} \varphi \ dS \quad \forall \varphi \in D(Q). \]
Table 1
Comparison between the classical and the weak formulations of the basic Stefan model, i.e., Problems 1.1.1 and 1.2.1

<table>
<thead>
<tr>
<th></th>
<th>Classical formulation (CSP)</th>
<th>Weak formulation (WSP)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy balance</td>
<td>heat equation in $Q_1$, $Q_2$</td>
<td>energy balance in $\mathcal{D}'(Q)$</td>
</tr>
<tr>
<td>Stefan condition on $S$</td>
<td>$\theta, S$</td>
<td>$\theta, \chi$</td>
</tr>
<tr>
<td>Unknowns</td>
<td>$\theta \equiv 0$ on $S$</td>
<td>$\chi \in \text{sign(\theta)}$ a.e. in $Q$</td>
</tr>
<tr>
<td>Phase characterization</td>
<td>global, via $S$</td>
<td>local, via sign(\theta)</td>
</tr>
<tr>
<td>Mushy regions</td>
<td>excluded</td>
<td>allowed</td>
</tr>
<tr>
<td>Metastable states</td>
<td>allowed</td>
<td>excluded</td>
</tr>
<tr>
<td>Analytical features</td>
<td>free boundary problem</td>
<td>degenerate PDE</td>
</tr>
<tr>
<td>Function spaces</td>
<td>classical $C^k$-spaces</td>
<td>Sobolev spaces</td>
</tr>
<tr>
<td>Well-posedness for $\Omega \subset \mathbb{R}$</td>
<td>for any time</td>
<td>for any time</td>
</tr>
<tr>
<td>Well-posedness for $\Omega \subset \mathbb{R}^3$</td>
<td>just for small time</td>
<td>for any time</td>
</tr>
</tbody>
</table>

The selection of the sign of the latter integral depends on the orientation of $\vec{v}$. The system (1.2.2), (1.2.4) is thus equivalent to (1.3.2). (Loosely speaking, in the latter equation the Dirac-type masses of the second and third addendum cancel each other.) □

Comparison of analytical properties. The CSP consists of nondegenerate equations set in unknown domains, hence it is a genuine free boundary problem. On the other hand in the WSP the domain is fixed but the equation is degenerate.

The one-dimensional CSP is well posed in any time interval, under natural assumptions. On the other hand in several space dimensions in general the CSP has a solution only in a small time interval. It is true that under suitable quantitative restrictions on the data a solution exists in any time interval, and may also be very regular. But if one excludes special configurations, in general the classical solution may fail after some time, even if the heat source term $f$ vanishes identically. Actually, discontinuities may occur in the temperature evolution as the topological properties of the phase interface change: e.g., a connected component may split into two components, or conversely the latter may merge into a single one. On the other hand under simple hypotheses the WSP is well posed in any time interval in any number of space dimensions, see Section 3.1, and may also be solved numerically by means of standard techniques. These differences are summarized in Table 1.

1.4. A Stefan-type problem arising in ferromagnetism

Phase transitions occur in many physical phenomena. Here we outline a macroscopic model of ferromagnetism without hysteresis that is reminiscent of the Stefan model, although in this case the unknown field is a vector and the equations have a different structure.

---

11 See e.g. Meirmanov [331].
12 See e.g. Borodin [73], Prüss, Saal and Simonett [380]. Compare also with Friedman and Kinderlehrer [236], Nochetto [358].
13 Problems of this sort were studied e.g. by Bossavit [75–78], Bossavit and Damlamian [79], Visintin [448,449].
Let the domain $\Omega$ be occupied by a ferromagnetic material, denote the magnetic field by $\vec{H}$, the magnetization by $\vec{M}$, and the magnetic induction by $\vec{B}$; in Gauss units, $\vec{B} = \vec{H} + 4\pi \vec{M}$. Let us also denote the electric field by $\vec{E}$, the electric displacement by $\vec{D}$, the electric current density by $\vec{J}$, the electric charge density by $\hat{\rho}$, and the speed of light in vacuum by $c$.

Dealing with electromagnetic processes, in general it is not natural to formulate a boundary-value problem in a Euclidean domain. In fact the exterior evolution may affect the interior process, and it is not easy to account for this interaction at a distance by prescribing appropriate boundary conditions. For this reason we rather set the Maxwell equations in the whole space $\mathbb{R}^3$, and assume different constitutive relations inside and outside $\Omega$. The Ampère–Maxwell, Faraday and Gauss laws respectively read

$$c \nabla \times \vec{H} = 4\pi \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \text{in } Q_\infty := \mathbb{R}^3 \times ]0, T[,$$

$$c \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{in } Q_\infty \quad (\nabla \times := \text{curl}),$$

$$\nabla \cdot \vec{B} = 0, \quad \nabla \cdot \vec{D} = 4\pi \hat{\rho} \quad \text{in } Q_\infty.$$  

These equations must be coupled with appropriate constitutive relations, with initial conditions for $\vec{D}$ and $\vec{B}$, and with suitable restrictions on the behaviour of $\vec{H}$ and $\vec{E}$ at infinity.\textsuperscript{14} We assume that the magnetic material is surrounded by air, and that $\vec{J}$ equals a prescribed time-dependent field, $\vec{J}_{\text{ext}}$, outside $\Omega$; this might be due for instance to an electric current circulating in an exterior conductor. We extend this field by setting $\vec{J}_{\text{ext}} := \vec{0}$ in $\Omega$. We also assume that a prescribed electromotive force $\vec{E}_{\text{app}}$, that may be due e.g. to an electric generator, is applied to the system. Denoting by $\sigma$ the electric conductivity, the Ohm law then reads

$$\vec{J} = \sigma (\vec{E} + \vec{E}_{\text{app}}) + \vec{J}_{\text{ext}} \quad \text{in } Q_\infty.$$  

We assume that $\sigma = 0$ outside $\Omega$, and that the field $\vec{E}$ does not vary too rapidly in $\Omega$.\textsuperscript{15} On the other hand in metals the conductivity $\sigma$ is very large. In Eq. (1.4.1) in $\Omega$ the Ohmic current $\vec{J}$ thus dominates the displacement current $\partial \vec{D}/\partial t$, which may then be neglected; this is named the eddy-current approximation. As

$$\vec{D} = \epsilon \vec{E} \quad \text{in } Q,$$  

with a constant electric permittivity $\epsilon$, (1.4.1) then yields

$$c \nabla \times \vec{H} = 4\pi \sigma (\vec{E} + \vec{E}_{\text{app}}) \quad \text{in } Q,$$

$$c \nabla \times \vec{H} = 4\pi \vec{J}_{\text{ext}} + \epsilon \frac{\partial \vec{E}}{\partial t} \quad \text{in } Q_\infty \setminus Q.$$  

By (1.4.2) we may eliminate the field $\vec{E}$, and thus get

\textsuperscript{14} In the functional formulation, the behaviour at infinity is implicit in the Sobolev spaces.

\textsuperscript{15} This is an a priori assumption on the unknown field $\vec{E}$, and should be verified a posteriori.
Fig. 3. Constitutive relation between the moduli of the colinear vectors $\vec{H}$ and $\vec{B}$, for an (isotropic) ferromagnetic material with negligible hysteresis.

\[ 4\pi\sigma \frac{\partial \vec{B}}{\partial t} + c^2 \nabla \times \nabla \times \vec{H} = 4\pi c \sigma \nabla \times \vec{E}_{\text{app}} \quad \text{in } Q, \]
\[ \epsilon \frac{\partial^2 \vec{B}}{\partial t^2} + c^2 \nabla \times \nabla \times \vec{H} = 4\pi c \sigma \nabla \times \vec{J}_{\text{ext}} \quad \text{in } Q_{\infty} \setminus Q. \] (1.4.7)

However, as we said, we shall not formulate the problem separately inside and outside the domain $\Omega$. Although ferromagnetic materials exhibit hysteresis, soft iron with high field-saturation and other metals are characterized by so narrow a hysteresis loop, that in first approximation this may be replaced by a maximal monotone graph; see Section 5.5. Let us then prescribe the following constitutive relation:

\[ \vec{B} \in \vec{H} + 4\pi\mathcal{M}\vec{\beta}(\vec{H}) \quad (=: \vec{\bar{F}}(\vec{H})) \quad \text{in } Q, \quad \vec{B} = \vec{\bar{F}} \quad \text{in } Q_{\infty} \setminus Q, \] (1.4.8)

where $\mathcal{M}$ is a positive constant and $\vec{\beta}$ is the subdifferential of the modulus function:

\[ \vec{\beta}(\vec{v}) := \begin{cases} \{ \frac{\vec{v}}{|\vec{v}|} \} & \text{if } \vec{v} \neq \vec{0}, \\
\{ \vec{v} \in \mathbb{R}^3 : |\vec{v}| \leq 1 \} & \text{if } \vec{v} = \vec{0}, \end{cases} \quad \forall \vec{v} \in \mathbb{R}^3; \] (1.4.9)

cf. Figure 3 (see Section 5.2). In this case, the unmagnetized and magnetically saturated phases are respectively characterized by $\vec{B} = \vec{0}$ and $|\vec{B}| \geq 4\pi\mathcal{M}$. In general, the occurrence of a mixed phase characterized by $0 < |\vec{B}| < 4\pi\mathcal{M}$ in a subdomain of $\Omega$ (a sort of magnetic mushy region) is not a priori excluded.

More generally, we may assume that $\vec{\bar{F}} : \mathbb{R}^3 \to 2\mathbb{R}^3$ is a (possibly multi-valued) maximal monotone mapping.

The system (1.4.2)–(1.4.6), (1.4.8), (1.4.9) is a vector parabolic–hyperbolic problem. More precisely, it is quasilinear parabolic in $Q$ and semilinear hyperbolic in $Q_{\infty} \setminus Q$. The former setting may be compared with Problem 1.1.1, namely the weak formulation of the Stefan problem: the vector fields $\vec{H}$, $\mathcal{M}$, and $\vec{B}$ play similar roles to those of the scalar variables $\theta$, $\chi$ and $u$, respectively.
If the system has planar symmetry, that is, if all variables only depend on two space coordinates \((x, y)\), and if the fields \(\vec{H}\) and \(\vec{B}\) are parallel to the orthogonal \(z\)-axis, then they may be represented by their \(z\)-components, \(H\) and \(B\). In this case in the second-order equation (1.4.7) the operator \(\nabla \times \nabla \times\) equals \(-\Delta\) (here the bidimensional Laplace operator), and Eqs. (1.4.7) are thus reduced to

\[
4\pi \sigma \frac{\partial B}{\partial t} - c^2 \Delta H = g_1 \quad \text{in } Q,
\]

\[
\varepsilon \frac{\partial^2 B}{\partial t^2} - c^2 \Delta H = g_2 \quad \text{in } Q_\infty \setminus Q,
\]

for prescribed scalar fields \(g_1\) and \(g_2\).

A vector free boundary problem. If \(\vec{F}\) is multi-valued, then formally the system (1.4.2)–(1.4.6), (1.4.8), (1.4.9) is the weak formulation of a free boundary problem. In general it is not obvious a priori that the magnetically saturated and unsaturated phases are separated by an interface, even under regularity hypotheses. However under appropriate restrictions (e.g., planar symmetry) we are reduced to a scalar problem, for which conditions are known that guarantee the existence of an interface. The next statement concerning the free boundary conditions may be compared with Proposition 1.3.1.

**PROPOSITION 1.4.1** (Discontinuity conditions). Let us assume that:

(i) \(S \subset Q\) is a smooth 3-dimensional manifold, and \(S_t := S \cap (\Omega \times \{t\})\) is a (possibly disconnected) smooth surface, for any \(t \in ]0, T[\);

(ii) \(\vec{B}, \vec{H}, \partial \vec{B}/\partial t, \nabla \times \nabla \times \vec{H} \in L^1(Q \setminus S)^3\);

(iii) the traces of \(\vec{B}\) and \(\nabla \times \vec{H}\) exist on both sides of \(S\).

For any \(t \in [0, T]\) let us denote by \(\vec{v} \in \mathbb{R}^3\) a unit vector field normal to \(S_t\), by \(v := \vec{v} \cdot \vec{v}\) the (normal) speed of \(S_t\), and by \([\ ]\) the difference between the traces on the two sides of \(S_t\). Let us also assume that:

(iv) \(\vec{v} \times [\vec{H}] = 0\) a.e. on \(S\).\(^{16}\)

Then Eq. (1.4.7) in the sense of distributions is equivalent to the same equation pointwise in \(Q \setminus S\), coupled with the Rankine–Hugoniot-type condition

\[
4\pi \sigma [\vec{B}] = c^2 \vec{v} \times [\nabla \times \vec{H}] \quad \text{a.e. on } S.
\]

This statement may be checked via a similar argument to that of Proposition 1.3.1, that we omit. Moreover, the Gauss law \(\nabla \cdot \vec{B} = 0\) and the identity \(\nabla \cdot (\nabla \times \vec{H}) = 0\) in the sense of distributions entail that

\[
\vec{v} \cdot [\vec{B}] = 0, \quad \vec{v} \cdot [\nabla \times \vec{H}] = 0 \quad \text{a.e. on } S.
\]

**1.5. Other Stefan-type problems**

*The quasi-steady Stefan problem and the Hele–Shaw problem.* If either the heat capacity \(C_V\) is very small or the temperature evolves very slowly, then one may replace the heat

\(^{16}\) By the Ampère law (1.4.1), this assumption is equivalent to the absence of surface currents.
equation (1.1.4) by the quasi-stationary equation
\[
- \nabla \cdot \left[ k(\theta, \chi) \cdot \nabla \theta \right] = f \quad \text{in } Q_i \ (i = 1, 2). \tag{1.5.1}
\]
The energy balance (1.1.4) is then reduced to
\[
\frac{L(\theta)}{2} \frac{\partial \chi}{\partial t} - \nabla \cdot \left[ k(\theta, \chi) \cdot \nabla \theta \right] = f \quad \text{in } \mathcal{D}'(Q), \tag{1.5.2}
\]
in place of (1.1.4). Of course as an initial condition here one must just specify \( \chi(\cdot, 0) \). This also applies to material diffusion in heterogeneous systems, for the time-scale of mass-diffusion is rather small (much smaller than that of heat-diffusion); see Section 2.3.

This setting is also known as the \textit{Hele–Shaw problem}, since in the two-dimensional case it represents the evolution of a Hele–Shaw cell. This consists of two slightly separated parallel plates partially filled with a viscous fluid. If some fluid is injected into the cell with a syringe the fluid expands, and the evolution of the pressure \( p \) may be represented by
\[
\begin{cases}
\frac{\partial \chi}{\partial t} - \nabla \cdot \left[ k(p, \chi) \cdot \nabla p \right] = f \quad \text{in } \mathcal{D}'(Q), \\
\chi \in \text{sign}(p) \quad \text{a.e. in } Q,
\end{cases}
\tag{1.5.3}
\]
with \( f \geq 0 \).\footnote{See e.g. Ambrose [21], Antontsev, Meirmanov and Yurinsky [27], DiBenedetto and Friedman [176], Elliott and Janovsky [192], Escher and Simonett [195,196], Lacey et al. [303], Howison [277], Kim [288,289], Richardson [386,387], Saffman and Taylor [416].} This model may also represent the industrial process of \textit{electrochemical machining}, by which a metal body is either machined or formed by using it as an anode in an electrolytic cell.\footnote{See e.g. Alexiades and Cannon [7], Elliott [189], McGeough [325], McGeough and Rasmussen [326].} A rather different setting is obtained if the fluid is extracted from the Hele–Shaw cell. In this case \( f \leq 0 \) and \( p \leq 0 \), and the condition (1.5.3)\textsubscript{2} must be replaced by
\[
\chi \in \text{sign}(-p) \quad \text{a.e. in } Q. \tag{1.5.4}
\]
This problem is known as the \textit{inverse Hele–Shaw problem}, since it is equivalent to a \textit{backward} Hele–Shaw problem, and is ill-posed.\footnote{See e.g. DiBenedetto and Friedman [176], Nie and Tian [351].}

\textbf{The hyperbolic Stefan problem.} It is well known that the heat equation represents instantaneous propagation of heat, at variance with one of the main issues of the Einstein theory of relativity. As most of applications of the Stefan problem do not involve relativistic velocities, this shortcoming has no practical relevance. Anyway this drawback may be eliminated by replacing the Fourier conduction law (1.1.3) by a suitable relaxation dynamics. Here we illustrate four alternatives, and refer to Joseph and Preziosi [281,282] for a detailed review of conduction laws and associated \textit{heat waves} (in the linear setting). For the sake of simplicity, throughout this discussion we shall assume that \( k \) is a positive constant scalar.\footnote{Here the relevant assumption is the independence of \( \chi \), for that on \( \theta \) may be treated via the \textit{Kirchhoff transformation} (3.1.15).}
(i) In alternative to (1.1.3), one may use the Cattaneo law
\[ \tau \frac{\partial \vec{q}}{\partial t} + \vec{q} = -k \nabla \theta \quad \text{in } Q, \] (1.5.5)
where \( \tau \) is a relaxation constant. The parabolic system (1.1.4), (1.1.8) is accordingly replaced by the quasilinear hyperbolic system
\[ \begin{cases} \tau \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - k \Delta \theta = \tau \frac{\partial f}{\partial t} + f & \text{in } D'(Q), \\ u \in \alpha(\theta) \quad \text{in } Q, \end{cases} \] (1.5.6)
still with \( \alpha \) a maximal monotone function \( R \to 2^R \). The analysis of this problem is rather challenging. However, usually \( \tau \) is so small that for most applications the Fourier approximation is acceptable.\(^{21}\)

(ii) One may also insert a further relaxation term into the Cattaneo law (1.5.5):
\[ \tau \frac{\partial \vec{q}}{\partial t} + \vec{q} = -k \nabla \theta - k_1 \frac{\partial \nabla \theta}{\partial t} \quad \text{in } Q. \] (1.5.7)
By coupling this equation with the energy balance law (1.1.2), we get a third-order differential equation:
\[ \tau \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - k \Delta \theta - k_1 \frac{\partial \Delta \theta}{\partial t} = \tau \frac{\partial f}{\partial t} + f \quad \text{in } Q, \] (1.5.8)
that must then be coupled with the inclusion (1.1.8). Here one may prove well-posedness of a weak formulation for an associated boundary- and initial-value problem.

(iii) After Gurtin and Pipkin [261] one may also consider a dynamics with memory:
\[ \vec{q}(x,t) = -\int_0^{+\infty} h(s) \nabla \theta(x,t-s) \, ds \quad \text{for } (x,t) \in Q, \] (1.5.9)
for a prescribed positive-definite, decreasing and integrable kernel \( h(s) \). This equation is more general than the Cattaneo law, that is retrieved for \( h(s) = (k/\tau^2) \exp\{-s/\tau\} \). The analysis of the integro-differential problem that is obtained by coupling the energy balance law (1.1.2) with the inclusion (1.1.8) and with (1.5.9) exhibits difficulties comparable to those of the quasilinear hyperbolic system (1.5.6).

(iv) On the other hand, if after Coleman and Gurtin [131] one also allows for the occurrence of a Dirac mass \( \delta_0 \) in the kernel \( h \), by assuming
\[ \vec{q}(x,t) = -\frac{k_1}{\tau} \nabla \theta(x,t) + \left( \frac{k_1}{\tau^2} - \frac{k}{\tau} \right) \int_0^{+\infty} e^{-s/\tau} \nabla \theta(x,t-s) \, ds \quad \text{for } (x,t) \in Q \] (1.5.10)
one then retrieves the (more feasible) third-order equation (1.5.7).

---

\(^{21}\) See e.g. Colli and Recupero [141]. The physical aspects of hyperbolic and parabolic models are discussed e.g. in Herrera and Pavón [265]. Incidentally notice that, although the occurrence of a relaxation term like \( \tau \partial \vec{q}/\partial t \) might be expected to have effects just on a (short) transient, in [265, p. 122] it is maintained that “transient phenomena may affect the way in which the system leaves the equilibrium, thereby affecting the future of the system even for time scales much larger than the relaxation time.”
Initial- and boundary-value problems for the above equations were analysed in several works.\textsuperscript{22}

We represented phase transitions in an extremely simplified way, neglecting physically relevant aspects like stress and deformation in the solid, convection in the liquid, change of density, microforces, and so on.\textsuperscript{23} In Section 2 we shall address several other extensions of the Stefan model.

1.6. Historical note

In 1831 Lamé and Clapeyron\textsuperscript{306} formulated what seems to be the first model of phase transition.\textsuperscript{24} However the basic mathematical model of this phenomenon is traditionally named after the Austrian physicist Josef Stefan, who in 1889 studied the melting of the polar ices, dealing with several aspects of the one- and two-phase problem in a single dimension of space, see [430].

The first result of existence of a solution (for a large class of data) is due to L. Rubinstei n, who formulated the one-dimensional two-phase Stefan problem in 1947 in terms of a system of integral equations, and proved existence and uniqueness of a solution in a small time interval [406–408]. Further integral formulations of the one-dimensional Stefan problem were then studied.\textsuperscript{25}

Several techniques were used to prove well-posedness, results of approximation, regularity, asymptotic behaviour, and other properties.\textsuperscript{26} Many physically motivated generalizations were also investigated, and a fairly satisfactory understanding of a large class of single-dimensional problems was thus achieved.

The early research on the Stefan problem concentrated on the classical formulation of the univariate model. The introduction of weak formulations for nonlinear partial differential equations in the 1950s provided the key tool for the extension of the Stefan problem to the multi-dimensional setting in the early 1960s. The first results in this direction were achieved by Kamenomostskaya\textsuperscript{283} and Oleı˘nık\textsuperscript{364}.\textsuperscript{27} Although these pioneering works were followed by an extensive research, for some time this new trend was somehow controversial, since for some researchers just the classical formulation was the genuine mathematical model of phase transitions. This also prompted the study of the regularity of the weak solution.

\textsuperscript{22} See e.g. Aizicovici and Barbu [5], Barbu [48], Colli and Grasselli [134–137], Friedman [235], Showalter and Walkington [422].
\textsuperscript{23} The convection in phase transitions was studied e.g. in Cannon and DiBenedetto [113], Cannon, DiBenedetto and Knightly [114,115], Casella [118], DiBenedetto and Friedman [177], DiBenedetto and O’Leary [178], Hoffmann and Starovoitov [273], Rodrigues [398], Rodrigues and Urbano [399,400], Wang [462], Xu and Shillor [473]. Thermodynamic theories of phase transitions in presence of microforces were developed e.g. in Bonetti and Frémond [72], Frémond [224], Fried and Gurtin [226–228], Gurtin [256].
\textsuperscript{24} The process of technical solidification is older. The first cast objects (in copper) date back to more than 6000 years ago, ...
\textsuperscript{25} See Evans [199], Sestini [420], Friedman [229], Kolodner [295], Jiang [280], and others.
\textsuperscript{26} See e.g. Cannon and Hill [116], Friedman [230, Chapter 8] and [231–233], Fasano and Primicerio [210,211], Fasano, Primicerio and Kamin [215], Rubinstein, Fasano and Primicerio [411], Schaeffer [418], and others.
\textsuperscript{27} See also Ladyženskaja, Solonnikov and Ural’ceva [305, Section V.9].
Results on the regularity of the solid–liquid interface for the multi-dimensional one-phase Stefan model were obtained by reformulating the problem as a variational inequality by means of a variable transformation due to Baiocchi [44,45], Duvaut [184], and Frémond [223]. Under appropriate restrictions Friedman and Kinderlehrer [236], Caffarelli [96,97], Kinderlehrer and Nirenberg [290,291] proved that the weak solution is also classical. Continuity of the temperature was showed by Caffarelli and Friedman [99] for the multi-dimensional one-phase problem. An analogous result was obtained by DiBenedetto [174,175], Ziemer [477], and Caffarelli and L.C. Evans [98] for the two-phase problem. In [323] Matano proved that any weak solution of a one-phase Stefan problem in an exterior region eventually becomes a classical solution after a finite time, and that the shape of the free boundary approaches that of a growing sphere as \( t \to +\infty \).

In 1979 Meirmanov [327,328] proved the existence of the classical solution of the multi-dimensional two-phase Stefan problem in a small time interval; see also [329,331]. An analogous result was also shown by Hanzawa [263] for the one-phase problem by means of the Nash–Moser regularity theory.

Mushy regions were first investigated for the one-dimensional Stefan problem by Atthey [38] (who introduced that denomination), Lacey and Tayler [304], Fasano and Primicerio [214], Meirmanov [329,330], Primicerio [379], and others. After the introduction of weak solutions, these regions were also studied in several space dimensions by Andreucci [23], Bertsch, De Mottoni and Peletier [62], Bertsch and Klaver [63], Götz and Zaltzman [247,248], Lacey and Herraiz [301,302], Rogers and Berger [403] (see also Berger, Brezis and Rogers [60]), and in several other papers. See also the survey [205] of Fasano.

Free boundary problems. We already pointed out that the Stefan problems is a free boundary problem (FBP). Many other FBPs were formulated and studied in the last decades. Examples also include more general models of phase transitions, see Section 2. Free boundaries also occur as fronts between saturated and unsaturated regions in filtration through porous media, between plastic and elastic phases in continuous mechanics, between conducting and superconducting phases in electromagnetism, just to mention few cases. Relevant examples also come from reaction–diffusion, fluid dynamics, biomathematics, and so on.

Since the early years, the research on Stefan-type problems stimulated and was paralleled by that on other FBPs. Several of these problems are of industrial interest, and offer opportunities of collaboration among mathematicians, physicists, engineers, material 28 This transformation is illustrated in Section 3.3.
29 The existence of regular viscosity solutions was proved by Athanasopoulos, Caffarelli and Salsa, see [33–36,100]. Regularity results were also obtained by DiBenedetto and Vespri [180], Koch [294], Borodin [73], Bizhanova [64], Bizhanova and Rodrigues [65], Bizhanova and Solonnikov [66], Prüss, Saal and Simonett [380], and others.
30 The development of the mathematical analysis of phase transitions would hardly have been conceivable without the achievements of mathematical-physicists and applied scientists. We just select a small sample from a huge literature: Cahn [106–108], Cahn and Hilliard [109], Collins and Levine [146], Frémond [224], Fried and Gurtin [226–228], Gurtin [250–258], Gurtin and Soner [262], Hilliard [268], Langer [308,309], Mullins and Sekerka [346,347], Penrose and Fife [373,374], Romano [404], Wang et al. [461].
scientists, biologists, and other researchers. A large community of mathematicians, engineers and applicative scientists spread over the world has been formulating and studying those problems for many years, and have regularly been meeting in major conferences. The proceedings of those conferences provide a comprehensive picture of the development of research on FBPs in the last decades; see the item (V) of the Bibliographical Note in Section 6.

2. More general models of phase transitions

As we saw, the classical and the weak formulations of the Stefan problem are both based on a number of simplifying assumptions. Even under favourable circumstances, these models should then be regarded just as first approximations of melting and freezing processes, both from the qualitative and quantitative viewpoint. Nevertheless the Stefan model is the basis for the construction of more refined models of phase transitions, since heat-diffusion and exchange of latent heat underlie any (first-order) phase transition.

In this second part we illustrate some physically justified extensions of the Stefan model. We amend the basic Stefan problem by inserting the Gibbs–Thomson law, and derive the latter by defining a suitable free energy functional. We then replace the equilibrium conditions (1.1.7) and (1.2.6) (for the weak and classical formulations of the Stefan model, respectively) by a kinetic law, that accounts for decay towards local equilibrium.

Next we concentrate our attention upon phase transitions in binary alloys. First, we outline a model that is often used in engineering, that essentially consists in coupling heat and mass-diffusion, and point out some physical and mathematical drawbacks. We then introduce an alternative and more satisfactory model, in which the constitutive laws are formulated consistently with the second principle, along the lines of the theory of nonequilibrium thermodynamics.

Finally, we outline the phase-field model and the Cahn–Hilliard equation for phase separation, and relate models set at different length-scales by means of De Giorgi’s notion of \( \Gamma \)-limit (cf. Section 5.8).

2.1. The Gibbs–Thomson law

**Undercooling and superheating.** So far we dealt with phase transitions in pure materials, assuming local thermodynamic equilibrium at the solid–liquid interface, and neglecting surface tension effects. If these restrictions are dropped, then the interface (relative) temperature, \( \theta \), need not vanish. In the framework of the classical formulation, the interface condition (1.2.6) is actually replaced by a more general law of the form

\[
\theta = \theta_{\text{s,l}} + \theta_{\text{n,e.}} + \theta_{\text{imp.}} \quad \text{on } S. 
\]  

(2.1.1)

The first term on the right accounts for surface tension, and is proportional to the mean curvature of the solid–liquid interface. The second contribution is related to deviations from local thermodynamic equilibrium, and depends on the rate of phase transition. The third one accounts for the presence of secondary components (so-called impurities). The two latter corrections are especially relevant for applications to metallurgy and to other engineering processes.
By the continuity of the temperature at the solid–liquid interface, Eq. (2.1.1) entails the onset of undercooling and/or superheating in the interior of the phases, so that here the temperature-phase rule (1.1.5) necessarily fails. Next we shall examine the above three terms separately.

**The Gibbs–Thomson law.** First we deal with the term $\theta_{sL}$. Let us assume that at any instant $t$ the solid–liquid interface $S_t$ is a surface of class $C^2$, denote by $\kappa$ its mean curvature (assumed positive for a convex solid phase). The interface condition (1.2.6) may then be replaced by the *Gibbs–Thomson law* 

$$\theta = -\frac{2\sigma \tau_E}{L} \kappa \text{ on } S.$$  

(2.1.2)

The quantities $\tau_E$ and $L$ were already introduced in Section 1.1. $\sigma$ is known as the coefficient of surface tension (or capillarity), and is equal to the surface density of the free energy at the solid–liquid interface, see Section 2.5. For the sake of simplicity, we shall assume that $L$ and $\sigma$ are constant.

For water at atmospheric pressure at about $0^\circ C$, $2\sigma \tau_E / L$ is of the order of $10^{-5}$ cm, so that the deviation from the null temperature is significant just for mesoscopic curvature radii. The effects of the Gibbs–Thomson law are nevertheless perceivable also at the macroscopic length-scale, for it accounts for the undercooling prior to solid nucleation.\(^{31}\)

**Contact angle condition.** The curvature condition (2.1.2) may be represented by a second-order elliptic equation for the (local) parametric formulation of the solid–liquid interface $S$. For any $t \in [0, T]$ it is natural to associate to this equation a condition at the line of contact between $S_t$ and the boundary $\Gamma$ of the domain $\Omega$. For any $(x, t) \in \bar{S} \cap (\Gamma \times [0, T])$, let us denote by $\omega(x, t)$ the angle formed by the normal to $S_t$, oriented towards the liquid phase $\Omega_1(t)$, and the outward normal to $\Omega$ at $x$. We thus prescribe the *contact angle condition* 

$$\cos \omega = \frac{\sigma_S - \sigma_L}{\sigma} \text{ on } \bar{S} \cap (\Gamma \times [0, T]),$$  

(2.1.3)

where $\sigma_L$ and $\sigma_S$ (here also assumed to be constant) are equal to the surface density of free energy at a surface separating the liquid and solid phases, resp., from an external material. Of course (2.1.3) makes sense only if 

$$|\sigma_S - \sigma_L| \leq \sigma.$$  

(2.1.4)

We may now formulate the *Stefan–Gibbs–Thomson Problem*, or *(Stefan Problem with Surface Tension)* just by replacing (1.2.9) by (2.1.2) and (2.1.3) in the formulation of Problem 1.2.1.\(^{32}\)

In general this problem cannot have a solution for large time, for discontinuities may occur at the solid–liquid interface, just as for the classical formulation of the basic Stefan

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\(^{31}\) For some materials this may even be of the order of hundreds of degrees. See e.g. the monographs quoted in the item (VI) of the Bibliographical Note in Section 6.

\(^{32}\) If the heat capacity $C_V = \partial u / \partial \theta$ vanishes in both phases, then this is often referred to as the Mullins–Sekerka problem. This especially applies to material diffusion in heterogeneous systems.
problem. However several properties are known to hold for small-time evolution, and for the weak solution for any time. An anisotropic variant representing crystal growth was also studied.

**Free energy.** Let us define the perimeter functional $P$ as in (5.7.2). In the framework of a mesoscopic model, the (Helmholtz) free energy of a solid–liquid system may be represented as follows, but for an additive contribution that depends on the temperature field:

\[
\Phi_{\theta,\sigma}(\chi) := \begin{cases} 
\sigma P(\chi) + \frac{\sigma L - \sigma_S}{2} \int_\Gamma \gamma_0 \chi \, d\Gamma - \frac{L}{2\tau_E} \int_\Omega \theta \chi \, dx & \forall \chi \in \text{Dom}(P), \\
+\infty & \forall \chi \in L^1(\Omega) \setminus \text{Dom}(P).
\end{cases}
\] (2.1.5)

By Proposition 5.7.1, whenever the inequality (2.1.4) is satisfied, $\Phi_{\theta,\sigma}$ is lower semicontinuous and has at least one absolute minimizer. Moreover, whenever $\theta \in L^p(\Omega)$ for some $p > 3$, by Theorem 5.7.2 any relative minimizer of $\Phi_{\theta,\sigma}$ fulfills the Gibbs–Thomson law (2.1.2) and the contact angle condition (2.1.3).

**Limit as $\sigma \to 0$.** On the macroscopic length-scale $\sigma = 0$. It is easily seen that, as $\sigma \to 0$ (whence $\sigma_S - \sigma_L \to 0$ by (2.1.4)), the functional $\Phi_{\theta,\sigma}$ $\Gamma$-converges in the sense of De Giorgi (cf. Section 5.8) to

\[
\Phi_{\theta}(\chi) := \begin{cases} 
-\frac{L}{2\tau_E} \int_\Omega \theta \chi \, dx & \text{if } |\chi| \leq 1 \text{ a.e. in } \Omega, \\
+\infty & \text{otherwise}.
\end{cases}
\] (2.1.6)

This functional is convex and lower semicontinuous in $L^1(\Omega)$, and its minimization is clearly equivalent to the temperature-phase rule (1.1.5). In Section 2.5 we shall further discuss the form of the free energy functional at different length-scales.

Surface tension plays an important role in several phase transition phenomena. For instance it accounts for phase nucleation, see e.g. Capillarity effects are also relevant for crystal growth.
2.2. Kinetic undercooling and phase relaxation

In this section we represent the decay of a liquid–solid system towards local equilibrium via two basic modes of evolution: a kinetic law at the solid–liquid interface, and so-called phase relaxation. These modes are respectively associated with the classical and weak formulations of the basic Stefan problem, cf. Sections 1.1 and 1.2. We also reformulate this process from the point of view of nonequilibrium thermodynamics, in which the local formulation of the second principle plays a central role.

First mode: Directional solidification (or Columnar Growth or Kinetic Undercooling). A close inspection of the process of solidification shows that this is driven by undercooling; see e.g. the monographs quoted in the item (VI) of the Bibliographical Note in Section 6. We shall assume that melting is also driven by superheating, consistently with the symmetry of the representation of these phenomena that characterizes the Stefan model.40

In the framework of the classical formulation in a univariate system, we may replace the equilibrium condition \( \theta(s(t), t) = 0 \), cf. (1.2.6), by the kinetic law

\[
\frac{ds}{dt}(t) + \gamma(\theta(s(t), t)) = 0, \tag{2.2.1}
\]

for a kinetic function \( \gamma \) that depends on the material; cf. Figure 4(a). By replacing (1.2.9) with (2.2.1) in Problem 1.1.2, one gets the one-dimensional two-phase Stefan problem with kinetic law.

In the metallurgical literature, this mode of solidification is named directional solidification, and the corresponding undercooling is often referred to as kinetic undercooling; see e.g. Visintin [446].

40 In general solidification is more relevant and exhibits a richer phenomenology than melting, as it is confirmed by the wealthy of morphologies that appear for instance in crystal growth. This asymmetry between solidification (or rather crystallization) and melting stems from the process of nucleation and growth: building the crystal structure is harder than destroying it.
For many materials $\gamma : \mathbb{R} \to \mathbb{R}$ may be assumed to be continuous and strictly increasing, with $\gamma(0) = 0$. In several cases one may also deal with the corresponding linearized law

$$\frac{ds}{dt}(t) + c\theta(s(t), t) = 0,$$

where $c$ is a positive constant. However we shall see that this does not apply to all substances.

**Second mode: Equiaxed solidification (or phase relaxation).** Dealing with the weak formulation of the Stefan problem, next we replace the condition of local equilibrium "$\chi \in \text{sign}(\theta)$ in $Q$,” cf. (1.1.7), by a nonequilibrium law. As this inclusion also reads $\text{sign}^{-1}(\chi) \ni \theta$ in $Q$, it is natural to consider the relaxation law

$$a \frac{\partial \chi}{\partial t} + \text{sign}^{-1}(\chi) \ni \theta \quad \text{in } Q,$$

or equivalently,

$$\begin{cases} -1 \leq \chi \leq 1 & \text{in } Q, \\ (a \frac{\partial \chi}{\partial t} - \theta)(\chi - v) \leq 0 & \forall v \in [-1, 1] \quad \text{in } Q, \end{cases}$$

for some relaxation coefficient $a$; cf. Figure 4(b). For materials that are characterized by an increasing kinetic function $\gamma$, we may replace the right-hand side of (2.2.3) by $\gamma(\theta)$. In the metallurgical literature this mode of phase transition is referred to as equiaxed solidification.41

**Comparison of the two modes.** The laws (2.2.1) and (2.2.3) describe different evolution modes, although both represent relaxation towards local equilibrium. Equation (2.2.1) accounts for motion of the interface separating two pure phases, without formation of any mushy region. On the other hand, the second mode represents phase transition by formation of a mushy region, and (2.2.3) describes the evolution of the liquid concentration in that zone. From an analytical viewpoint, these two modes are naturally associated with the classical and weak formulations of the Stefan problem, respectively. The extension of the first mode to several space dimensions actually requires a revision of the mathematical model.

Directional and equiaxed growth are the basic modes of solidification, and may also combine to form a hybrid mode. For instance, in casting metal an equiaxed zone is at first formed in contact with the wall of the mould, and gives soon raise to a columnar region that advances towards the interior. Solid nucleation also occurs in the bulk, and an equiaxed solid phase grows in the remainder of the liquid. Eventually the two solid phases impinge on, and occupy the whole volume; see Figure 5. These physical aspects are illustrated e.g. in Flemings [222, Chapter 5], Kurz and Fisher [299, Section 1.1.2].

**Glass formation.** As we anticipated, for some materials the kinetic function is not monotone. For steel, polymers, and materials capable of forming a glass, the viscosity

41 See e.g. the monographs quoted in the item (VI) of the Bibliographical Note in Section 6.
Fig. 5. Part (a) illustrates how columnar and equiaxed solidification may interact in a univariate system. Part (b) represents the grain structure of a crystal that grew from an undercooled liquid in a vessel: the solid columns advanced from the border, and impinged on the equiaxed grains which formed in the bulk.

Fig. 6. Kinetic function for an amorphous material, e.g. a polymer.

increases so much with the undercooling, that the mobility of particles in their migration to reach the crystal sites is strongly impaired.

Although a glass apparently behaves like a solid, it has the fine-scale structure of a highly-viscous undercooled liquid, and indeed retains a large part of the latent heat of phase transition. Its crystal structure is largely incomplete, and the material is accordingly said to be amorphous. For these materials the kinetic function has the qualitative behaviour of Figure 6. A glass is formed by quenching (i.e., very rapidly cooling) the liquid material to a temperature below \( \tilde{\theta} \). The glass will eventually crystallize, but this may easily need geological time-scales. A similar process occurs in the austenite-pearlite transformation in eutectoid carbon steel.

The entropy balance. We shall represent the density of internal energy, \( u \), as a convex and lower semicontinuous function of the density of entropy, \( s \), and of the phase function, \( \chi \).

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42 Phase transitions in polymers and related industrial processes were studied e.g. by Andreucci et al. [24], Astarita and Sarti [32], Fasano [207], Fasano and Mancini [208], Fasano, Meyer and Primicerio [209].

43 See e.g. Agarwal and Brimacombe [4], Cahn [106], Hawboldt, Chau and Brimacombe [264], Scheil [419] for an outline of the phenomenon, and Brokate and Sprekels [91, Chapter 8], Hönberg [275], Verdi and Visintin [444], Visintin [450] for its mathematical analysis.
Thus \( u = \hat{u}(s, \chi) \), i.e., more explicitly \(^{44}\)
\[
\hat{u}(x, t) = \hat{u}(s(x, t), \chi(x, t)) \quad \text{for } (x, t) \in Q.
\]
By the very definition of the absolute (Kelvin) temperature, \( \tau \), the function \( \hat{u} \) is differen-
tiable w.r.t. \( s \) and
\[
\tau = \frac{\partial \hat{u}}{\partial s}(s, \chi) \quad (=: \hat{\tau}(s, \chi)).
\]
As \( \hat{\tau} > 0 \), the entropy may equivalently be represented as a concave function of \( u \) and \( \chi \), that is, \( s = \hat{s}(u, \chi) \). Because of the constraint \(-1 \leq \chi \leq 1\), the functions \( \hat{u} \) and \( \hat{s} \) cannot be differentiable at \( \chi = \pm 1 \). We shall accordingly use the notion of (partial) subdifferential, see Section 5.2. The differential notation is however too convenient for being dropped without a second thought, especially considering that \( \hat{u} \) and \( \hat{s} \) may be differentiable where \(-1 < \chi < 1\); we shall actually assume them to be so. We shall thus write differential formulas only for the restriction to these values of \( \chi \). For instance, we define the potential \( \lambda \) by setting
\[
(2.2.5) \quad d\hat{u} = \tau \, ds + \lambda \, d\chi,
\]
where \(-1 < \chi < 1\). More precisely, distinguishing between the potential \( \lambda \) and its functional representations \( \hat{\lambda}_1(s, \chi) \) and \( \hat{\lambda}_2(u, \chi) \), and denoting the partial subdifferential w.r.t. \( \chi \) by \( \partial \chi \), we have \(^{45}\)
\[
(2.2.6) \quad \hat{\lambda}_1(s, \chi) \in \partial \chi \hat{u}(s, \chi), \quad \frac{\hat{\lambda}_2(u, \chi)}{\tau} \in \partial \chi \hat{s}(u, \chi) \quad \text{for } -1 \leq \chi \leq 1.
\]
The energy balance (1.1.2) and (2.2.5) yield the entropy balance equation
\[
\frac{\partial s}{\partial t} = \frac{1}{\tau} \frac{\partial u}{\partial t} - \frac{\lambda}{\tau} \frac{\partial \chi}{\partial t} = -\nabla \cdot \vec{q} + \vec{q} \cdot \nabla \frac{1}{\tau} - \frac{\lambda}{\tau} \frac{\partial \chi}{\partial t} + \frac{f}{\tau} = -\nabla \cdot \vec{j}_s + \pi + \frac{f}{\tau} \quad \text{in } Q,
\]
where we set
\[
(2.2.8) \quad \vec{j}_s := \frac{\vec{q}}{\tau}: \text{ entropy flux (per unit surface),}
\]
\[
(2.2.9) \quad \pi := \vec{q} \cdot \nabla \frac{1}{\tau} - \frac{\lambda}{\tau} \frac{\partial \chi}{\partial t}: \text{ entropy production rate (per unit volume).}
\]
Thus
\[
(2.2.10) \quad \pi := \vec{J} \cdot \vec{G}, \quad \text{where } \vec{J} := \left( \frac{\vec{q}}{\tau}, \frac{\partial \chi}{\partial t} \right), \quad \vec{G} := \left( \nabla \frac{1}{\tau}, -\frac{\lambda}{\tau} \right).
\]
The quantity \( f/\tau \) is the rate of entropy production per unit volume, due to an external
source or sink of heat. By the local formulation of the second principle of thermodynam-
icstics,\(^ {46}\)

\(^{44}\) By this “hat notation” we shall distiguish between the physical field, \( u = u(x, t) \), and the function that represents how it depends on other variables, \( u = \hat{u}(s, \chi) \).

\(^{45}\) Here the symbol of inclusion is needed, for the subdifferential may be multivalued, see Section 5.2.

\(^{46}\) See e.g. the monographs quoted in the item (VII) of the Bibliographical Note in Section 6.
\[ \pi \geq 0, \quad \pi = 0 \text{ at local equilibrium} \]  

(Clausius–Duhem inequality). \hfill (2.2.11)

Because of the arbitrariness of \( \mathbf{G} \), assuming a linear dependence between \( \mathbf{J} \) and \( \mathbf{G} \), this inequality we infer\(^{47}\) the following linearized conduction and phase relaxation laws:

\[ \hat{q} = K \cdot \nabla \frac{1}{\tau}, \quad \text{that is,} \quad \hat{q} = -\frac{K}{\tau^2} \cdot \nabla \tau, \]  

\hfill (2.2.12)

\[ a \frac{\partial \chi}{\partial t} = -\frac{\lambda}{\tau}, \quad \text{that is,} \quad a \frac{\partial \chi}{\partial t} \in \partial \chi \hat{s}; \]  

\hfill (2.2.13)

here the tensor \( K \) and the scalar \( a \) are positive-definite functions of the state variables. These two equations may respectively be compared with the Fourier law (1.1.3) and with the phase relaxation dynamics (2.2.3). Thus

\[ \pi \geq 0; \quad \pi = 0 \iff \nabla \tau = \mathbf{0}, \quad \lambda = 0. \]  

(2.2.14)

**Linearization.** The occurrence of the term \( 1/\tau \) in the above formulas raises the need of granting that \( \tau > 0 \). This has been a source of technical difficulties in the analysis, that were overcome only at the expense of a certain effort.\(^{48}\) That achievement is valuable in itself, and a result which allows for extreme temperatures is clearly of interest. However, one might wonder whether in practice the risk of getting \( \tau \) close to zero is physically significant, and if so whether it is legitimate to extrapolate our models to those temperatures. Actually, constitutive relations typically just have a limited range of validity. This leads us to introduce a simplified model, that we shall study in Section 4.

Let us first define the function\(^{49}\)

\[ \varphi := u - \tau E s, \quad \text{that is,} \quad \varphi = \hat{\varphi}(u, \chi) := u - \tau E \hat{s}(u, \chi), \]  

\hfill (2.2.15)

and notice that by (2.2.5)

\[ d\varphi = du - \frac{\tau E}{\tau} d\varphi - \frac{\tau E}{\tau} \frac{\partial \hat{s}}{\partial \chi} d\chi = \frac{\theta}{\tau} du - \tau E \frac{\partial \hat{s}}{\partial \chi} d\chi. \]  

\hfill (2.2.16)

Thus \( \partial \hat{\varphi}/\partial \chi = -\tau E \partial \hat{s}/\partial \chi \) for \(-1 < \chi < 1\), and more generally in terms of partial subdifferentials \( \partial \hat{\varphi}/\partial \chi = -\tau E \partial \hat{s}/\partial \chi \) for \(-1 \leq \chi \leq 1\).

The energy balance (1.1.2) and the entropy balance (2.2.7) yield the balance of the function \( \varphi \):

\[ \frac{\partial \varphi}{\partial t} = \frac{\partial u}{\partial t} - \tau E \frac{\partial s}{\partial t} = -\nabla \cdot \hat{q} + f + \tau E \nabla \cdot \frac{\hat{q}}{\tau} - \frac{\tau E}{\tau} \pi - \frac{\tau E}{\tau} f \]  

\[ = -\nabla \cdot \left( \frac{\theta}{\tau} \hat{q} \right) - \tau E \pi + \frac{\theta}{\tau} f, \]  

\hfill (2.2.17)

---

\(^{47}\) In Section 2.4 we shall illustrate this derivation in a more general framework.

\(^{48}\) See Sprekels and Zheng [429], Zheng [476].

\(^{49}\) This potential is named available free energy and tends to a minimum as equilibrium is approached in a source-free isolated system, cf. (2.2.17) below; see Müller and Weiss [345, Chapter 7]. In [255,257] Gurtin referred to it as a Gibbs function; actually, this function was first introduced by Gibbs dealing with uniform fields. It may also be noticed that relaxation towards thermal equilibrium is much faster than other relaxation processes, so that the difference between \( \varphi \) and \( \psi \) is not quantitatively relevant.
with \( \pi \) defined as in (2.2.9):

\[
\tau_E \pi := \vec{q} \cdot \nabla \tau_E = \frac{\lambda \tau_E \partial \chi}{\tau} = -\vec{q} \cdot \nabla \left( \frac{\theta}{\tau} - \lambda \left( 1 - \frac{\theta}{\tau} \right) \frac{\partial \chi}{\partial t} \right).
\] (2.2.18)

Next we linearize the system (2.2.17) and (2.2.18) w.r.t. \( \theta \). For any function \( \sigma \), let us set \( \sigma(\rho) = o(\rho) \) whenever \( \sigma(\rho)/\rho \to 0 \) as \( \rho \to 0 \). As

\[
\frac{\theta}{\tau} = \frac{\theta}{\tau_E} + o\left( \frac{\theta}{\tau_E} \right),
\]

\[
\nabla \frac{\theta}{\tau} = \tau_E \frac{\nabla \theta}{\tau^2} = \nabla \frac{\theta}{\tau_E} \left( 1 + \frac{\theta}{\tau_E} \right)^{-2} = \nabla \frac{\theta}{\tau_E} \left[ 1 - 2 \frac{\theta}{\tau_E} + o\left( \frac{\theta}{\tau_E} \right) \right],
\]

we get

\[
d\varphi = \left[ \frac{\theta}{\tau_E} + o\left( \frac{\theta}{\tau_E} \right) \right] d\mu - \tau_E \frac{\partial \hat{s}}{\partial \chi} d\chi,
\] (2.2.19)

\[
\frac{\partial \varphi}{\partial t} = -\nabla \cdot \left[ \left[ \frac{\theta}{\tau_E} + o\left( \frac{\theta}{\tau_E} \right) \right] \vec{q} \right] - \tau_E \pi + \left[ \frac{\theta}{\tau_E} + o\left( \frac{\theta}{\tau_E} \right) \right] f,
\] (2.2.20)

\[
\tau_E \pi = -\vec{q} \cdot \nabla \frac{\theta}{\tau_E} \left[ 1 - 2 \frac{\theta}{\tau_E} + o\left( \frac{\theta}{\tau_E} \right) \right] - \lambda \left[ 1 - \frac{\theta}{\tau_E} + o\left( \frac{\theta}{\tau_E} \right) \right] \frac{\partial \chi}{\partial t}.
\] (2.2.21)

Notice that in these formulas the terms in square brackets are all positive. Neglecting infinitesima, we thus get

\[
d\varphi = \frac{\theta}{\tau_E} d\mu - \tau_E \frac{\partial \hat{s}}{\partial \chi} d\chi,
\] (2.2.22)

\[
\frac{\partial \varphi}{\partial t} = -\nabla \cdot \left( \frac{\theta}{\tau_E} \vec{q} \right) - \tau_E \pi + \frac{\theta}{\tau_E} f,
\] (2.2.23)

\[
\tau_E \pi = -\vec{q} \cdot \nabla \frac{\theta}{\tau_E} \left( 1 - 2 \frac{\theta}{\tau_E} \right) - \lambda \left( 1 - \frac{\theta}{\tau_E} \right) \frac{\partial \chi}{\partial t}.
\] (2.2.24)

### 2.3. Phase transitions in heterogeneous systems

In this section we extend the Stefan model to phase transitions in mixtures of two materials. The diffusion of heat is here coupled with that of the constituents, so that we have a system of equations instead of a single parabolic equation. In this case the interface is characterized by a discontinuity not only of the heat flux, but also of the mass flux and of the composition of the mixture. We shall first introduce a classical formulation, and then derive a weak one.\(^{50}\) Some physical and mathematical drawbacks will also arise, and

\(^{50}\) This procedure is opposite to that we followed in Sections 1.1 and 1.2, that however might be applied here, too.
these will induce us to reformulate this phenomenon by a different model in the next section.

**Mass diffusion.** We confine ourselves to a composite of two constituents. We might also deal with a larger number of species, but even in this simple setting we shall encounter some difficulties in the analysis. More precisely, we shall deal with a *binary alloy*, that is, a homogeneous mixture of two substances, that are soluble in each other in all proportions in each phase, outside a critical range of temperatures. Here *homogeneity* means that the constituents are intermixed on the atomic length-scale to form a single phase, either solid or liquid. We shall regard one of the two components as the *solute*, for instance that with the lower solid–liquid equilibrium temperature, and the other one as the *solvent*. We shall also use the following notation:

- $c$: concentration of the solute (per unit volume),
- $\mathbf{j}_c$: flux of the solute (per unit surface),
- $D_1$ ($D_2$, resp.): mass diffusivity of the solute in the liquid (in the solid, resp.).

Although the coefficient $D_2$ is much smaller than $D_1$, it need not vanish.\(^{51}\)

In a simplified formulation, we assume that the specific heat and the heat conductivity may depend on the temperature and on the phase, but not on the (solute) concentration. We also assume that the mass diffusivity may depend on the concentration and on the phase, but not on the temperature. Thus

$$CV_i = CV_i(\theta), \quad k_i = k_i(\theta), \quad D_i = D_i(c) \quad \text{for } i = 1, 2. \tag{2.3.1}$$

If the two constituents have different temperatures of phase transition, then that of the mixture depends on the concentration. The latent heat is then a prescribed function of the temperature: $L = L(\theta)$.

As for pure substances, the heat equation is here fulfilled in the interior of each phase

$$CV_i(\theta)\frac{\partial \theta}{\partial t} - \nabla \cdot [k_i(\theta) \cdot \nabla \theta] = f \quad \text{in } Q_i \ (i = 1, 2), \tag{2.3.2}$$

and is complemented by the Stefan condition at the solid–liquid interface $S$:

$$k_1(\theta) \cdot \frac{\partial \theta_1}{\partial n} - k_2(\theta) \cdot \frac{\partial \theta_2}{\partial n} = -L(\theta)\mathbf{v} \cdot \mathbf{n} \quad \text{on } S. \tag{2.3.3}$$

The principle of mass conservation, $\partial c/\partial t + \nabla \cdot \mathbf{j}_c = 0$, and the Fick law, $\mathbf{j}_c = -D_i(c)\nabla c$, yield the *equation of mass-diffusion* in each phase:

$$\frac{\partial c}{\partial t} - \nabla \cdot [D_i(c)\nabla c] = 0 \quad \text{in } Q_i \ (i = 1, 2). \tag{2.3.4}$$

Let us introduce some further notation:

- $\mathbf{j}_{ci}$: mass flux (per unit surface) across $S$ contributed by the phase $i$,
- $c_i$: limit of $c$ on $S$ from the phase $i$.

\(^{51}\) Moreover the liquid diffusivity $D_1$ is much smaller than the heat conductivity $k$ in either phase. One might also assume that $D_1 = D_2 = 0$, as in the Mullins–Sekerka problem, where however the capillarity is also accounted for.
Fig. 7. The graph of $\eta_1$ and $\eta_2$ (respectively named liquidus and solidus) represent states of stable thermodynamic equilibrium at the solid–liquid interface for a noneutectic composite. The states outside the lens-shaped region (and with $0 \leq c \leq 1$) are also stable, whereas those inside are either metastable or unstable.

---

$\vec{v}$: (normal) velocity of $S_t$, 
$
\vec{n} \in \mathbb{R}^3$: unit vector field normal to $S_t$ oriented from the liquid to the solid.

By mass conservation we have

\[
\vec{j}_c^2 \cdot \vec{n} - \vec{j}_c^1 \cdot \vec{n} = (c_2 - c_1) \vec{v} \cdot \vec{n} \quad \text{on } S_t.
\]

The Fick law then yields another discontinuity condition:

\[
D_1(c_1) \frac{\partial c_1}{\partial n} - D_2(c_2) \frac{\partial c_2}{\partial n} = (c_2 - c_1) \vec{v} \cdot \vec{n} \quad \text{on } S_t.
\]

The reader will notice the analogy between the balance laws (2.3.2) and (2.3.4) in the interior of the phases, and the difference between the discontinuity conditions (2.3.3) and (2.3.6) at the solid–liquid interface: the field $c$ is discontinuous across $S$, at variance with $\theta$.

Actually, the concentration, $c$, should be compared with the density of internal energy, $u$, rather than with the temperature, $\theta$. Ahead we shall introduce a further field, $w$, that is continuous at the solid–liquid interface, and plays an analogous role to that of $\theta$.

**Phase separation.** At the solid–liquid interface, the temperature and the concentration fulfill an equilibrium relation of the form

\[
\theta = \eta_1(c_1) = \eta_2(c_2) \quad \text{on } S_t,
\]

where $\eta_1$ and $\eta_2$ are known functions. Their graphs are traditionally named liquidus and solidus, for obvious reasons. For a noneutectic composite,\(^\text{52}\) we may also assume that

\[
\eta_i \in C^1([0, 1]), \quad \eta_i' < 0 \ (i = 1, 2), \quad \eta_1 > \eta_2 \text{ in } [0, 1],
\]

\[
\eta_1(0) = \eta_2(0) = 0, \quad \eta_1(1) = \eta_2(1) \quad (\text{see Figure 7}).
\]

At local thermodynamic equilibrium, the temperature-phase rule (1.1.5) is here replaced by a temperature-concentration-phase rule:

\[
\theta \geq \eta_1(c) \quad \text{in } Q_1, \quad \theta \leq \eta_2(c) \quad \text{in } Q_2.
\]

\(^\text{52}\) A composite is named a eutectic if $\eta_1(\bar{c}) = \eta_2(\bar{c})$ for some eutectic concentration $\bar{c} \in [0, 1]$. 
The states where \( \eta_2(c) < \theta < \eta_1(c) \) are in either metastable or unstable thermodynamic equilibrium. Whenever the variables are forced to attain those intermediate values, e.g. by rapid cooling of a liquid system, one or more nuclei of the secondary phase are formed, and grow until the two phases have attained the respective concentrations of equilibrium: \( c_i = \eta^{-1}_i(\theta) \) \((i = 1, 2)\). Under isothermal conditions, this process of phase separation (also known as spinodal decomposition) is represented by the Cahn–Hilliard equation, that we briefly illustrate in Section 2.5.

Next we introduce a classical formulation, that extends that of the basic Stefan model, cf. Problem 1.2.1.

**Problem 2.3.1** (Classical formulation of the multi-dimensional problem of phase transition in binary alloys). Find \( \theta, c : Q \to \mathbb{R} \) and a partition \( \{Q_1, Q_2, S\} \) of \( Q \) such that:

(i) \( Q_1 \) and \( Q_2 \) are open sets;
(ii) \( S \subset Q \) is a regular 3-dimensional manifold, and \( S := S \cap (\Omega \times \{t\}) \) is a regular surface, for any \( t \in [0, T] \);
(iii) \( \theta, c, \partial \theta / \partial t, \partial c / \partial t, \partial^2 \theta / \partial x_i \partial x_j, \partial^2 c / \partial x_i \partial x_j \) \((i, j \in \{1, 2, 3\})\) exist and are continuous in \( Q_1 \) and in \( Q_2 \);
(iv) the normal derivatives \( \partial \theta / \partial n \) and \( \partial c / \partial n \) exist on the respective sides of \( S \);
(v) Eqs. (2.3.2)–(2.3.4), (2.3.6), (2.3.7) are fulfilled;
(vi) \( \theta \) and \( c \) attain prescribed values on \( \Omega \times \{0\} \) and on \( \Gamma_D \times [0, T] \);
(vii) the normal derivatives \( \partial \theta / \partial n \) and \( \partial c / \partial n \) attain prescribed values on \( \Gamma_N \times [0, T] \);
(viii) \( \bar{S} \cap (\Omega \times \{0\}) \) is also prescribed.

Here the occurrence of metastable states is not excluded, just as for Problem 1.2.1.

*A transformation of variable.* In view of deriving a weak formulation of **Problem 2.3.1**, let us introduce the new variable\(^{53} \)

\[
w := \eta_i(c) \left( \in [\tilde{\theta}, 0] \right) \quad \text{in } Q_i \quad (i = 1, 2),
\]

so that by (2.3.7) and (2.3.9)

\[
w \text{ is continuous across } S, \quad w = \theta \text{ on } S, \quad \theta \geq w \text{ in } Q_1, \quad \theta \leq w \text{ in } Q_2.
\]

Setting \( \tilde{\zeta}_i := \eta_i^{-1} \) for \( i = 1, 2 \), we have

\[
c = \tilde{\zeta}_i(w), \quad \nabla c = \tilde{\zeta}'_i(w) \nabla w \quad \text{in } Q_i \quad (i = 1, 2).
\]

Let us also set

\[
\tilde{D}_i(w) := -D_i(\tilde{\zeta}_i(w)) \tilde{\zeta}'_i(w) \quad (\geq 0) \quad \forall w \in [\tilde{\theta}, 0] \quad (i = 1, 2),
\]

so that the Fick law also reads

\[
j_c := -D_i(c) \nabla c = \tilde{D}_i(w) \nabla w \quad \text{in } Q_i \quad (i = 1, 2).
\]

\(^{53}\) In Section 2.4 we shall see that \( \nabla w \) is proportional to \( -\nabla \mu \), where by \( \mu \) we denote the relative chemical potential, namely the difference between the chemical potentials of the two components.
Defining the phase function $\chi$ as above (i.e., $\chi = 1$ in the liquid, $\chi = -1$ in the solid), the phase rule (1.1.7) is here replaced by

$$\chi \in \text{sign}(\theta - w) \quad \text{in} \quad Q \quad \text{(cf. Figure 8).} \tag{2.3.16}$$

Let us now set

$$C_V(\theta, \chi) := C_{V1}(\theta) \frac{1+\chi}{2} + C_{V2}(\theta) \frac{1-\chi}{2},$$

$$k(\theta, \chi) := k_1(\theta) \frac{1+\chi}{2} + k_2(\theta) \frac{1-\chi}{2},$$

$$\tilde{D}(w, \chi) := \tilde{D}_1(w) \frac{1+\chi}{2} + \tilde{D}_2(w) \frac{1-\chi}{2},$$

$$\forall (\theta, w, \chi) \in \mathbb{R} \times [\tilde{\theta}, 0] \times [-1, 1].$$

By Proposition 1.3.1, the heat equations (2.3.2) and the Stefan condition (2.3.3) may be expressed in weak form by the single equation

$$C_V(\theta, \chi) \frac{\partial \theta}{\partial t} + \frac{L(\theta)}{2} \frac{\partial \chi}{\partial t} - \nabla \cdot [k(\theta, \chi) \cdot \nabla \theta] = 0 \quad \text{in} \quad \mathcal{D}'(Q). \tag{2.3.18}$$

That argument yields the analogous statement for mass-diffusion.

**Proposition 2.3.1.** Let the pair $(u, S)$ fulfill the regularity conditions of Problem 2.3.1, define $w$ as in (2.3.10), and set $\chi := -1$ in $Q_2$, $\chi := 1$ in $Q_1$. The system (2.3.4), (2.3.6) is then equivalent to

$$\frac{\partial c}{\partial t} + \nabla \cdot [\tilde{D}(w, \chi) \nabla w] = 0 \quad \text{in} \quad \mathcal{D}'(Q). \tag{2.3.19}$$

Notice that by (2.3.13)

$$c = \zeta_1(w) \frac{1+\chi}{2} + \zeta_2(w) \frac{1-\chi}{2} \quad \text{in} \quad Q. \tag{2.3.20}$$

Thus $c$ is a decreasing function of $w$, and Eq. (2.3.19) is forward parabolic.
Table 2
Comparison between heat and mass-diffusion in binary alloys. By $[v_i] := v_1 - v_2$ we denote the jump of a quantity $v$ across the solid–liquid interface $S$.

<table>
<thead>
<tr>
<th></th>
<th>Heat diffusion</th>
<th>Mass diffusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>continuous variable</td>
<td>temperature $\theta$</td>
<td>new variable $w$</td>
</tr>
<tr>
<td>discontinuous variable</td>
<td>internal energy $u$</td>
<td>solute concentration $c$</td>
</tr>
<tr>
<td>equation in each phase</td>
<td>$C_Vi(\theta)\frac{\partial \theta}{\partial t} - \nabla \cdot [k_i(\theta)\nabla \theta] = f$</td>
<td>$\frac{\partial c}{\partial t} - \nabla \cdot [D_i(c)\nabla c] = 0$</td>
</tr>
<tr>
<td>jump condition across $S$</td>
<td>$[k_i(\theta)\frac{\partial \theta_i}{\partial n}] = -L(\theta)\vec{v} \cdot \vec{n}$</td>
<td>$[D_i(c)\frac{\partial c_i}{\partial n}] = (c_2 - c_1)\vec{v} \cdot \vec{n}$</td>
</tr>
<tr>
<td>equation in $D'(Q)$</td>
<td>$\frac{\partial u}{\partial t} - \nabla \cdot [k(\theta, \chi)\nabla \theta] = f$</td>
<td>$\frac{\partial c}{\partial t} + \nabla \cdot \tilde{D}(w, \chi)\nabla w = 0$</td>
</tr>
</tbody>
</table>

Equations (2.3.16), (2.3.18), (2.3.19) and (2.3.20) for the unknown functions $\theta$, $c$, $w$, $\chi$, coupled with appropriate initial and boundary conditions, constitute the weak formulation of the problem of phase transition in binary alloys. We derived these equations from the classical formulation; alternatively one might also derive them directly from the laws of heat and mass conservation set in the whole space–time domain $Q$.

**Linearized constitutive laws.** If the solute concentration $c$ is small (as it often occurs in practice), it is possible to linearize the functions $\eta_1$ and $\eta_2$, that is, to replace (2.3.7) by

$$\theta = \eta'_i(0)c_i =: -\frac{1}{r_i}c_i \quad \text{on} \ S \quad (i = 1, 2),$$

where $0 \leq r_2 < r_1$; cf. Figure 9. By setting

$$w := -\frac{1}{r_i}c_i \quad (\leq 0) \quad \text{in} \ Q \quad (i = 1, 2),$$

from (2.3.20) we thus get

$$c = -r_1w\frac{\chi + 1}{2} - r_2w\frac{1 - \chi}{2} \quad \text{in} \ Q.$$  

Although the linearization only applies for small values of $c$, here the range of $c$ is assumed to be the whole $\mathbb{R}^+$, which corresponds to $w \leq 0$.

**A nonparabolic system of equations.** The model above has extensively been used by material scientists and engineers, and their numerical approximation provided quantitatively acceptable results. However, as far as this author knows, even existence of a weak solution is not known for this problem in the multi-variate setting, in spite of the simplifications that are inherent in this model. Actually Problem 2.3.1 does not seem prone to analysis. The equations of heat and mass-diffusion (2.3.18) and (2.3.19) are separately parabolic;

---

54 See e.g. the monographs quoted in the item (VI) of the Bibliographical Note in Section 6. This model was also investigated by mathematicians, see e.g. Alexiades and Cannon [7], Alexiades and Solomon [8], Alexiades, Solomon and Wilson [9,425,464,465], Bermudez and Saguez [61], Crowley [151], Crowley and Ockendon [152], Fix [219], Ockendon and Tayler [363], Tayler [437].
however, coupled with (2.3.16), as a system they miss this property, for the multi-valued mapping

\[ \mathbb{R}^2 \rightarrow 2\mathbb{R}^2 : \begin{pmatrix} \theta \\ w \end{pmatrix} \mapsto \begin{pmatrix} u \\ c \end{pmatrix} = \begin{pmatrix} C_V(\theta, \chi)\theta + L(\theta) \text{sign}(\theta - w) \\ \zeta_1(w) \frac{1 + \text{sign}(\theta - w)}{2} + \zeta_2(w) \frac{1 - \text{sign}(\theta - w)}{2} \end{pmatrix} \] (2.3.24)

fails to be monotone. This is easily checked, as this property fails for the discontinuous part of that mapping:

\[ \mathbb{R}^2 \rightarrow 2\mathbb{R}^2 : \begin{pmatrix} \theta \\ w \end{pmatrix} \mapsto \begin{pmatrix} L(\theta) \text{sign}(\theta - w) \\ \frac{1}{2} [\zeta_1(w) - \zeta_2(w)] \text{sign}(\theta - w) \end{pmatrix} . \] (2.3.25)

This analytical issue has a physical counterpart: the model of this section does not account for cross effects between heat and mass-diffusion: a temperature gradient induces a mass flux (Soret effect), and in turn a gradient of chemical potential causes a heat flux (Dufour effect). Although in several cases the omitted terms do not seem to be quantitatively very significant, their absence impairs the analytical structure of the problem. Another physical drawback of this model was pointed out in Alexiades and Cannon [7], Alexiades, Wilson and Solomon [9].

These physical and mathematical drawbacks are overcome by the theory of nonequilibrium thermodynamics, that we illustrate in the next section. There the constitutive relations are dictated by the second principle, rather than being just extrapolated from the uncoupled heat and mass-diffusion, as above.

2.4. Approach via nonequilibrium thermodynamics

The physical and mathematical drawbacks that emerged in the last section induce us to represent phase transitions in binary alloys via an alternative and more successful approach, that also applies to more general heterogeneous systems. This is based on the following main elements:

(i) the first principle of thermodynamics and the principle of mass conservation (i.e., two balance laws),

\[ \text{See e.g. the monographs quoted in the item (VII) of the Bibliographical Note in Section 6.} \]
(ii) a constitutive relation for the entropy density (namely, the *Gibbs formula*),
(iii) three further constitutive relations, the so-called *phenomenological laws*, that are consistent with a local formulation of the second principle of thermodynamics, and include a dynamics of phase-relaxation.

The first two issues lead to the formulation of a doubly-nonlinear second-order system of PDEs. The third requirement accounts for dissipation, which in analytical terms corresponds to the forward parabolicity of the problem. This approach is based on the theory of nonequilibrium thermodynamics.56

Although one might deal with any number of constituents and also allow for chemical reactions, see e.g. Luckhaus and Visintin [317], in this simplified presentation we still confine ourselves to a nonreacting (noneutectic) binary system. As we did in Section 2.2, but at variance with the model of Section 2.3 and with previous works, here we also account for phase nonequilibrium by including the phase function among the state variables.

*Balance laws and Gibbs formula.* In view of extending the derivation of the entropy balance of Section 2.2 to binary alloys, let us define some further notation:
- \( \mu \): difference between the chemical potential of the two constituents,
- \( j_u \): flux of energy (per unit surface), due to flux of heat and mass,
- \( h \): intensity of a prescribed energy source or sink, due to injection or extraction either of heat or mass.

In the absence of chemical reactions, the principles of energy and mass conservation yield

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -\nabla \cdot j_u + h \quad \text{in } Q, \\
\frac{\partial c}{\partial t} &= -\nabla \cdot j_c \quad \text{in } Q.
\end{align*}
\]

We shall assume that the internal energy is a prescribed convex function of the entropy density, \( s \), of the solute concentration, \( c \), and of the phase function, \( \chi \); that is, \( u = \hat{u}(s, c, \chi) \). The specific form of the function \( \hat{u} \) obviously depends on the constituents.

Consistently with the discussion of Section 2.2, because of the constraints \( 0 \leq c \leq 1 \) and \(-1 \leq \chi \leq 1\), we may assume \( \hat{u} \) to be differentiable for \((c, \chi) \in ]0, 1[ \times ]-1, 1[\), but not on the boundary of this set. Let us extend \( \hat{u} \) with value \(+\infty\) for \((c, \chi) \notin ]0, 1[ \times ]-1, 1[\). We may thus assume this function to be differentiable for \((c, \chi) \in \text{open rectangle}\), but not on its boundary. We must then deal with the partial subdifferentials \( \partial_c \hat{u}, \partial_\chi \hat{u} \) (see Section 5.2), which coincide with the respective derivatives only in \([0, 1[ \times ]-1, 1[\). Reminding the definition of the absolute temperature, \( \tau = \partial_s \hat{u}(s, c, \chi) \), for any selection

\[
\begin{align*}
\mu &\in \partial_c \hat{u}(s, c, \chi), \\
\lambda &\in \partial_\chi \hat{u}(s, c, \chi),
\end{align*}
\]

we thus have

\[
\begin{align*}
u = \hat{u}(s, c, \chi), \\
du = \tau \, ds + \mu \, dc + \lambda \, d\chi \quad \forall (s, c, \chi) \in (\text{Dom } \hat{u})^0.
\end{align*}
\]

56 See e.g. the monographs quoted in the item (VII) of the Bibliographical Note in Section 6. The mathematical aspects of this formulation were studied e.g. in Alexiades, Wilson and Solomon [9], Donnelly [182], Luckhaus [314], Luckhaus and Visintin [317].
As $\frac{\partial}{\partial s} \hat{s}(s, c, \chi) = \tau > 0$, this constitutive relation may also be written in the equivalent form $s = \hat{s}(u, c, \chi)$. Moreover (2.4.4) is equivalent to the Gibbs-type formula

$$s = \hat{s}(u, c, \chi),$$

(2.4.5)

and more generally

$$\frac{1}{\tau} \in \partial_s \hat{s}(u, c, \chi), \quad -\frac{\mu}{\tau} \in \partial_c \hat{s}(u, c, \chi), \quad -\frac{\lambda}{\tau} \in \partial_{\chi} \hat{s}(u, c, \chi)$$

(2.4.6)

Consistently with a basic postulate of nonequilibrium thermodynamics, we assume that (2.4.4) and (2.4.5) also apply to systems that are not too far from equilibrium. Actually, the limits of validity of the whole theory strongly depend on those of the Gibbs formula (2.4.5).

**Entropy balance.** Let us multiply (2.4.1) by $1/\tau$ and (2.4.2) by $-\mu/\tau$. By (2.4.6) we thus get the entropy balance equation

$$\frac{\partial s}{\partial t} = \frac{1}{\tau} \frac{\partial u}{\partial t} - \frac{\mu}{\tau} \frac{\partial c}{\partial t} - \frac{\lambda}{\tau} \frac{\partial \chi}{\partial t}$$

$$= -\frac{1}{\tau} \nabla \cdot \vec{j}_u + \frac{h}{\tau} + \frac{\mu}{\tau} \nabla \cdot \vec{j}_c - \frac{\lambda}{\tau} \frac{\partial \chi}{\partial t}$$

$$= -\nabla \cdot \frac{\vec{j}_u - \mu \vec{j}_c}{\tau} + \frac{\vec{j}_u}{\tau} \cdot \nabla 1 - \frac{\vec{j}_c}{\tau} \cdot \nabla \mu - \frac{\lambda}{\tau} \frac{\partial \chi}{\partial t} + \frac{h}{\tau}$$

$$= -\nabla \cdot \vec{j}_s + \pi + \frac{h}{\tau} \quad \text{in} \ Q,$$

(2.4.7)

where we set

$$\vec{j}_s := \frac{\vec{j}_u - \mu \vec{j}_c}{\tau}, \quad \text{entropy flux (per unit surface),}$$

(2.4.8)

$$\pi := \vec{j}_u \cdot \nabla 1 - \vec{j}_c \cdot \nabla \mu - \frac{\lambda}{\tau} \frac{\partial \chi}{\partial t}, \quad \text{entropy production rate (per unit volume).}$$

(2.4.9)

The quantity $h/\tau$ is the rate at which entropy is either provided to the system or extracted from it by an external source or sink of heat. Note that $\vec{j}_u = \vec{q} + \mu \vec{j}_c$, where $\vec{q}$ is the heat flux; (2.4.8) and (2.4.9) then also read

$$\vec{j}_s = \frac{\vec{q}}{\tau}, \quad \pi = \frac{\vec{q} \cdot \nabla 1 - \vec{j}_c \cdot \nabla \mu - \frac{\lambda}{\tau} \frac{\partial \chi}{\partial t}}{\tau}.$$  

(2.4.10)

According to the local formulation of the second principle of thermodynamics, the entropy production rate is pointwise nonnegative, and vanishes only at equilibrium. This is
tantamount to the Clausius–Duhem inequality:

$$\pi \geq 0$$ for any process, and

$$\pi = 0$$ only if $\nabla \tau = \nabla \mu = \vec{0}$ and $\partial \chi / \partial t = 0.$ \hfill (2.4.11)

Moreover, $\pi = 0$ ($\pi > 0$, resp.) corresponds to a reversible (irreversible, resp.) process. We must then formulate constitutive laws consistent with (2.4.11).

**Phenomenological laws.** Let us introduce some further definitions:

$\vec{J} := (\vec{j}_u, \vec{j}_c, \partial \chi / \partial t)$: generalized fluxes,

$z := (1/\tau, -\mu / \tau, -\lambda / \tau)$ (\in Dom(s^*)): dual state variables, $z' := (1/\tau, -\mu / \tau)$,

$\vec{G} := (\nabla (1/\tau), -\nabla (\mu / \tau), -\lambda / \tau)$: generalized forces.$^{57}$

Along the lines of nonequilibrium thermodynamics, we assume that the generalized fluxes are functions of the dual state variables and of the generalized forces, via constitutive relations called *phenomenological laws*,

$$\vec{J} = \vec{F}(z, \vec{G}) \quad \forall z \in \text{Dom}(s^*), \quad \forall \vec{G} \in (\mathbb{R}^3)^2 \times \mathbb{R},$$ \hfill (2.4.12)

that must be consistent with the second principle, cf. (2.4.11). The mapping $\vec{F}$ must thus be positive-definite w.r.t. $\vec{G}$. Close to thermodynamic equilibrium, namely, for small generalized forces, one may also assume that this dependence is linear. As the first two components of $\vec{J}$ and $\vec{G}$ are vectors and the third one is a scalar, the above linearized relations uncouple, because of the *Curie principle*: “generalized forces cannot have more elements of symmetry than the generalized fluxes that they produce.”$^{58}$ Thus

$$(j_u, j_c) = \mathcal{L}(z) \cdot \left( \begin{array}{c} \frac{1}{\tau} \\ -\nabla \mu / \tau \end{array} \right) \left( = \mathcal{L}(z) \cdot \nabla z' \right),$$ \hfill (2.4.13)

$$a(z) \frac{\partial \chi}{\partial t} = -\frac{\lambda}{\tau}, \text{ that is, } a(z) \frac{\partial \chi}{\partial t} \in \partial \hat{s}.$$ \hfill (2.4.14)

(In (2.4.13) the dot denotes the rows by columns product of a tensor of $(\mathbb{R}^3)^2 \times 2$ by a vector of $(\mathbb{R}^3)^2$.) Consistently with (2.4.11), for any $z$ the tensor $\mathcal{L}(z)$ is assumed to be positive-definite, and $a(z) > 0$. A fundamental result of nonequilibrium thermodynamics due to Onsager states that the tensor $\mathcal{L}(z)$ is symmetric:

$$\mathcal{L} = \left( \begin{array}{cc} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{array} \right), \quad \mathcal{L}_{12}(z) = \mathcal{L}_{21}(z) \quad (\in \mathbb{R}^3) \quad \forall z \in \text{Dom}(s^*).$$ \hfill (2.4.15)

The phenomenological laws (2.4.13) and (2.4.14) may then be represented in gradient form, for a suitable potential $\Phi$:

$$\vec{J} = \nabla \Phi(z, \vec{G}) \quad \forall z \in \text{Dom}(s^*), \quad \forall \vec{G} \in (\mathbb{R}^3)^2 \times \mathbb{R},$$ \hfill (2.4.16)

where by $\nabla \Phi$ we denote the gradient w.r.t. the second argument, $\vec{G}$.

$^{57}$ Notice that $-\lambda / \tau$ occurs as a dual state variable and also as a generalized force. See Section 5.2 for the definition of the conjugate concave function $s^*$.

$^{58}$ See e.g. the monographs quoted in the item (VII) of the Bibliographical Note in Section 6.
In conclusion, we have represented processes in two-phase composites by the quasi-linear parabolic system (2.4.1), (2.4.2), (2.4.6), (2.4.13), (2.4.14). This system is doubly nonlinear, and the techniques of Alt and Luckhaus [17], DiBenedetto and Showalter [179], and others may be used.

A transformation of the state variables. The vector of state variables $z := (1/\tau, -\mu/\tau, -\lambda/\tau)$ may equivalently be replaced by $\tilde{z} := (1/\tau, \mu, -\lambda/\tau)$; let us also set $\tilde{z}' := (1/\tau, \mu, -\lambda/\tau)$. As the corresponding transformation $\nabla z' \rightarrow \nabla \tilde{z}'$ is linear and $\vec{j}_u = \vec{q} + \mu \vec{j}_c$, by a simple calculation the linearized phenomenological laws (2.4.13) may equivalently be reformulated in terms of these new variables:

$$\begin{pmatrix} \bar{q} \\ \bar{j}_c \end{pmatrix} = \mathcal{M}(\tilde{z}) \cdot \begin{pmatrix} \nabla \frac{1}{\tau} \\ -\frac{1}{\tau} \nabla \mu \end{pmatrix} \equiv \tilde{\mathcal{M}}(\tilde{z}) \cdot \nabla \tilde{z}'. \quad (2.4.17)$$

(The latter equality represents the definition of the tensor $\tilde{\mathcal{M}}$.) Like $\mathcal{L}$, the tensor $\mathcal{M}$ is also positive-definite for any $\tilde{z}$. By the symmetry of $\mathcal{L}$ it is easily checked that $\mathcal{M}$ is also symmetric, at variance with $\tilde{\mathcal{M}}$. These conclusions may also be attained by applying the above argument based on the second principle to (2.4.10), instead of (2.4.9).

If the tensor $\mathcal{M}(\tilde{z})$ is diagonal and depends continuously on its arguments, then by the developments of the final part of Section 2.2 we retrieve Fourier- and Fick-type laws:

$$\begin{align*}
\bar{q} &= \mathcal{M}_{11}(\tilde{z}) \nabla \frac{1}{\tau} = -\frac{1}{\tau E} \mathcal{M}_{11}(\tilde{z}) \nabla \theta \left[ 1 - 2 \frac{\theta}{\tau E} + o \left( \frac{\theta}{\tau E} \right) \right], \\
\bar{j}_c &= -\frac{1}{\tau} \mathcal{M}_{22}(\tilde{z}) \nabla \mu = -\frac{1}{\tau E} \mathcal{M}_{22}(\tilde{z}) \nabla \mu \left[ 1 - \frac{\theta}{\tau E} + o \left( \frac{\theta}{\tau E} \right) \right],
\end{align*} \quad (2.4.18)$$

with $\mathcal{M}_{11}$ and $\mathcal{M}_{22}$ positive scalars (more generally, positive-definite $3 \times 3$-tensors). On the other hand, a nondiagonal tensor $\mathcal{M}$ would also account for the Soret and Dufour cross effects.

As $\mathcal{M}_{22}$ and $\tilde{D}_{ij}(w)$ (cf. (2.3.14)) are both positive, by comparing (2.3.15) with (2.4.18) we see that $\nabla w$ is proportional to $-\nabla \mu$.

REMARKS. (i) It is possible to define a function analogous to (2.2.15), with the further dependence on $c$:

$$\varphi := u - \tau E s, \quad \text{that is, } \varphi = \hat{\varphi}(u, c, \chi) := u - \tau E \hat{s}(u, c, \chi). \quad (2.4.19)$$

Along the lines of (2.2.15)–(2.2.21), one may reformulate the entropy balance in terms of $\varphi$, $\theta$, $\mu$ and $\lambda$, and then linearize the state variables $1/\tau$, $-\lambda/\tau$, and $-\mu/\tau$ in a neighborhood of $\tau = \tau E$.

(ii) An approach based on nonequilibrium thermodynamics may also be applied to other coupled phenomena with phase transition, e.g. thermal and electromagnetic processes in a ferromagnetic body (with negligible hysteresis), see Visintin [449].

2.5. Diffuse-interface models and length-scales

In this section we introduce the Landau–Ginzburg representation of the free energy of a two-phase system, the associated Cahn–Hilliard and Allen–Cahn dynamics, and the
Penrose–Fife and phase-field models. As these models are set at a finer length-scale than the (macroscopic) scale that we considered in Section 1, we relate these free energy functionals and that of Section 2.1 by means of De Giorgi’s notion of $\Gamma$-limit.59

**Double wells.** Along the lines of the Landau–Ginzburg theory of phase transitions, see e.g. Landau and Lifshitz [307], here we fix two positive parameters $b$ and $\varepsilon$, and represent the free energy of a solid–liquid system by a functional of the form

$$F_{\theta, \varepsilon}(\chi) := \int_\Omega \left( \varepsilon b |\nabla \chi|^2 + \frac{1}{\varepsilon} (1 - \chi^2)^2 - \frac{L}{2\tau E} \theta \chi \right) \, dx \quad \forall \chi \in H^1(\Omega), \quad (2.5.1)$$

plus a constant that may depend on the temperature field. By the direct method of the calculus of variations,60 for any $\varepsilon > 0$ this functional has an absolute minimizer. The terms $\varepsilon b |\nabla \chi|^2$ and the double well potential $(1 - \chi^2)^2/\varepsilon$ compete for the minimization of $F_{\theta, \varepsilon}(\chi)$. As the second term is minimized by $\chi = \pm 1$, a temperature having nonuniform sign may induce sharp variations of $\chi$ between $-1$ to $1$; but high gradients of $\chi$ are penalized by the first term. Compromising between these two exigences, for small $\varepsilon$ any relative minimizer of $F_{\theta, \varepsilon}(\chi)$ attains values that are close to $\pm 1$ in the whole $\Omega$, but for thin transition layers. The actual physical value of the coefficients $b, \varepsilon$ is so small that the layer thickness is typically of the order of nanometers.

The functional (2.5.1) is Fréchet differentiable, and its functional derivative reads

$$DF_{\theta, \varepsilon}(\chi) = -\frac{2 \varepsilon b}{\varepsilon^2} \Delta \chi + \frac{4}{\varepsilon^2} \chi (\chi^2 - 1) - \frac{L \theta}{2\tau E} \forall \chi \in H^1(\Omega). \quad (2.5.2)$$

Because of the nonconvexity, a stationary point of $F_{\theta, \varepsilon}$ may either be an absolute minimizer, or a relative minimizer (namely, the absolute minimizer of the restriction of $F_{\theta, \varepsilon}$ to some neighbourhood of that point), or a saddle point, or even a relative maximizer. These points may respectively be interpreted as states of stable, metastable, and for the two latter cases) unstable equilibrium.61

**Two relaxation dynamics.** The phase function $\chi$ may be regarded as an order parameter; the same applies to the solute concentration in alloys, the magnetization in ferromagnetics, the polarization in ferroelectrics, and so on. One may distinguish between phenomena in which for an isolated system the integral of the order parameter is conserved, and those in which it is not. Phase separation in alloys belongs to the first class, phase transition in solid–liquid systems to the second one.

Conserved-integral dynamics. Along the lines of Hohenberg and Halperin [274], in the first case one typically represents processes by a relaxation dynamics of the form

$$a \frac{\partial \chi}{\partial t} - \nabla \cdot \left\{ K \cdot \nabla \left[ DF_{\theta, \varepsilon}(\chi) \right] \right\} = 0 \quad \text{in} \ Q; \quad (2.5.3)$$

59 See e.g. the monographs quoted in the item (XIV) of the Bibliographical Note in Section 6.

60 See e.g. Braides and Defranceschi [82], Buttazzo, Giavunia and Hildebrand [94], Carbone and De Arcangelis [117], Dacorogna [154], Dal Maso [155], Evans and Gariery [201], Giusti [245].

61 The notion of metastability is (implicitly) referred to a time-scale. A (nonabsolute) relative minimizer will appear as stable at a sufficiently fine time-scale, and as unstable at a sufficiently long time-scale. Steel, polymers and glasses are examples of these state-stable relative minimizers, up to geological time-scales.
here $a$ is a positive relaxation coefficient, and $K$ is a positive-definite tensor that may depend on the state variables. For instance let us consider a binary alloy and denote by $c$ the concentration of one component; in this case $L = 0$. By coupling Eq. (2.5.3) with the homogeneous boundary condition $\{ K \cdot \nabla [DF_{\theta,\varepsilon}(c)] \} \cdot \nu = 0$, one then sees that $\int_{\Omega} c(x,t) \, dx$ is constant in time. After Cahn [107,108], Cahn and Hilliard [109], for isothermal processes one thus gets the Cahn–Hilliard equation of phase separation:

$$a \frac{\partial c}{\partial t} + \nabla \cdot \left\{ K \cdot \nabla \left[ -2\varepsilon b \Delta c + \frac{4}{\varepsilon} c(c^2 - 1) \right] \right\} = \lambda \quad \text{in } Q,$$  

(2.5.4)

where $\lambda$ is the (unknown) Lagrange multiplier associated to the constraint on the integral of $\chi$. The phenomenon of phase separation and the analytical properties of the Cahn–Hilliard equation were studied in a large literature.62

Nonconserved-integral dynamics. On the other hand for solid–liquid systems the typical dynamics of the phase function $\chi$ (and more generally that of nonconserved order parameters) reads

$$a \frac{\partial \chi}{\partial t} + DF_{\theta,\varepsilon}(\chi) = 0 \quad \text{in } Q.$$  

(2.5.5)

For $F_{\theta,\varepsilon}(\chi)$ as in (2.5.1), this yields the Allen–Cahn (or Landau–Ginzburg) equation, see Allen and Cahn [13]:

$$a \frac{\partial \chi}{\partial t} - 2\varepsilon b \Delta \chi + \frac{4}{\varepsilon} \chi(\chi^2 - 1) = \frac{L \theta}{2\tau_E} \quad \text{in } Q.$$  

(2.5.6)

The Penrose–Fife and phase-field models. So far we dealt with the representation of the free energy functional and with its dynamics. For nonisothermal processes equation (2.5.6) must be coupled with the energy balance. As we saw, the theory of nonequilibrium thermodynamics leads one to formulate the Fourier law as the proportionality between the heat flux and $-\nabla (1/\tau)$, cf. (2.2.12). An approach of this sort was proposed by Penrose and Fife in [373,374], and then studied in many works.63

By linearizing $1/\tau$ in a neighbourhood of $1/\tau_E$ we have $\nabla (1/\tau) \simeq -\nabla \theta / \tau_E^2$, and the Fourier law is reduced to the form (1.1.3). By coupling this law with the energy balance (1.1.2) and with the free energy dynamics (2.5.6), one obtains the so-called phase-field model, which was first proposed by Fix [219–221] and Collins and Levine [146], and was then extensively studied by Caginalp and others.64

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62 See e.g. Alikakos, Bates and Chen [10], Bates and Fife [50,51], Blowey and Elliott [69], Caginalp [104], Chen [126], Chen, Hong and Yi [127], Elliott [190], Elliott and Garke [191], Elliott and Zheng [194], Escher and Simonett [197], Kessler et al. [287], Novick-Cohen and Segel [361], Pego [372], Rappaz and Scheid [385]. See also Alt and Pawlow [18–20], Fabrizio, Giorgi and Morro [204], for the extension to nonisothermal processes.

63 See e.g. Bonetti et al. [71], Chen and Fife [125], Colli and Laurençot [139], Colli and Plotnikov [140], Colli and Sprekels [142,143], Fife [217], Hilliard [268], Kenmochi and Kubo [286], Miranville, Yin and Showalter [333], Sprekels and Zheng [429], Wang et al. [461], Zheng [476].

64 See e.g. Aizicovici and Barbuc [5], Caginalp [101–104], Caginalp and Xie [105], Colli, Gilardi and Grasselli [132], Colli et al. [133], Fried and Gurtin [226–228], Miranville, Yin and Showalter [333], Krejčí, Rocca and Sprekels [297], Novick-Cohen [359,360], Plotnikov and Starovoitov [375].
A limit procedure. As we already pointed out, the parameter $\epsilon$ defines a nanoscopic length-scale. It is then of interest to study the limit of the free energy $F_{\theta,\epsilon}$ as $\epsilon$ vanishes, in order to retrieve a model at a larger length-scale. The notion of $\Gamma$-limit in the sense of De Giorgi, see Section 5.8, is especially appropriate to represent the asymptotic behaviour of the absolute minimizers of $F_{\theta,\epsilon}$, see Proposition 5.8.2.65

**PROPOSITION 2.5.1 (Γ-limit)**66. Let $\theta \in L^1(\Omega)$ and $\sigma = 4\sqrt{b}/3$. As $\epsilon \to 0$, the family of functionals $\{F_{\theta,\epsilon}\} \Gamma$-converges to $\Phi_{\theta,\sigma}$ in $L^1(\Omega)$ (here with $\sigma_L = \sigma_S$), cf. (2.1.5).

This result directly follows from Theorem 5.8.4. By Corollary 5.8.3 the latter statement entails that, if $u_\epsilon$ is an absolute minimizer of the functional $F_{\theta,\epsilon}$ for any $\epsilon$, then there exists a state $u \in L^1(\Omega)$ such that, as $\epsilon$ vanishes along a suitable sequence (not relabelled),

$$u_\epsilon \to u \quad \text{in} \quad L^1(\Omega), \quad F_{\theta,\epsilon}(u_\epsilon) \to \Phi_{\theta,\sigma}(u) = \inf \Phi_{\theta,\sigma}.$$ (2.5.7)

**Macroscopic-mesoscopic, and microscopic length-scales.** We represented the free energy of solid–liquid systems at three length-scales, see Table 3:

(i) At the macroscopic scale the functional $\Phi_\theta$ is convex, cf. (2.1.6), and processes may be described by the weak formulation of the Stefan problem, i.e., Problem 1.1.1. At this length-scale a mushy region may appear, corresponding to the condition $|\chi| < 1$ a.e.. The solid–liquid interface may accordingly be either sharp or diffuse.

(ii) At the mesoscopic scale $\sigma \ll 1$ the functional $\Phi_{\theta,\sigma}$ is nonconvex, cf. (2.1.5), and evolution may be represented by the Stefan–Gibbs–Thomson problem. Here $|\chi| = 1$ a.e., and thus one distinguishes solid from liquid parts, also in the mushy region. In other terms, what at the macroscopic scale appears as a mushy region is here resolved in its liquid and solid constituents.

(iii) At the microscopic scale $\epsilon \ll \sigma$ the functional $F_{\theta,\epsilon}$ is also nonconvex, cf. (2.5.1), and evolution may be described by the phase-field model. Here $\chi$ varies smoothly: the interface is represented by a nanoscopic transition layer, and may thus be regarded as diffuse. In this case $|\chi| < 1$ a.e., but intermediate values of $\chi$ represent a transition layer rather than a mushy region.

For instance, length-scales of the order of the millimeter, of the micrometer and of the nanometer may loosely be labelled as macroscopic, mesoscopic and microscopic, respectively. The process of zooming out from the microscopic to the mesoscopic scale is here represented by the $\Gamma$-limit as $\epsilon \to 0$. On the other hand the $\Gamma$-limit as $\sigma \to 0$ accounts for the passage from the mesoscopic to the macroscopic scale.

For evolution problems the asymptotic analysis is more delicate. A large literature was devoted to the models of Cahn–Hilliard, Allen–Cahn, Mullins–Sekerka, Stefan–Gibbs–Thomson, in particular to establish asymptotic relations among them.67

65 The analysis of the limit behaviour of the relative minimizers would also be of interest, but needs a different approach; see e.g. Dal Maso and Modica [156,157].

66 See Luckhaus and Modica [315], Modica [335,336] for related results.

67 See e.g. Alikakos, Bates and Chen [10], Alikakos, Bates, Chen and Fusco [11], Alikakos, Fusco and Kari-ali [12], Caginalp [103,104], Escher and Simonett [197], Evans, Soner and Souganidis [202], Garroni and
Table 3
Comparison among some properties of the solid–liquid interface at different length-scales

<table>
<thead>
<tr>
<th>Macroscopic scale</th>
<th>Mesoscopic scale</th>
<th>Microscopic scale</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stefan model</td>
<td>Stefan–Gibbs–Thomson model</td>
<td>Phase-field model</td>
</tr>
<tr>
<td>Sharp/diffuse interface</td>
<td>Sharp interface</td>
<td>Diffuse interface</td>
</tr>
<tr>
<td>Convex free energy</td>
<td>Nonconvex free energy</td>
<td>Nonconvex free energy</td>
</tr>
<tr>
<td>$</td>
<td>\chi</td>
<td>\leq 1$</td>
</tr>
<tr>
<td>No $\nabla \chi$ in free energy</td>
<td>$\sigma \int_{\Omega}</td>
<td>\nabla \chi</td>
</tr>
<tr>
<td>$\chi \in L^\infty(\Omega)$</td>
<td>$\chi \in L^\infty(\Omega) \cap BV(\Omega)$</td>
<td>$\chi \in L^\infty(\Omega) \cap H^1(\Omega)$</td>
</tr>
</tbody>
</table>

Discussion. The Landau–Ginzburg potential $F_{\theta,\varepsilon}$ may look rather appealing to the mathematical analyst: its principal part is quadratic in the derivatives, the nonconvexity is confined to the null-order term and is smooth. This functional is Frechét differentiable, and the differential part of the derivative is linear, so that the Euler equation is semilinear. The Cahn–Hilliard and Allen–Cahn relaxation dynamics are easily formulated, and are also semilinear. What better?

On the other hand, its $\Gamma$-limit as $\varepsilon \to 0$, namely the functional $\Phi_{\theta,\sigma}$, cf. (2.1.5), misses all these nice features. Its principal part is nonquadratic (even worse: it has critical growth of degree 1), and is not integrable: it is just a Borel measure. Here the nonconvexity is highly nonsmooth: it is the characteristic constraint, namely $| \chi | = 1$. The functional $\Phi_{\theta,\sigma}$ is nondifferentiable, and the associated dynamics is also nontrivial: it consists of the mean curvature flow with forcing term. What worse?

All elements seem to indicate that $F_{\theta,\varepsilon}$ should be preferred to $\Phi_{\theta,\sigma}$. In favour of the latter there are however two features: $\Phi_{\theta,\sigma}(\chi)$ is more appropriate for mesoscopic models, and it may be discretized by means of a coarser mesh, for it is set at a larger length-scale than $F_{\theta,\varepsilon}$.

3. Analysis of the weak formulation of the Stefan model

In Section 1.1 we represented the weak formulation of the basic Stefan model as an initial- and boundary-value problem for the quasilinear parabolic system

$$
\begin{align*}
\left\{ 
    u & \in \alpha(\theta), \\
    \frac{\partial u}{\partial t} - \nabla \cdot [k(\theta) \cdot \nabla \theta] &= f \\
\end{align*}
$$

(3.0.1)

Niethammer [239], Luckhaus and Sturzenhecker [316], Krejčí, Rocca and Sprekels [297] Miranville, Yin and Showalter [333], Niethammer [352,353], Plotnikov and Starovoitov [375], Röger [402], Soner [427,428], Stoth [432], and the detailed review of Soner [426].

68 See e.g. the monographs Almgren and Wang [16], Buttazzo and Visintin [95], Damlamian, Spruck and Visintin [163], Evans and Spruck [203], Giga [242] and references therein.

69 One might also question the use of a continuous model at a microscopic (actually, nanoscopic) length-scale: is it really justified to apply differential calculus at a scale at which the discrete structure of matter starts becoming perceivable?
for a multi-valued maximal monotone function $\alpha$, cf. Problem 1.1.1. In this part we illustrate a number of classical methods that may be used for the analysis of this problem. Specifically, we deal with:

(i) approximation by implicit time-discretization (Section 3.1),
(ii) a priori estimates in $L^2$-spaces (Section 3.1),
(iii) passage to the limit via compactness, monotonicity and lower semicontinuity techniques (Section 3.1),
(iv) a contraction procedure in $L^1(\Omega)$ (Section 3.2),
(v) a priori estimates in $L^\infty(\Omega)$ (Section 3.2),
(vi) a priori estimates in $L^q(\Omega)$ (Section 3.2),
(vii) a variable transformation via time-integration (Section 3.3),
(viii) a variable transformation via inversion of the elliptic operator (Section 3.3),
(ix) nonlinear semigroups of contractions in $H^{-1}(\Omega)$ and in $L^1(\Omega)$ (Section 3.4).70

3.1. $L^2$-techniques

In this section we prove the existence of a solution of the weak formulation of the basic Stefan problem in any prescribed time interval, show its structural stability, and derive some regularity results. This gives us the opportunity to illustrate some basic techniques for the analysis of quasilinear parabolic equations in Sobolev spaces.71

As above we shall assume that $\Omega$ is a bounded domain of $\mathbb{R}^3$ of Lipschitz class, denote its boundary by $\Gamma$, fix any $T > 0$, and set $Q := \Omega \times ]0, T]$, $\Sigma := \Gamma \times ]0, T]$. We also fix a subset $\Gamma_D$ of $\Gamma$ of positive bidimensional Hausdorff measure, and set $\Gamma_N := \Gamma \setminus \Gamma_D$,

$$V := \{v \in H^1(\Omega) : \gamma_0 v = 0 \text{ on } \Gamma_D\},$$

where $\gamma_0$ denotes the trace operator.72 This is a Hilbert space equipped with the customary $H^1$-norm, which by the Friedrichs-Poincaré inequality is equivalent to

$$\|v\|_V := \left(\int_\Omega |\nabla v|^2 \, dx\right)^{1/2}.$$

We shall identify the space $L^2(\Omega)$ with its dual $L^2(\Omega)'$. As $V$ is a dense subspace of $L^2(\Omega)$, the dual space $L^2(\Omega)'$ may in turn be identified with a subspace of $V'$. This yields the Hilbert triplet

$$V \subset L^2(\Omega) = L^2(\Omega)' \subset V', \quad \text{with dense and compact injections. \quad (3.1.2)}$$

We shall denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $V'$ and $V$, and define the linear, continuous and coercive operator

$$A : V \to V', \quad \langle Au, v \rangle := \int_\Omega \nabla u \cdot \nabla v \, dx \quad \forall u, v \in V,$$

70 This part rests upon the classical theory of linear and nonlinear PDEs, see e.g. the monographs quoted in the items (X)–(XIII) of the Bibliographical Note in Section 6.

71 See e.g. the monographs quoted in the item (XIII) of the Bibliographical Note in Section 6, Alt and Luckhaus [17], Brezis [83], Damlamian [158], Damlamian and Kenmochi [161], and many others.

72 See e.g. Lions and Magenes [312].
whence $Au = -\Delta u$ in $\mathcal{D}'(\Omega)$. We also assume that\footnote{This function $\varphi$ should not be confused with that of Section 2.2.}

$$\varphi : \mathbb{R} \to \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$$

is lower semicontinuous and convex, and $\varphi \not\equiv +\infty$, \hspace{1cm} (3.1.3)

denote its convex conjugate function by $\varphi^*$, and notice that $\partial \varphi^* = (\partial \varphi)^{-1}$, cf. (5.2.11).

Finally, we fix any

$$u^0 \in L^2(\Omega), \quad f \in L^2(0, T; V'),$$

and introduce our weak formulation.

**PROBLEM 3.1.1.** Find $u \in L^2(Q)$ and $\theta \in L^2(0, T; V)$ such that

$$u(\theta - v) \geq \varphi(\theta) - \varphi(v) \quad \forall v \in \text{Dom}(\varphi), \text{ a.e. in } Q, \hspace{1cm} (3.1.5)$$

\begin{align*}
\iint_Q \left[ (u^0 - u) \frac{\partial v}{\partial t} + \nabla \theta \cdot \nabla v \right] \, dx \, dt &= \int_0^T \langle f, v \rangle \, dt \\
\forall v &\in L^2(0, T; V) \cap H^1(0, T; L^2(\Omega)), \quad v(\cdot, T) = 0. \hspace{1cm} (3.1.6)
\end{align*}

**Interpretation.** By Proposition 5.2.5 the variational inequality (3.1.5) is tantamount to the inclusion

$$u \in \partial \varphi(\theta) \quad \text{a.e. in } Q. \hspace{1cm} (3.1.7)$$

The variational equation (3.1.6) yields

$$\frac{\partial u}{\partial t} + A\theta = f \quad \text{in } V' \text{ a.e. in } ]0, T[, \hspace{1cm} (3.1.8)$$

whence $\frac{\partial u}{\partial t} = f - A\theta \in L^2(0, T; V')$. Thus $u \in H^1(0, T; V')$, and by integrating \hspace{1cm} (3.1.6) by parts in time we get

$$u|_{t=0} = u^0 \quad \text{in } V' \text{ (in the sense of the traces of } H^1(0, T; V')) \hspace{1cm} (3.1.9)$$

In turn (3.1.8) and (3.1.9) yield (3.1.6). In view of interpreting Eq. (3.1.8), let us now take

$$g \in L^2(Q), \quad h \in L^2(\Gamma_N \times ]0, T[),$$

and define $f \in L^2(0, T; V')$ by setting

$$\langle f(t), v \rangle := \int_\Omega g(x, t)v(x) \, dx + \int_{\Gamma_N} h(x, t)\gamma_0 v(x) \, d\sigma$$

$$\forall v \in V, \text{ for a.a. } t \in ]0, T[. \hspace{1cm} (3.1.11)$$

In this case Eq. (3.1.8) then yields the differential equation

$$\frac{\partial u}{\partial t} - \Delta \theta = g \quad \text{in } \mathcal{D}'(Q), \hspace{1cm} (3.1.12)$$
where \( \Delta \theta = \partial u / \partial t - g \in H^{-1}(0, T; L^2(\Omega)) \). Denoting by \( \partial / \partial v \) the external normal derivative, the trace \( \partial \theta / \partial v \) is then an element of \( H^{-1}(0, T; H^{1/2}_0(\Gamma_N)) \). By partial integration of (3.1.6) we then retrieve the Neumann condition
\[
\frac{\partial \theta}{\partial v} = h \quad \text{in} \quad H^{-1}(0, T; H^{1/2}_0(\Gamma_N')).
\] (3.1.13)

Moreover the definition of \( V \) obviously yields the Dirichlet condition
\[
\gamma_0 \theta = 0 \quad \text{on} \quad \Gamma_D \times ]0, T[.
\] (3.1.14)

In conclusion, if \( f \) is as in (3.1.10) and (3.1.11), then (3.1.6) is a weak formulation of the system (3.1.12)–(3.1.14).

**Remark.** Let \( k \) be a continuous, bounded and positive function \( \mathbb{R} \to \mathbb{R} \). If we replace \( \Delta \theta \) by \( \nabla \cdot [k(\theta) \nabla \theta] \) in Eq. (3.1.12), then by the Kirchhoff transformation
\[
K : V \to V : \theta \mapsto \tilde{\theta} := \int_0^\theta k(\xi) \, d\xi
\] (3.1.15)
we get \( \nabla \cdot [k(\theta) \nabla \theta] = \Delta \tilde{\theta} \). As \( K \) is invertible, the inclusion (3.1.7) may then be replaced by \( u \in \partial \varphi(K^{-1}(\vartheta)) \) a.e. in \( Q \), which is also of the form \( u \in \partial \tilde{\varphi}(\tilde{\theta}) \), for a lower semicontinuous and convex function \( \tilde{\varphi} : \mathbb{R} \to \tilde{\mathbb{R}} \). A formulation like Problem 3.1.1 for the unknown pair \( (u, \tilde{\theta}) \) is thus retrieved in this case, too.

**Theorem 3.1.1 (Existence).** Assume that the hypotheses (3.1.3) and (3.1.4) are satisfied. If
\[
\exists L, M > 0: \forall (\xi, \eta) \in \text{graph}(\partial \varphi), \quad |\xi| \leq L|\eta| + M,
\] (3.1.16)
\[
\varphi^*(u^0) \in L^1(\Omega),
\] (3.1.17)
then Problem 3.1.1 has a solution such that \( u \in L^\infty(0, T; L^2(\Omega)) \).

**Proof.** (i) Approximation. We shall prove existence of a solution by means of a classic procedure: time-discretization, derivation of a priori estimates, and passage to the limit.\(^{75}\)

We fix any \( m \in \mathbb{N} \), set
\[
k := \frac{T}{m}, \quad u^0_m := u^0,
\] (3.1.18)
and approximate our problem by the following implicit time-discretization scheme.

\(^{74}\) By \( H^{1/2}_0(\Gamma_N) \) we denote the Hilbert space of the restrictions to \( \Gamma_N \) of the functions of \( H^{1/2}(\Gamma) \), that vanish a.e. in \( \Gamma \setminus \Gamma_N \). \( H^{1/2}(\Gamma) \) being the space of the traces of the functions of \( H^1(\Omega) \). See e.g. Lions and Magenes [312, vol. I] and other monographs on Sobolev spaces, that are quoted in the item (XII) of the Bibliographical Note in Section 6.

\(^{75}\) See e.g. the discussion in the introduction of Lions [311].
PROBLEM 3.1.1. Find $u^n_m \in L^2(\Omega)$ and $\theta^n_m \in V$ for $n = 1, \ldots, m$, such that for $n = 1, \ldots, m$
\begin{align}
\frac{u^n_m - u^{n-1}_m}{k} + A\theta^n_m &= f^n_m \quad \text{in } V',
\tag{3.1.19} \\
u^n_m &\in \partial \varphi(\theta^n_m) \quad \text{a.e. in } \Omega.
\tag{3.1.20}
\end{align}

In view of solving this problem step by step, let us fix any $n \in \{1, \ldots, m\}$, assume that $u^{n-1}_m \in L^2(\Omega)$ is known, and define the lower semicontinuous, convex, coercive functional

\[ J^n_m(v) := \int_{\Omega} \left[ \varphi(v) + \frac{k}{2} |\nabla v|^2 - u^{n-1}_m v \right] \, dx - k \langle f^n_m, v \rangle \quad \forall v \in V. \]

By the direct method of the calculus of variations, this functional has a minimizer $\theta^n_m \in V$. Hence $0 \in \partial J^n_m(\theta^n_m)$ in $V'$, whence by Theorem 5.2.3

\[ 0 \in \partial \varphi(\theta^n_m) + kA\theta^n_m - u^{n-1}_m - k f^n_m \quad \forall n \in \{1, \ldots, m\}. \tag{3.1.21} \]

This inclusion is equivalent to the system (3.1.19) and (3.1.20).

(ii) A priori estimates. First notice that by (3.1.20) and (5.2.11), for $n = 1, \ldots, m$
\[ \theta^n_m \in (\partial \varphi)^{-1}(u^n_m) = \partial \varphi^*(u^n_m) \quad \text{a.e. in } \Omega, \]
whence
\[ \theta^n_m(u^n_m - u^{n-1}_m) \geq \varphi^*(u^n_m) - \varphi^*(u^{n-1}_m) \quad \text{a.e. in } \Omega. \tag{3.1.22} \]

Let us now multiply (3.1.19) by $k \theta^n_m$ and sum for $n = 1, \ldots, \ell$, for any $\ell \in \{1, \ldots, m\}$. By (3.1.22) this yields
\begin{align}
\int_{\Omega} \left[ \varphi^*(u^n_\ell) - \varphi^*(u^0_m) \right] \, dx + k \sum_{n=1}^{\ell} \int_{\Omega} |\nabla \theta^n_m|^2 \, dx \\
\leq k \sum_{n=1}^{\ell} \|f^n_m\|_V \|\theta^n_m\|_V \leq \|f\|_{L^2(0,T;V')} \left( k \sum_{n=1}^{\ell} \|\theta^n_m\|_V^2 \right)^{1/2}. \tag{3.1.23}
\end{align}

By (3.1.16) it is easy to check that
\[ \varphi^*(\eta) \geq \frac{1}{2L}(\eta - M)^2 \quad \forall \eta \in \mathbb{R} \text{ such that } |\eta| \geq M. \tag{3.1.24} \]

The inequality (3.1.23) then yields
\[ \max_{n=1,\ldots,m} \left\| u^n_m \right\|_{L^2(\Omega)} \leq C_1. \tag{3.1.25} \]

(By $C_1, C_2, \ldots$, we shall denote several constants that are independent of $m$.)

For any family $\{v^n_m\}_{n=0,\ldots,m}$ of functions $\Omega \to \mathbb{R}$, let us now set
\begin{align}
v_m &:= \text{piecewise linear time-interpolate of } v^0_m, \ldots, v^m_m, \text{ a.e. in } \Omega,
\bar{v}_m(\cdot, t) &:= v^n_m \quad \text{a.e. in } \Omega, \forall t \in [(n-1)h, nh[ \text{ for } n = 1, \ldots, m. \tag{3.1.26}
\end{align}
Defining \( u_m, \bar{u}_m, \bar{\theta}_m, \bar{f}_m \) in this way, the system (3.1.19), (3.1.20) and the estimates (3.1.25) also read

\[
\frac{\partial u_m}{\partial t} + A\bar{\theta}_m = \bar{f}_m \quad \text{in } V', \quad \text{a.e. in } [0, T], \tag{3.1.27}
\]

\[
\bar{u}_m \in \partial \varphi(\bar{\theta}_m) \quad \text{a.e. in } Q, \tag{3.1.28}
\]

\[
\|u_m\|_{L^\infty(0,T;L^2(\Omega))}, \|\bar{\theta}_m\|_{L^2(0,T;V)} \leq C_2. \tag{3.1.29}
\]

Hence \( A\bar{\theta}_m \) is uniformly bounded in \( L^2(0,T;V') \), and by comparing the terms of (3.1.27) we get

\[
\|u_m\|_{H^1(0,T;V')} \leq C_3. \tag{3.1.30}
\]

(iii) Limit procedure. By the uniform estimates (3.1.29) and (3.1.30), there exist \( \theta, u \) such that, possibly taking \( m \to \infty \) along a subsequence (not relabelled),

\[
\bar{\theta}_m \to \theta \quad \text{weakly in } L^2(0,T;V), \tag{3.1.31}
\]

\[
u_m \to u \quad \text{weakly star in } L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;V'). \tag{3.1.32}
\]

By passing to the limit in (3.1.27) we then get (3.1.8). In view of proving (3.1.5), first notice that by the compactness of the injection \( V \subset L^2(\Omega) \) and by Sobolev-space interpolation\(^76\)

\[
L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;V') \subset C^0([0,T];V')
\]

so that by (3.1.32)

\[
u_m \to u \quad \text{strongly in } C^0([0,T];V'). \tag{3.1.34}
\]

Hence \( \|\bar{u}_m - u_m\|_{V'} \to 0 \) uniformly in \([0, T]\), and we get

\[
\iint_Q \bar{u}_m \bar{\theta}_m \, dx \, dt = \int_0^T (\bar{u}_m - u_m, \bar{\theta}_m) \, dt + \iint_Q u_m \bar{\theta}_m \, dx \, dt \to \iint_Q u\theta \, dx \, dt. \tag{3.1.35}
\]

By (3.1.20), (3.1.31), (3.1.32), (3.1.35), applying Corollary 5.5.5 we then get (3.1.5).

Finally, by (3.1.34) the initial condition for \( u \) (cf. (3.1.18)) is preserved in the limit. \( \Box \)

**REMARK.** By the self-adjointness of the operator \(-\Delta\), existence of a solution might also be proved without exploiting the compactness of the injection \( V \subset L^2(\Omega) \), by a procedure that is only based on the convexity and lower semicontinuity of the function \( \varphi \).\(^77\)

\(^76\) See e.g. Lions and Magenes [312].

\(^77\) See e.g. Visintin [453, Section II.3].
Next we show that our problem is structurally stable.

**Theorem 3.1.2 (Weakly-continuous dependence on the data).** Let us assume that \( \{ \varphi_n \} \), \( \{ f_n \} \), \( \{ u^0_n \} \) are sequences that fulfill the assumptions of Theorem 3.1.1, with \( L \) and \( M \) independent of \( n \) in (3.1.16). Let us also assume that the sequence \( \{ \varphi^*_n(u^0_n) \} \) is bounded in \( L^1(\Omega) \), and that

\[
\varphi_n \to \varphi \quad \text{uniformly in } \mathbb{R}, \quad (3.1.36)
\]
\[
f_n \to f \quad \text{weakly in } L^2(0, T; V'), \quad (3.1.37)
\]
\[
u^0_n \to u^0 \quad \text{weakly in } L^2(\Omega). \quad (3.1.38)
\]

For any \( n \), let \( (u_n, \theta_n) \) be a solution of the corresponding Problem 3.1.1.\(^{78}\) Then there exist \( \theta \) and \( u \) such that, as \( n \to \infty \) along a suitable sequence (not relabelled),

\[
u_n \to u \quad \text{weakly star in } L^\infty(0, T; L^2(\Omega)) \cap H^1(0, T; V'), \quad (3.1.39)
\]
\[
\theta_n \to \theta \quad \text{weakly in } L^2(0, T; V). \quad (3.1.40)
\]

This entails that \( (u, \theta) \) is a solution of Problem 3.1.1.

We shall see that the solution of Problem 3.1.1 is unique, so that it is not necessary to extract any subsequence in (3.1.39) and (3.1.40).

**Outline of the proof.**\(^{79}\) By multiplying Eq. (3.1.8) by \( \theta_n \) and then using the procedure of the proof of Theorem 3.1.1, one may easily derive uniform estimates like (3.1.29) and (3.1.30). One can then pass to the limit as above as \( n \to \infty \) along a subsequence, and show that \( (u, \theta) \) is the solution of Problem 3.1.1 by mimicking the argument of Theorem 3.1.1. \( \square \)

**Regularity.** Several regularity properties may be proved for the solution of Problem 3.1.1, by deriving further a priori estimates on the approximate solution. Next we just illustrate three examples, all based on the symmetry of the elliptic operator (i.e., \( \Delta \)).

**Proposition 3.1.3 (First regularity result).** If the assumptions of Theorem 3.1.2 are satisfied and

\[
\exists c > 0: \forall (\xi_i, \eta_i) \in \text{graph}(\partial \varphi) (i = 1, 2), \quad (\xi_1 - \xi_2)(\eta_1 - \eta_2) \geq c(\xi_1 - \xi_2)^2, \quad (3.1.41)
\]

then Problem 3.1.1 has a solution such that

\[
\theta \in H^1(0, T; L^2(\Omega)) \quad \forall r < \frac{1}{2}, \quad (3.1.42)
\]

\(^{78}\) By this we denote the problem that is obtained from Problem 3.1.1 by replacing the functions \( u, \theta, \ldots \) by \( u^*_n, \theta^*_n, \ldots \). We shall use this sort of notation several repeatedly.

\(^{79}\) See Visintin [453, Section II.3] for details.
OUTLINE OF THE PROOF. The first part of the proof of Theorem 3.1.1 yields (3.1.29) and (3.1.30). In view of deriving a further uniform estimate, let us first fix any $h \in ]0, T[$ and define the shift operator $(\tau_h v)(t) := v(t + h)$ for any $t \in \mathbb{R}$ and any function $v : \mathbb{R} \to \mathbb{R}$. Let us then set $\theta_m(\cdot, t) := \theta_m(\cdot, 0)$ for any $t < 0$, multiply the approximate equation (3.1.27) by $\theta_m - \tau_h \theta_m$, and then integrate with respect to $t$. This procedure provides a uniform estimate for $\theta_m$ in $H^r(0, T; L^2(\Omega))$. □

PROPOSITION 3.1.4 (Second regularity result). If the assumptions of Theorem 3.1.2 and (3.1.41) are satisfied and moreover

$$\theta^0 := \partial \varphi^*(u^0) \in V,$$

$$f \in L^2(Q) + W^{1,1}(0, T; V'),$$

then Problem 3.1.1 has a solution such that

$$u \in L^\infty(0, T; L^2(\Omega)), \quad \theta \in H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; V).$$

OUTLINE OF THE PROOF. By (3.1.41) we have

$$\int_\Omega \left( u_m^n - u_m^{n-1} \right) \left( \theta_m^n - \theta_m^{n-1} \right) dx \geq c \int_\Omega \left( \theta_m^n - \theta_m^{n-1} \right)^2 dx$$

for $n = 1, \ldots, \ell$. (3.1.46)

Multiplying (3.1.19) by $\theta_m^n - \theta_m^{n-1}$ and summing for $n = 1, \ldots, \ell$, for any $\ell \in \{1, \ldots, m\}$, one then gets uniform estimates on $u_m$ and $\theta_m$ that yield (3.1.45) in the limit.

PROPOSITION 3.1.5 (Third regularity result). If (3.1.3), (3.1.16), (3.1.41) are satisfied and

$$f = f_1 + f_2, \quad \sqrt{t} f_1 \in L^2(Q), \quad \sqrt{t} \frac{\partial f_2}{\partial t} \in L^1(0, T; V'),$$

then Problem 3.1.1 has a solution such that

$$\sqrt{t} \frac{\partial \theta}{\partial t} \in L^2(Q), \quad \sqrt{t} \theta \in L^\infty(0, T; V).$$

Hence $\theta \in H^1(\delta, T; L^2(\Omega)) \cap L^\infty(\delta, T; V)$ for any $\delta > 0$, without any hypothesis for the initial datum $u^0$.

OUTLINE OF THE PROOF. The further regularity (3.1.48) stems from an estimation procedure that follows the lines of the argument that we used above to derive (3.1.25); the main difference is that here one multiplies (3.1.19) by $nk(\theta_m^n - \theta_m^{n-1})$, instead of $\theta_m^n - \theta_m^{n-1}$.

80 See Visintin [453, Section II.2] for details.
81 Notice that (3.1.41) entails that the mapping $\partial \varphi^*$ is single-valued.
82 See Visintin [453, Section II.2] for details.
83 See Visintin [453, Section II.2] for details.
Notice that
\[
\sum_{n=1}^{\ell} (A\theta^n_m \cdot nk (\theta^n_m - \theta^{n-1}_m)) \geq \frac{\ell k}{2} \int_{\Omega} |\nabla \theta^\ell_m|^2 \, dx - \frac{k}{2} \sum_{n=0}^{\ell-1} \int_{\Omega} |\nabla \theta^n_m|^2 \, dx,
\]
and the latter sum is uniformly estimated because of (3.1.25). One thus gets a uniform estimate for \(\theta_m\) that corresponds to (3.1.48).

By a judicious choice of the test function, one may also prove results of local regularity in space and time, see e.g. Visintin [453, Section II.4].

3.2. \(L^1\)- and \(L^\infty\)-techniques

In this section we prove the well-posedness of Problem 3.1.1 by using an \(L^1\)-contraction technique. We then show the essential boundedness of the solution via cut-off and approximation procedures.

An \(L^1\)-result. Next we prove that the solution of Problem 3.1.1 depends monotonically and Lipschitz-continuously on the data in the \(L^1\)-metric. This technique is at the basis of the semigroup approach that we shall illustrate in Section 3.4.

**THEOREM 3.2.1** (Monotone and \(L^1\)-Lipschitz-continuous dependence on the data). Assume that the assumptions of Theorem 3.1.1 are satisfied. For \(i = 1, 2\), let
\[
u_i^0 \in L^2(\Omega), \quad f_i \in L^2(0, T; V'), \quad f_1 - f_2 \in L^1(Q),
\]
and \((u_i, \theta_i)\) be a solution of the corresponding Problem 3.1.1. Setting \(\tilde{u} := u_1 - u_2\), \(\tilde{u}^0 := u_1^0 - u_2^0\), \(\tilde{f} := f_1 - f_2\), we then have
\[
\int_{\Omega} \tilde{u}^+(x,t) \, dx \\
\leq \int_{\Omega} (\tilde{u}^0)^+(x) \, dx + \int_0^t d\tau \int_{\Omega} \tilde{f}^+(x, \tau) \, dx \quad \text{for a.a. } t \in ]0, T[,
\]
\[
\int_{\Omega} |\tilde{u}(x,t)| \, dx \\
\leq \int_{\Omega} |\tilde{u}^0(x)| \, dx + \int_0^t d\tau \int_{\Omega} |\tilde{f}(x, \tau)| \, dx \quad \text{for a.a. } t \in ]0, T[.
\]

**PROOF.** (i) At first we assume that
\[\partial \varphi \text{ is Lipschitz-continuous and fulfills (3.1.41)};\]
\[u_i^0 \text{ and } f_i \text{ fulfill (3.1.43) and (3.1.44), for } i = 1, 2;\]

---

84 See also e.g. Bénilan [54], Damlamian [158,160].

85 By \(\xi^+\) we denote the positive part of any real number \(\xi\).
afterwards we shall drop these restrictions. By Proposition 3.1.4 then \( \theta_i, u_i \in H^1(0, T; L^2(\Omega)) \) for \( i = 1, 2 \). Let us also define the Heaviside graph \( H \) and its Yosida regularized function \( H_\lambda(\eta) := \lambda^{-1}[1 - (I + \lambda H)^{-1}] \) (by \( I \) we denote the identity operator) for any \( \lambda > 0 \):

\[
H(\eta) := \begin{cases} 
0 & \text{if } \eta < 0, \\
[0, 1] & \text{if } \eta = 0, \\
\lambda \eta & \text{if } 0 \leq \eta \leq \lambda, \\
1 & \text{if } \eta > \lambda.
\end{cases}
\]

Let us then write (3.1.8) for \( i = 1, 2 \), take the difference between these equations, multiply it by \( H_\lambda(\tilde{\theta}) \), and integrate it in \( \Omega \). As

\[
\int_\Omega \nabla \tilde{\theta} \cdot \nabla H_\lambda(\tilde{\theta}) \, dx = \int_\Omega H'_\lambda(\tilde{\theta})|\nabla \tilde{\theta}|^2 \, dx \geq 0 \quad \text{a.e. in } [0, T[.
\]

we get

\[
\int_\Omega \frac{d}{dt} H_\lambda(\tilde{\theta}) \, dx \leq \int_\Omega \tilde{f} H_\lambda(\tilde{\theta}) \, dx \leq \int_\Omega \tilde{f}^+ \, dx \quad \text{a.e. in } [0, T[.
\]

Let us then pass to the limit as \( \lambda \to 0^+ \). Note that

\[
H_\lambda(\tilde{\theta}) \to \psi := \begin{cases} 
0 & \text{where } \tilde{\theta} \leq 0, \\
1 & \text{where } \tilde{\theta} > 0
\end{cases} \quad \text{a.e. in } Q,
\]

so that \( \psi \in H(\tilde{\theta}) \) a.e. in \( Q \). Moreover \( H(\tilde{\theta}) = H(\tilde{u}) \) a.e. in \( Q \), for by the auxiliary assumption (3.2.4) \( \partial \varphi \) and \( (\partial \varphi)^{-1} \) are both monotone and single-valued. Hence \( \psi \in H(\tilde{u}) \) a.e. in \( Q \). By the Lebesgue dominated-convergence theorem, we thus get

\[
\frac{d}{dt} \int_\Omega \tilde{u}^+ \, dx = \int_\Omega \frac{d}{dt} \tilde{u}^+ \, dx = \int_\Omega \frac{d}{dt} \psi \, dx = \int_\Omega \frac{d}{dt} \lim_{\lambda \to 0} H_\lambda(\tilde{\theta}) \, dx
\]

\[
= \lim_{\lambda \to 0} \int_\Omega \frac{d}{dt} H_\lambda(\tilde{\theta}) \, dx \leq \int_\Omega \tilde{f}^+ \, dx \quad \text{a.e. in } [0, T[.
\]

This yields (3.2.2) by time integration.

(ii) Let us now drop the auxiliary hypothesis (3.2.4), and approximate \( \varphi, u_0, f \) \( (i = 1, 2) \) by means of sequences \( \{\varphi_n\}, \{u_0^0\}, \{f_n\} \) that fulfill (3.2.4) for any \( n \). For any \( n \) the inequality (3.2.2) thus holds for the difference of the corresponding solutions, \( \tilde{u}_n \). By Theorem 3.1.2,

\[
\tilde{u}_n \to \tilde{u} \quad \text{weakly in } L^2(\Omega), \end{lim}_{t \in [0, T]},
\]

whence, by the convexity of the positive-part mapping,

\[
\liminf_{n \to \infty} \int_\Omega (\tilde{u}_n)^+(x, t) \, dx \geq \int_\Omega \tilde{u}^+(x, t) \, dx \quad \forall t \in [0, T].
\]

By writing (3.2.2) for \( \tilde{u}_n \) and then passing to the inferior limit as \( n \to \infty \), we then get (3.2.2) for \( \tilde{u} \). The inequality (3.2.3) is finally obtained by exchanging \( u_1 \) and \( u_2 \) in (3.2.2), and then adding the two inequalities.

\[ \square \]

**Corollary 3.2.2.** Under the assumptions of Theorem 3.1.1, the solution of Problem 3.1.1 is unique and depends monotonically and Lipschitz-continuously on the data \( u^0 \) and \( f \).
The technique of Theorem 3.2.1 also allows one to derive some results of time regularity, that here we omit.\footnote{See e.g. Visintin [453, Section II.3].}

$L^\infty$-results. $L^\infty$-estimates may be derived in two ways, multiplying the equation either by a cut-off of the solution, or by a power of the solution and then letting the exponent diverge. The second procedure also provides $L^q$-estimates for any fixed $q > 2$, but, at variance with the first one, it does not need the uniqueness of the solution.

**Proposition 3.2.3 (Maximum and minimum principles).** Assume that (3.1.3), (3.1.4), (3.1.16) and (3.1.17) hold. If

\[ \exists M > 0, \exists \theta^0 \in L^1(\Omega): \theta^0 \in \partial \varphi^*(u^0), \]

\[ \theta^0 \leq M \quad (\theta^0 \geq -M, \text{ resp.}) \text{ a.e. in } \Omega, \]

\[ f \leq 0 \quad (f \geq 0, \text{ resp.}) \text{ in the sense of } \mathcal{D}'(Q), \]

then the solution of Problem 3.1.1 is such that

\[ \theta \leq M \quad (\theta \geq -M, \text{ resp.}) \text{ a.e. in } Q. \]

**Proof.** By Theorem 3.1.1 and Corollary 3.2.2, Problem 3.1.1 has one and only one solution. Let us assume that $\theta^0 \leq M, f \leq 0$. For any measurable selection $b$ of $\partial \varphi^*$ ($= (\partial \varphi)^{-1}$), let us also set

\[ \Phi(v) := \int_0^t [b(\xi) - M]^+ \, d\xi \quad (\geq 0) \quad \forall v \in \text{Dom}(\partial \varphi^*); \]

notice that this integral is independent of the selection. Let us then multiply (3.1.8) by $(\theta - M)^+$ ($\in L^2(0, T; V)$), and integrate in time. Note that (3.2.9) and (3.2.10), respectively, yield

\[ \Phi(u^0) = 0 \quad \text{a.e. in } \Omega, \quad \langle f, (\theta - M)^+ \rangle \, d\tau \leq 0 \quad \text{a.e. in } [0, T[. \]

Moreover by Proposition 5.2.7

\[ \int \Phi(u) \, dx \in W^{1,1}(0, T), \quad \left\{ \frac{\partial u}{\partial t}, (\theta - M)^+ \right\} = \frac{d}{dt} \int \Phi(u) \, dx. \]

We thus get

\[ \int \Phi(u(x, T)) \, dx + \int_0^T \int \nabla(\theta - M)^+ \, dx \leq 0, \]

whence $\nabla(\theta - M)^+ = 0$ a.e. in $Q$. As $\theta \in V$ this yields $(\theta - M)^+ = 0$ a.e. in $Q$, namely $\theta \leq M$.

The case of $\theta^0 \geq -M$ and $f \geq 0$ may be dealt with similarly, using $-(\theta - M)^-$ in place of $(\theta - M)^+$. \qed
PROPOSITION 3.2.4 ($L^q$-estimate). Assume that (3.1.3) and (3.1.16) are fulfilled, and that for some $q > 2$
\begin{equation}
    u^0 \in L^q(\Omega), \quad f \in L^q(Q) \cap L^2(0, T; V').
\end{equation}
Then the solution of Problem 3.1.1 is such that $u \in L^\infty(0, T; L^q(\Omega))$, and
\begin{equation}
    \|u(\cdot, t)\|_{L^q(\Omega)} \leq \left[ 1 + \left( q \|f\|_{L^q(Q)} \right)^{1/q} + \|u^0\|_{L^q(\Omega)} \right] \exp \left( t \|f\|_{L^q(Q)} \right)
    \forall t \in [0, T].
\end{equation}

PROOF. By Theorem 3.1.1 and Corollary 3.2.2, Problem 3.1.1 has one and only one solution. Let us at first assume that $\partial \varphi$ is Lipschitz-continuous; afterwards we shall drop this auxiliary hypothesis. Let us also fix any $M > 0$, and set
\begin{align*}
    \alpha_q(v) &= |v|^{q-2} v, \\
    \beta_M(v) &= \min \left\{ \max \left\{ v, -M \right\}, M \right\} \quad \forall v \in \mathbb{R}, \\
    \gamma_qM &= \alpha_q \circ \beta_M \circ \partial \varphi \quad \text{in} \ \mathbb{R}, \\
    u_M &= \beta_M(u) \quad \text{a.e. in} \ Q.
\end{align*}
Hence $\alpha_q(u_M) = \gamma_qM(\theta)$ by (3.1.7). Note that the function $\gamma_qM$ is Lipschitz-continuous and nondecreasing. Let us multiply Eq. (3.1.8) by $\alpha_q(u_M) = \gamma_qM(\theta)(\in L^2(0, T; V))$ and integrate in time. Note that $\langle A\theta, \gamma_qM(\theta) \rangle = \int_\Omega \nabla \theta \cdot \nabla \gamma_qM(\theta) \ dx \geq 0$. By the Schwarz–Hölder inequality we then get
\begin{equation}
    \int_\Omega \left( \left| u_M(x, t) \right|^q - \left| u^0(x) \right|^q \right) \ dx \\
    = q \int_0^t \left( \left| \frac{\partial u}{\partial t} \right|, \alpha_q(u_M) \right) \ d\tau \\
    \leq q \|f\|_{L^q(Q \times [0, t])} \left\| u_M \right\|_{L^{q/(q-1)}(Q \times [0, t])}^{q-1} \left( q \int_0^t \int_\Omega |u_M(x, \tau)|^q \ d\tau \right)^{(q-1)/q} \\
    \leq q \|f\|_{L^q(Q)} \left( \int_0^t \int_\Omega \left( 1 + \left| u_M(x, \tau) \right|^q \right) \ dx \right) \forall t \in [0, T].
\end{equation}
The Gronwall Lemma 3.2.5 yields
\begin{equation}
    \int_\Omega |u_M(x, t)|^q \ dx \leq \left( q \|f\|_{L^q(Q)} + \int_\Omega |u^0(x)|^q \ dx \right) \exp \left( q t \|f\|_{L^q(Q)} \right),
\end{equation}
for any $t \in [0, T]$. By passing to the limit as $M \to +\infty$ and then taking the $q$th root of both members, we finally get (3.2.13).

If $\partial \varphi$ is not Lipschitz-continuous, one can approximate it via Yosida approximation and then apply Theorem 3.1.2.

□

LEMMA 3.2.5 (Gronwall).\(^{87}\) Let $g, a, b : [0, T] \to \mathbb{R}$ be continuous functions, with $a$ nondecreasing and $b \geq 0$. If
\begin{equation}
    g(t) \leq a(t) + \int_0^t b(\tau)g(\tau) \ d\tau \quad \forall t \in [0, T],
\end{equation}

\(^{87}\) See e.g. Walter [460, Section I.1].
then
\[ g(t) \leq a(t) \exp \left( \int_0^t b(\tau) \, d\tau \right) \quad \forall t \in [0, T]. \] (3.2.16)

**COROLLARY 3.2.6.** Assume that (3.1.3) and (3.1.16) are fulfilled, and that
\[ u^0 \in L^\infty(\Omega), \quad f \in L^\infty(Q) \cap L^2(0, T; V'). \] (3.2.17)
Then the solution of Problem 3.1.1 is such that \( u \in L^\infty(Q) \).

**PROOF.** It suffices to apply Proposition 3.2.4 for any \( q > 2 \), and then to pass to the limit as \( q \to +\infty \) in (3.2.13). □

### 3.3. Two integral transformations

In this section we discuss two natural transformations of Problem 3.1.1. By integrating Eq. (3.1.8) in time, we eliminate the time derivative that acts on \( u \). Because of (3.1.7), Problem 3.1.1 may then be formulated as a variational inequality. A similar conclusion may be attained by applying the operator \( A^{-1} \) to (3.1.8). Actually, as it is the case for several parabolic problems, these two transformations are essentially equivalent, and yield analogous regularity properties. The theory of variational inequalities has extensively been applied to PDEs.88

**Time-integral transformation.** The transformation that here we illustrate was independently introduced by Duvaut [184,185] and Frémond [223]. This technique was inspired by an integral transformation, that Baiocchi introduced for a free boundary problem arising in porous medium filtration, see Baiocchi [44], and Baiocchi and Capelo [45]. Let us set
\[ z(\cdot, t) := \int_0^t \theta(\cdot, \tau) \, d\tau, \quad F(t) := \int_0^t f(\tau) \, d\tau + u^0 \quad \forall t \in [0, T], \] (3.3.1)
and note that, by integrating (3.1.8) in time and coupling it with (3.1.7), we get
\[ \partial\phi \left( \frac{\partial z}{\partial t} \right) + Az \ni F \quad \text{in } V', \quad \forall t \in [0, T]. \] (3.3.2)
By definition of subdifferential, cf. (5.2.5), this inclusion is equivalent to the following variational inequality:
\[
\left( Az - F, \frac{\partial z}{\partial t} - v \right) + \int_\Omega \left[ \phi \left( \frac{\partial z}{\partial t} \right) - \phi(v) \right] \, dx \leq 0
\quad \forall v \in V, \text{ a.e. in } ]0, T[. \] (3.3.3)

88 See e.g. the monographs quoted in the item (XI) of the Bibliographical Note in Section 6.
Notice that, as \( z(\cdot, 0) = 0 \) a.e. in \( \Omega \),
\[
\int_0^t \left[ Az - F, \frac{\partial z}{\partial t} \right] \, dt
= \frac{1}{2} \int_\Omega \left| \nabla z(\cdot, \tilde{t}) \right|^2 \, dx + \int_0^t \left\{ \frac{\partial F}{\partial t}, z \right\} \, dt - \left\{ F(\tilde{t}), z(\cdot, \tilde{t}) \right\} \quad \forall \tilde{t} \in [0, T].
\]

Although in terms of the new variable \( z \) the regularity that is prescribed for \( u \) in Problem 3.1.1 corresponds to \( z \in H^1(0, T; V) \), here we reformulate the variational inequality (3.3.3) under weaker regularity requirements for \( z \). Let us first assume that \( f \in L^2(0, T; V') \) and \( u^0 \in V' \), so that
\[
F \in W^{1,1}(0, T; V').
\]

**PROBLEM 3.3.1.** Find \( z \in L^\infty(0, T; V) \cap H^1(0, T; L^2(\Omega)) \) such that
\[
\frac{1}{2} \int_\Omega \left| \nabla z(\cdot, \tilde{t}) \right|^2 \, dx + \int_0^t \left\{ \frac{\partial F}{\partial t}, z \right\} \, dt - \left\{ F(\tilde{t}), z(\cdot, \tilde{t}) \right\}
+ \int_0^t d\tilde{t} \int_\Omega \left[ \varphi \left( \frac{\partial z}{\partial t} \right) - \varphi(v) \right] \, dx \leq \int_0^t \left\langle Az - F, v \right\rangle \, dt
\quad \forall v \in V, \text{ for a.a. } \tilde{t} \in [0, T],\]
\[
z(\cdot, 0) = 0 \quad \text{a.e. in } \Omega.
\]

**THEOREM 3.3.1 (Existence and uniqueness).** If (3.1.3), (3.1.4), (3.1.16), (3.1.17) are satisfied and
\[
\exists \hat{L}, \hat{M} > 0: \forall (\xi, \eta) \in \text{graph}(\partial \varphi), \left| \eta \right| \leq \hat{L} \left| \xi \right| + \hat{M},
\]
then Problem 3.3.1 has one and only one solution.

**OUTLINE OF THE PROOF.**\(^{89}\) Let us set \( z_m^0 := 0 \) a.e. in \( \Omega \), \( F_m^n := F(nk) \) in \( V' \) for any \( n \in \{1, \ldots, m\} \), and
\[
\hat{J}_m(v) := \int_\Omega \left[ k \varphi \left( \frac{v - z_m^{n-1}}{k} \right) + \frac{1}{2} |\nabla v|^2 \right] \, dx - \left\langle F_m^n, v \right\rangle \quad \forall v \in V.
\]
These functionals are convex, lower semicontinuous and coercive, and may thus be minimized recursively. Each of them has a (unique) minimum point \( z_m^n \), which thus solves the implicit time-discretization scheme:
\[
\partial \varphi \left( \frac{z_m^n - z_m^{n-1}}{k} \right) + Az_m^n \ni F_m^n \quad \text{in } V', \text{ for } n = 1, \ldots, m.
\]

Let us define the time-interpolate functions \( z_m \) as in (3.1.26). Multiplying (3.3.8) by \( z_m^n - z_m^{n-1} \), one may easily derive a uniform estimate for \( z_m \) in \( L^\infty(0, T; V) \cap L^2(\Omega) \)

\(^{89}\) See Visintin [453, Section II.5].
\( H^1(0, T; L^2(\Omega)) \). Therefore there exists \( z \) such that, possibly extracting a subsequence,

\[
z_m \rightarrow z \quad \text{weakly star in } L^\infty(0, T; V) \cap H^1(0, T; L^2(\Omega)).
\]

By passing to the limit in the approximate variational inequality, by lower semicontinuity one then obtains (3.3.5). The proof of uniqueness is straightforward. \( \square \)

**Inversion of the Laplace operator.** In view of introducing our second integral transformation, let us first revisit the functional setting of Problem 3.1.1. We shall use the inverse operator \( A^{-1} : V' \rightarrow V \). For instance, if \( g \in L^2(\Omega) \), \( h \in L^2(\Gamma_N) \) and

\[
\langle f, v \rangle = \int_\Omega g(x)v(x) \, dx + \int_{\Gamma_N} h(x)\gamma_0 v(x) \, d\sigma \quad \forall v \in V,
\]

then

\[
u = A^{-1}f \iff u \in V, \quad -\Delta u = g \text{ a.e. in } \Omega, \quad \frac{\partial u}{\partial \nu} = h \text{ a.e. on } \Gamma_N, \quad (3.3.9)
\]

for in this case the normal trace \( \partial u/\partial \nu \) is an element of \( L^2(\Gamma_N) \).

By applying the operator \( A^{-1} \) to (3.1.8), we have

\[
A^{-1} \frac{\partial u}{\partial t} + \theta = A^{-1}f =: F \quad \text{in } V, \quad \text{a.e. in } [0, T[. \quad (3.3.10)
\]

By coupling this equation with (3.1.7) we then get the inclusion

\[
A^{-1} \frac{\partial u}{\partial t} + \partial \phi^*(u) \ni F \quad \text{in } V, \quad \text{a.e. in } [0, T[. \quad (3.3.11)
\]

which is equivalent to the following variational inequality:

\[
\left\langle A^{-1} \frac{\partial u}{\partial t} - F, u - v \right\rangle + \int_\Omega \left[ \phi^*(u) - \phi^*(v) \right] \, dx \leq 0
\]

\[
\forall v \in L^2(\Omega), \quad \text{a.e. in } [0, T[. \quad (3.3.12)
\]

Assuming (3.1.4), namely

\[
u^0 \in V', \quad F \in L^2(0, T; V), \quad (3.3.13)
\]

we can now introduce a further weak formulation of our problem.

**Problem 3.3.2.** Find \( u \in L^2(Q) \cap H^1(0, T; V') \) such that (3.3.12) is satisfied, and

\[
u|_{t=0} = \nu^0 \quad \text{in } V' \text{ (in the sense of the traces of } H^1(0, T; V')). \quad (3.3.14)
\]

**Theorem 3.3.2 (Existence and uniqueness).** If (3.1.3), (3.1.4), (3.1.16), (3.1.17) and (3.3.13) are satisfied, then Problem 3.3.1 has one and only one solution.
OUTLINE OF THE PROOF. \(^{90}\) One may approximate the inclusion (3.3.11) via implicit time-discretization, and then derive a priori estimates by multiplying the approximate equation by the approximate solution \(u^m_n\). This yields a uniform estimate for the linear interpolate function \(u_m\) in \(L^2(Q) \cap H^1(0, T; V')\). Hence a suitable subsequence of \(\{u_m\}\) weakly converges in this space. By passing to the limit in the approximate variational inequality, Eq. (3.3.12) follows by lower semicontinuity. The proof of uniqueness is straightforward. \(\square\)

3.4. Semigroup techniques

In this section we apply to Problem 3.1.1 methods of the theory of nonlinear semigroups of contractions in Hilbert and Banach spaces, cf. Section 5.6. We shall see that in this framework the spaces \(H^{-1}(\Omega)\) and \(L^1(\Omega)\) play special roles.\(^{91}\) In the first case we shall retrieve the method of inversion of the Laplace operator that we just illustrated, whereas in the second one we shall exploit the \(L^1\)-contraction procedure of Theorem 3.2.1.

Change of pivot space. Here we continue our discussion on the inversion of the operator \(A\), under the assumption that \(\Gamma_D\) has positive \((N - 1)\)-dimensional Hausdorff measure. Let us first notice that the bilinear forms

\[
(u, v)_V := \int_{\Omega} \nabla u \cdot \nabla v \, dx \quad \forall u, v \in V, \tag{3.4.1}
\]

\[
(u, v)_{V'} := (A^{-1}u, A^{-1}v)_V = \langle A^{-1}u, v \rangle \quad \forall u, v \in V', \tag{3.4.2}
\]

are scalar products in the Hilbert spaces \(V\) and \(V'\), respectively, and that

\[
(Au, v)_{V'} := \langle u, v \rangle = \int_{\Omega} uv \, dx \quad \forall u \in V, \forall v \in L^2(\Omega). \tag{3.4.3}
\]

Let us now denote the space \(V'\) by \(\mathcal{H}\), in order to avoid any possible confusion with the dual spaces that we are going to introduce, and define the Riesz operator

\[
\mathcal{R} : \mathcal{H} \to \mathcal{H}', \quad \langle \mathcal{R}u, v \rangle = (u, v)_{\mathcal{H}} \quad \forall u, v \in \mathcal{H}.
\]

As \(L^2(\Omega) \subset \mathcal{H}\) with continuous and dense injection, we can identify \(\mathcal{H}'\) with a subspace of \(L^2(\Omega)'\). This yields

\[
\mathcal{R}L^2(\Omega) \subset \mathcal{R}\mathcal{H} = \mathcal{H}' \subset L^2(\Omega)' \quad \text{with dense and compact injections.}
\]

The space \(\mathcal{H}\) is thus identified with \(\mathcal{R}^{-1}(\mathcal{H}')\), and accordingly plays the role of pivot space. This approach is at variance from the more usual procedure of identifying \(L^2(\Omega)\) with its dual, cf. (3.1.2). Henceforth we shall omit to display the operator \(\mathcal{R}\).

\(^{90}\) See Visintin [453, Section II.5].

\(^{91}\) See e.g. Bénilan [55], Bénilan and Crandall [57], Bénilan, Crandall and Pazy [58], Bénilan, Crandall and Sacks [59], Berger, Brezis, and Rogers [60], Brezis [83,84], Brezis and Pazy [88], Crandall and Pierre [148], Magenes, Verdi and Visintin [321], Rogers and Berger [403].
By (3.4.3), in the space $\mathcal{H}$ the variational equation (3.1.8) also reads

$$
\left( \frac{\partial u}{\partial t}, v \right)_{\mathcal{H}} + \int_\Omega \theta v \, dx = (f, v)_{\mathcal{H}} \quad \forall v \in \mathcal{H},
$$

which is equivalent to (3.3.10). In this way we have thus retrieved Problem 3.3.2.

This technique was studied by Lions [311, Section 2.3].

$L^2$-semigroups. Whenever the mapping $\partial \varphi^*$ is nonlinear, the operator $u \mapsto -\Delta \varphi^*(u)$ is nonmonotone in $L^2(\Omega)$. Next we shall see that, if properly defined, this operator is maximal and cyclically monotone in $\mathcal{H}$, in the sense of Section 5.5. Let us first assume that

$$
\exists c, \tilde{M} > 0: \forall v \in \mathbb{R}^3, \varphi^*(v) \leq c|v|^2 + \tilde{M},
$$

so that $\varphi^*(v) \in L^1(\Omega)$ for any $v \in L^2(\Omega)$. One may see that this condition is equivalent to (3.3.7). Let us then define the (possibly multivalued) operator

$$
\{ \Lambda_2 : \text{Dom}(\Lambda_2) := L^2(\Omega) \subset \mathcal{H} \to 2^\mathcal{H} : u \mapsto A\partial \varphi^*(u) := \{ A\theta : \theta \in V, \theta \in \partial \varphi^*(u) \text{ a.e. in } \Omega \}. \tag{3.4.6}
$$

Note that, for any $u \in L^2(\Omega)$ and any $\theta \in V$, $\theta \in \partial \varphi^*(u)$ a.e. in $\Omega$, if and only if

$$
(A\theta, u - v)_{\mathcal{H}} = \int_\Omega \theta(u - v) \, dx \geq \int_\Omega \varphi^*(u) \, dx - \int_\Omega \varphi^*(v) \, dx \quad \forall v \in L^2(\Omega).
$$

By this variational inequality $\Lambda_2$ coincides with the subdifferential of the proper, convex, and lower semicontinuous functional

$$
\mathcal{H} \to \mathbb{R} : v \mapsto \begin{cases} 
\int_\Omega \varphi^*(v) \, dx & \text{if } v \in L^2(\Omega), \\
+\infty & \text{if } v \in \mathcal{H} \setminus L^2(\Omega). 
\end{cases} \tag{3.4.7}
$$

By Theorem 5.5.3, the operator $\Lambda_2$ is then maximal and cyclically monotone. One may then apply the classical theory of semigroups of nonlinear contractions in Hilbert spaces, see Section 5.6, to the equation

$$
\frac{\partial u}{\partial t} + \Lambda_2(u) = f \quad \text{in } \mathcal{H}, \text{ a.e. in } [0, T[. \tag{3.4.8}
$$

In this way one retrieves Theorem 3.3.2 and several other results. In fact this semigroup approach is essentially equivalent to the inversion of the Laplace operator.

$L^1$-semigroups. The $L^1$-contraction technique that we used in the proof of Theorem 3.2.1 suggests investigating the accretiveness of the multivalued operator $u \mapsto -\Delta \varphi^*(u)$ in $L^1(\Omega)$, in the framework of the theory of nonlinear semigroups of contractions in Banach spaces. We still assume (3.3.7), but there we also require that $\Gamma_D = \Gamma$, namely that the homogeneous Dirichlet condition is fulfilled on the whole boundary.\footnote{More general boundary conditions are considered e.g. in Bénilan [55], Magenes, Verdi and Visintin [321].} We then define the
operator

\[
\begin{align*}
A_1 : \text{Dom}(A_1) \subset L^1(\Omega) & \to L^1(\Omega) : u \mapsto -\Delta \varphi^*(u) := \\
\{ -\Delta \theta \in L^1(\Omega) : \theta \in L^1(\Omega), \quad \gamma_0 \theta = 0 \text{ on } \Gamma, \quad \theta \in \partial \varphi^*(u) \text{ a.e. in } \Omega \}. 
\end{align*}
\]

(3.4.9)

For any

\[
u^0 \in L^1(\Omega), \quad f \in L^1(Q),
\]

(3.4.10)

we now reformulate the weak Stefan problem as follows.

**PROBLEM 3.4.1.** Find a continuous vector-valued function \(u : [0, T] \to L^1(\Omega)\), that is absolutely continuous in \(]0, T[\) and such that

\[
\begin{align*}
\frac{\partial u}{\partial t} + A_1(u) & \ni f \quad \text{in } L^1(\Omega), \text{ a.e. in } ]0, T[, \\
u(., 0) & = u^0 \quad \text{a.e. in } \Omega.
\end{align*}
\]

(3.4.11)

(3.4.12)

In general this problem has no solution. Actually, the occurrence of phase interfaces is not consistent with the condition \(u(., t) \in \text{Dom}(A_1)\), for \(\Delta \theta = \Delta \varphi^*(u)\) is a nonintegrable Borel measure whenever the Stefan condition \((1.2.4)\) is fulfilled. We shall then investigate a weaker notion of solution.

**LEMMA 3.4.1.**\(^{93}\) Assume that \(\alpha\) is a maximal monotone mapping \(\mathbb{R} \to 2^{\mathbb{R}}\) such that \(\alpha(0) \ni 0\). Let \(p \in [1, +\infty[\) and set \(p' = p/(p - 1)\) if \(p \neq 1\), \(p' = +\infty\) if \(p = 1\). If

\[
\begin{align*}
u & \in L^p(\Omega), \quad \Delta u \in L^p(\Omega), \quad \gamma_0 u = 0 \quad \text{a.e. on } \Gamma, \\
h & \in L^{p'}(\Omega), \quad h \in \alpha(u) \quad \text{a.e. in } \Omega,
\end{align*}
\]

(3.4.13)

then

\[-\int_{\Omega} h \Delta u \, dx \geq 0.\]

**THEOREM 3.4.2.** Assume that \((3.1.3), (3.1.16)\) and \((3.3.7)\) hold, and that \(\Gamma_D = \Gamma\). The operator \(A_1\) is then \(T\)- and \(m\)-accretive in \(L^1(\Omega)\), that is,

\[
\begin{align*}
\forall u_i & \in \text{Dom}(A_1), \quad \forall -\Delta \theta_i \in A_1(u_i) (i = 1, 2), \quad \forall \lambda > 0, \\
\| (u_1 - u_2)^+ \|_{L^1(\Omega)} & \leq \| [u_1 - u_2 - \lambda \Delta(\theta_1 - \theta_2)]^+ \|_{L^1(\Omega)}, \\
\forall \lambda & > 0, \quad \forall f \in L^1(\Omega), \exists u \in \text{Dom}(A_1) : u + \lambda A_1(u) \ni f \text{ a.e. in } \Omega.
\end{align*}
\]

(3.4.14)

(3.4.15)

**PROOF.** (i) In view of proving \((3.4.14)\), let us first fix any \(u_i \in \text{Dom}(A_1)\), and select any \(-\Delta \theta_i \in A_1(u_i)\) for \(i = 1, 2\). Let us then set

\[
h(x) := \begin{cases} 
1 & \text{if either } u_1(x) > u_2(x) \text{ or } \theta_1(x) > \theta_2(x), \\
0 & \text{otherwise}
\end{cases}
\]

for a.a. \(x \in \Omega\).

\(^{93}\) See Brezis and Strauss [89, p. 566], where this result is stated in more general form, for a class of unbounded \(m\)-accretive operators in \(L^1(\Omega)\) that fulfill a maximum principle.
Note that $h$ is measurable and, defining the Heaviside graph $H$ as in (3.2.5), $h \in H(u_1 - u_2) \cap H(\theta_1 - \theta_2)$ a.e. in $\Omega$. Lemma 3.4.1 then yields $-\int_{\Omega} h \Delta (\theta_1 - \theta_2) \, dx \geq 0$. Hence
\[
\int_{\Omega} [u_1 - u_2 - \lambda \Delta (\theta_1 - \theta_2)]^+ \, dx \geq \int_{\Omega} [u_1 - u_2 - \lambda \Delta (\theta_1 - \theta_2)] \, dx \geq \int_{\Omega} (u_1 - u_2) \, dx = \int_{\Omega} (u_1 - u_2)^+ \, dx,
\]
that is (3.4.14). As an analogous statement is fulfilled with the negative part in place of the positive part, we get
\[
\forall u_i \in \text{Dom}(A_1), \ \forall -\Delta \theta_i \in A_1(u_i) \ (i = 1, 2), \ \forall \lambda > 0,
\]
\[
\|u_1 - u_2\|_{L^1(\Omega)} \leq \|u_1 - u_2 + \lambda \Delta (\theta_1 - \theta_2)\|_{L^1(\Omega)}; \tag{3.4.16}
\]
namely, $A_1$ is accretive in $L^1(\Omega)$.

(ii) Next we prove (3.4.15). Let us first assume that $f \in L^2(\Omega)$. The functional
\[
J : V \to \overline{\mathbb{R}} : v \mapsto \int_{\Omega} \left( \varphi(v) + \frac{\lambda}{2} |\nabla v|^2 - f v \right) \, dx
\]
is convex, lower semicontinuous, and coercive; hence it has a minimum point $\theta \in V$. Thus $\partial J(\theta) \ni 0$ in $V'$, that is,
\[
u := f + \lambda \Delta \theta \in \partial \varphi(\theta) \quad \text{in } V';
\]
hence $u \in L^2(\Omega)$, by (3.1.16). For any $f \in L^2(\Omega)$ thus there exists a pair $(\theta, u) \in V \times L^2(\Omega)$ such that $\Delta \theta \in L^2(\Omega)$ and
\[
\theta \in \partial \varphi^*(u), \quad u - \lambda \Delta \theta = f \quad \text{a.e. in } \Omega. \tag{3.4.17}
\]

Let us now consider any $f \in L^1(\Omega)$ and any sequence $\{f_n\}$ in $L^2(\Omega)$ that converges to $f$ strongly in $L^1(\Omega)$. For any $n$ let $(\theta_n, u_n)$ solve (3.4.17)$_n$. By the accretiveness of $A_1$, $\{u_n\}$ is a Cauchy sequence in $L^1(\Omega)$; hence it converges to some $u$ strongly in this space. By comparing the terms of (3.4.17)$_n$, we see that the sequence $\{\Delta \theta_n\}$ is then uniformly bounded in $L^1(\Omega)$. Possibly extracting a subsequence, $\theta_n$ then converges to some $\theta$ strongly in $L^1(\Omega)$. Therefore, possibly extracting further subsequences, $\theta_n$ and $u_n$ converge a.e. in $\Omega$. By passing to the limit in (3.4.17)$_n$ a.e. in $\Omega$, we then infer (3.4.17) for the pair $(\theta, u)$.

**Theorems 3.4.2 and 5.6.1 yield the next statement.**

**Theorem 3.4.3.** Assume that (3.1.3), (3.1.16) and (3.3.7) are satisfied and that $\Gamma_D = \Gamma$.

For any $u^0 \in L^1(\Omega)$ and any $f \in L^1(\Omega)$, **Problem 3.4.1** then has one and only one mild solution (in the sense of Section 5.6).

This solution depends Lipschitz-continuously and monotonically on the data.

**Remarks.** (i) If $f \in BV(0, T; L^1(\Omega))$ and $u^0 \in \text{Dom}(A_1)$, then $u : [0, T] \to L^1(\Omega)$ is Lipschitz-continuous. However, consistently with our previous remark, $u$ need not be a strong solution, for the space $L^1(\Omega)$ does not fulfill the Radon–Nikodým property, cf. Section 5.6.
(ii) So far we studied quasilinear parabolic P.D.E.s containing a single nonlinear term. One may also deal with doubly nonlinear equations, namely equations that contain two nonlinear functions, for instance of the form

$$\frac{\partial}{\partial t} \alpha(u) - \nabla \cdot \gamma(\nabla u) \ni f \quad \text{in } Q,$$

$$\alpha \left( \frac{\partial u}{\partial t} \right) - \nabla \cdot \gamma(\nabla u) \ni f \quad \text{in } Q,$$

with $\alpha$ and $\gamma$ given (possibly multi-valued) maximal monotone mappings. In general these two equations are not mutually equivalent. For instance an equation of the form (3.4.18) is met in nonequilibrium thermodynamics: the Gibbs relation and the phenomenological laws provide the two nonlinearities, see the system (2.4.1), (2.4.2), (2.4.6), (2.4.13), (2.4.14). Equations of the form (3.4.19) arise in a number of diffusion problems, with $\alpha$ equal to the subdifferential of a so-called dissipation potential, see e.g. Germain [241].

4. Phase relaxation with nonlinear heat diffusion

In this part we deal with an initial- and boundary-value problem for a quasilinear (actually, multi-nonlinear) parabolic P.D.E., that represents phase transition coupled with nonlinear heat-diffusion and with phase relaxation; cf. Section 2.2. We provide the weak formulation of an initial- and boundary-value problem in the framework of Sobolev spaces, and prove existence of a solution in any prescribed time interval.

This part includes some elements of novelty. A rather general constitutive relation is assumed between internal energy, temperature, and phase; this also allows for (possibly nonlinear) dependence of the heat capacity on the phase. Existence of an approximate solution is here proved via a saddle point formulation. Compactness by strict convexity (cf. Section 5.4) is applied in the limit procedure, besides more standard techniques of compactness, convexity and lower semicontinuity. This approach might also be extended to coupled heat- and mass-diffusion, as well as to other models.

4.1. Weak formulation

In this section we formulate our problem in the framework of Sobolev spaces.

Let the sets $\Omega$, $\Gamma$, $Q$, the space $V$, the operator $A$, and the duality pairing $\langle \cdot, \cdot \rangle$ be defined as in Section 3.1. Let us assume that the following functions are also given:

$$\phi : \mathbb{R}^2 \to \tilde{\mathbb{R}} := \mathbb{R} \cup \{ +\infty \} \text{ proper, convex and lower semicontinuous,}$$

$$\text{Dom}(\phi) = \mathbb{R} \times [-1, 1],$$

94 This mode of phase transition was studied in many works, see e.g. Bénilan, Blanchard and Ghidouche [56], Blanchard, Damalamian and Ghidouche [67], Blanchard and Ghidouche [68], Frémond and Visintin [225], Visintin [446,447] and [453, Chapter V].

95 See Visintin [459]. This part rests upon the classical theory of linear and nonlinear PDEs, see e.g. the monographs quoted in the items (XI), (XII), (XIII) of the Bibliographical Note in Section 6.
\[ \Phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \] such that
\[ \Phi(\cdot, \vec{\xi}) \text{ is continuous } \forall \vec{\xi} \in \mathbb{R}^3, \]
\[ \Phi(\theta, \cdot) \text{ is convex and lower semicontinuous } \forall \theta \in \mathbb{R}. \]

We shall deal with the system
\[ (\theta, r) \in \partial \varphi(u, \chi), \]
\[ \vec{q} \in -\partial_2 \Phi(\theta, \nabla \theta), \]
\[ \frac{\partial u}{\partial t} + \nabla \cdot \vec{q} = g, \]
\[ a \frac{\partial \chi}{\partial t} + r = 0, \]

in \( Q \), coupled with initial conditions for \( u \) and \( \chi \), and with boundary conditions for \( u \) and \( \vec{q} \). Here \( a \) is a positive constant. By \( \partial_2 \Phi \) we denote the partial subdifferential of \( \Phi \) w.r.t. the second argument, here \( \nabla \theta \); see Section 5.2. The inclusion (4.1.3) accounts for a dependence of the heat capacity on the phase. In the next section we shall see that the system (4.1.3)–(4.1.6) is consistent with the second principle of thermodynamics, provided that the function \( \varphi \) represents the function of (2.2.15), cf. (2.2.20) and (2.2.21).

Let us assume that
\[ u^0, \chi^0 \in L^2(\Omega), \quad (u^0, \chi^0) \in \text{Dom}(\partial \varphi) \text{ a.e. in } \Omega, \]
\[ f \in L^2(0, T; V'), \]

and introduce our weak formulation.

**Problem 4.1.1.** Find \( u, \theta, \chi, r, \vec{q} \) such that
\[ u, r \in L^2(Q), \quad \chi \in H^1(0, T; L^2(\Omega)), \]
\[ \theta \in L^2(0, T; V), \quad \vec{q} \in L^2(Q)^3, \]
\[ (u, \chi) \in \text{Dom}(\varphi) \quad \text{and} \quad \forall (\vec{u}, \vec{\chi}) \in \text{Dom}(\varphi), \]
\[ \theta(u - \vec{u}) + r(\chi - \vec{\chi}) \geq \varphi(u, \chi) - \varphi(\vec{u}, \vec{\chi}) \text{ a.e. in } Q, \]
\[ \vec{q} \cdot (\vec{\xi} - \nabla \theta) \geq \Phi(\theta, \nabla \theta) - \Phi(\theta, \vec{\xi}) \quad \forall \vec{\xi} \in L^2(\Omega)^3, \text{ a.e. in } Q, \]
\[ \int_Q \left[ (u^0 - u) \frac{\partial v}{\partial t} - \vec{q} \cdot \nabla v \right] \, dx \, dt = \int_0^T \langle f, v \rangle \, dt \quad \forall v \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; V), v(\cdot, T) = 0, \]
\[ a \frac{\partial \chi}{\partial t} + r = 0 \text{ a.e. in } Q, \]
\[ \chi(\cdot, 0) = \chi^0 \text{ a.e. in } \Omega. \]

---

96 In Section 1 we had no difficulty in dealing with this dependence under the hypothesis of local equilibrium, for if in that case the phase is determined by the temperature. On the other hand, if that hypothesis is dropped, the analysis is less obvious.

97 This constant factor is here included in \( \varphi \), in order to simplify the lay-out of formulas. This rescaling is immaterial for the present analysis.
If \( f \) is defined as in (3.1.11), this problem is the weak formulation of an initial- and boundary-value problem for the system (4.1.3)–(4.1.6). The system (4.1.13) and (4.1.14) is trivially integrated: 
\[
\chi(\cdot, t) = \chi_0 - \frac{1}{a} \int_0^t r(\cdot, \tau) \, d\tau \text{ a.e. in } Q.
\]

**Remark.** Setting 
\[
U = (u, \chi), \quad V = (\theta, r), \quad \gamma(V, \nabla V) = (\partial_2 \Phi(\theta, \nabla \theta), r), \quad F = (f, 0)
\]
the system (4.1.3)–(4.1.6) reads as a doubly nonlinear system:
\[
V \in \partial \varphi (U), \quad W \in \gamma(V, \nabla V), \quad \frac{\partial U}{\partial t} + \Lambda W = F.
\]

Because of the multiple nonlinearity of this problem, here the results of Section 3 are not directly applicable. Nevertheless those techniques are at the basis of the theorem of existence, that we prove in the next section.98

**4.2. Existence of a weak solution**

In view of stating our existence result, let us first define the partial conjugate \( \psi : \mathbb{R} \times [-1, 1] \to \tilde{\mathbb{R}} := \mathbb{R} \cup \{+\infty\} \) of the function \( \varphi \) w.r.t. \( u \), cf. (5.2.3):

\[
\psi(\theta, \chi) := \sup_{u \in \mathbb{R}} [u \theta - \varphi(u, \chi)] \quad \forall (\theta, \chi) \in \mathbb{R} \times [-1, 1].
\]  

By Theorem 5.3.3,

\[
\psi(\cdot, \chi) \text{ is convex and lower semicontinuous } \forall \chi \in [-1, 1],
\]

\[
\psi(\theta, \cdot) \text{ is concave and upper semicontinuous } \forall \theta \in \mathbb{R}.
\]

**Theorem 4.2.1 (Existence).** Assume that (4.1.7)–(4.1.8) are satisfied, and that

the function \( \psi \) may be represented in the form

\[
\psi(\theta, r) = \psi_1(\theta) + \psi_2(\theta, r) \quad \forall (\theta, r) \in \mathbb{R} \times [-1, 1],
\]

where \( \psi_1 \) is strictly convex and everywhere finite,

\[
\psi_2(\cdot, \chi) \text{ is convex and lower semicontinuous } \forall \chi \in [-1, 1],
\]

\[
\psi_2(\theta, \cdot) \text{ is concave and upper semicontinuous } \forall \theta \in \mathbb{R},
\]

\[
\exists a_1, a_2 > 0: \forall (u, \chi) \in \text{Dom}(\varphi), \quad \varphi(u, \chi) \geq a_1 |u|^2 - a_2,
\]

\[
\exists a_3, \ldots, a_6 > 0: \forall (\theta, \tilde{\xi}) \in \mathbb{R} \times \mathbb{R}^3,
\]

\[
a_3 |\tilde{\xi}|^2 - a_4 \leq \Phi(\theta, \tilde{\xi}) \leq a_5 |\tilde{\xi}|^2 + a_6,
\]

\[
\exists a_7 > 0: \forall \theta \in \mathbb{R}, \quad \Phi(\theta, \tilde{0}) \leq a_7.
\]

Then Problem 4.1.1 has a solution such that \( u \in L^\infty(0, T; L^2(\Omega)) \).

98 Boundary- and initial-value problems for doubly-nonlinear PDEs were studied in many works; see e.g. Alt and Luckhaus [17], Colli and Visintin [145], DiBenedetto and Showalter [179], Damlamian, Kenmochi and Sato [162], Visintin [453, Chapter III], and references therein.
The reader will notice that in (4.2.3) the function $\psi_1$ is independent of $\chi$, so that strict convexity is only assumed for the dependence of $\psi_1$ on $\theta$. We do not assume $\psi$ to be strictly concave w.r.t. the phase variable $\chi$, for this would exclude the occurrence of a sharp interface between the phases. See the remarks after the proof of this theorem.

PROOF. We proceed via approximation by time-discretization, derivation of a priori estimates, and passage to the limit, as we did for Theorem 3.1.1.

(i) Approximation. Let us fix any $m \in \mathbb{N}$, set

$$k := \frac{T}{m}, \quad u_0 := u^0, \quad \chi_0 := \chi^0 \quad \text{a.e. in } \Omega,$$

$$f_n := \frac{1}{k} \int_{(n-1)k}^{nk} f(\tau) \, d\tau \quad \text{in } V^{'}, \quad \text{for } n = 1, \ldots, m,$$

(4.2.7)

fix any $\theta_0 \in L^2(\Omega)$, and introduce the following approximate problem.99

PROBLEM 4.1.1$_m$. Find $u^n_m, \theta^n_m, \chi^n_m, r^n_m, q^n_m$ for $n = 1, \ldots, m$ such that

$$u^n_m, \chi^n_m, r^n_m \in L^2(\Omega), \quad \theta^n_m \in V, \quad q^n_m \in L^2(\Omega)^3,$$

$$\left( \theta^n_m, r^n_m \right) \in \partial \psi(u^n_m, \chi^n_m) \quad \text{a.e. in } \Omega,$$

$$q^n_m \in -\partial_2 \Phi(\theta^n_m, \nabla \theta^n_m) \quad \text{a.e. in } \Omega,$$

(4.2.8)

$$\int_{\Omega} \left( \frac{u^n_m - u^{n-1}_m}{k} v - q^n_m \cdot \nabla v \right) \, dx = \langle f^n_m, v \rangle \quad \forall v \in V,$$

(4.2.11)

Let us define the double subdifferential $\tilde{\partial} \psi(\theta, \chi) := (\partial_\theta \psi(\theta, \chi), \partial_\chi (-\psi(\theta, \chi)))$ as in (5.3.17). By Theorem 5.3.3, the inclusion (4.2.9) is equivalent to the system

$$\left\{ \begin{array}{l}
\left( u^n_m, r^n_m \right) \in \tilde{\partial} \psi(\theta^n_m, \chi^n_m),
\theta^n_m \in \partial \psi(u^n_m, \chi^n_m),
r^n_m \in \partial \chi (-\psi(\theta^n_m, \chi^n_m))
\end{array} \right.$$

a.e. in $\Omega.$

(4.2.12)

We shall prove the existence of a solution of Problem 4.1.1$_m$ recursively. Let us fix any $n \in \{1, \ldots, m\}$, assume that $u^{n-1}_m, \theta^{n-1}_m, \chi^{n-1}_m$ are known, define the closed and convex set $X := V \times L^2(\Omega; [1, 1])$ and set

$$J^n_m(\theta, \chi) := \int_{\Omega} \left[ \psi(\theta, \chi) - \frac{\alpha}{2} \chi^2 - u^{n-1}_m \theta + a \chi^{n-1} \chi + k \Phi(\theta^{n-1}_m, \nabla \theta) \right] \, dx$$

$$- k \langle f^n_m, \theta \rangle \quad \forall (\theta, \chi) \in X.$$

(4.2.14)

Notice that

99 It is necessary to select a value for $\theta_0$, because this function occurs in (4.2.10) for $n = 1$. Nevertheless this function is not prescribed among the data, and it turns out that it is immaterial which value is selected, for the function $\theta$ need not have a trace for $t = 0$. 


\[ V \to \mathbb{R} : \theta \mapsto J^n_m(\theta, \chi) \]
is convex and lower semicontinuous, \( \forall \chi \in L^\infty(\Omega; [1, 1]) \).
\[
J^n_m(\theta, \chi) \to +\infty \quad \text{as} \quad \|\theta\|_V \to +\infty, \quad \forall \chi \in (\Omega; [-1, 1]).
\] (The coerciveness of \( J^n_m \) w.r.t. \( \chi \) makes no sense, for this variable is confined to the interval \([-1, 1]\).) By Theorem 5.3.2, the functional \( J^n_m \) has then a saddle point in \( X \), namely
\[
J^n_m(\theta, \chi) \leq J^n_m(\theta_n^m, \chi^m_n) \quad \forall (\theta, \chi) \in X.
\]
Hence \( \partial J^n_m(\theta_n^m, \chi^m_n) \ni (0, 0) \), that is
\[
\partial \theta J^n_m(\theta_n^m, \chi^m_n) \ni 0 \quad \text{in} \quad V',
\]
\[
\partial \chi (-J^n_m)(\theta_n^m, \chi^m_n) \ni 0 \quad \text{a.e. in} \quad \Omega.
\]
For a suitable selection of the fields
\[
u^n_m \in \partial \psi(\theta^n_m, \chi^n_m), \quad r^n_m \in \partial (-\psi)(\theta^n_m, \chi^n_m), \quad q^n_m \in -\partial_2 \Phi(\theta^n_m, \nabla \theta^n_m),
\]
a.e. in \( \Omega \), the inclusions (4.2.19) and (4.2.20) yield (4.2.11) and (4.2.12). The functions \( u^n_m, \theta^n_m, \chi^n_m, r^n_m, q^n_m \) thus solve Problem 4.1.1.

(ii) A priori estimates. In view of deriving the balance of the function \( \varphi \), let us first notice that by (4.2.9)
\[
\sum_{n=1}^{\ell} \left[[u^n_m - u^{n-1}_m] \theta^n_m + (\chi^n_m - \chi^{n-1}_m) r^n_m \right]
\]
\[
\geq \sum_{n=1}^{\ell} \left[ \varphi(u^n_m, \chi^n_m) - \varphi(u^{n-1}_m, \chi^{n-1}_m) \right]
\]
\[
= \varphi(u_{\ell}^m, \chi_{\ell}^m) - \varphi(u^0, \chi^0) \quad \text{a.e. in} \quad \Omega.
\]

Moreover, by the inclusion (4.2.10) and by the hypotheses (4.2.5), (4.2.6),
\[
-\bar{q}^n_m \cdot \nabla \theta^n_m \geq \Phi(\theta^{n-1}_m, \nabla \theta^n_m) - \Phi(\theta^{n-1}_m, 0)
\]
\[
\geq a_3 |\nabla \theta^n_m|^2 - a_4 - a_7 \quad \text{a.e. in} \quad \Omega, \forall n.
\]
Let us now take \( v = k\theta^n_m \) in (4.2.11), multiply (4.2.12) by \( \chi^n_m - \chi^{n-1}_m \), and sum these formulas for \( n = 1, \ldots, \ell \), for any \( \ell \in \{1, \ldots, m\} \). By (4.2.21) we get
\[
\int_{\Omega} \left[ \varphi(u_{\ell}^m, \chi_{\ell}^m) - \varphi(u^0, \chi^0) \right] \, dx - k \sum_{n=1}^{\ell} \int_{\Omega} \bar{q}^n_m \cdot \nabla \theta^n_m \, dx
\]
\[
+ ak \sum_{n=1}^{\ell} \int_{\Omega} \frac{\chi^n_m - \chi^{n-1}_m}{k}^2 \, dx \leq k \sum_{n=1}^{\ell} f^n_m \theta^n_m,
\]
and then by (4.2.22)
\[
\int_{\Omega} \left[ \varphi(u_m^\ell, \chi_m^\ell) - \varphi(u_0^0, \chi_0^0) \right] \, dx + a_3 k \sum_{n=1}^{\ell} \int_{\Omega} |\nabla \theta_n^m|^2 \, dx \\
+ a k \sum_{n=1}^{\ell} \int_{\Omega} \frac{\| \chi_n^m - \chi_{n-1}^m \|^2}{k} \, dx \leq (a_4 + a_7) \ell k |\Omega| + k \sum_{n=1}^{\ell} \| f_n^m \|_V \| \theta_n^m \|_V.
\]
(4.2.24)

By (4.2.4) we then get
\[
\max_{n=1,\ldots,m} \| u_n^m \|_{L^2(\Omega)} + k \sum_{n=1}^{m} \| \theta_n^m \|_V, k \sum_{n=1}^{m} \left\| \chi_n^m - \chi_{n-1}^m \right\|_{L^2(\Omega)} \leq C_1.
\]
(4.2.25)

(By \( C_1, C_2, \ldots \) we shall denote constants independent of \( m \).) Using the notation (3.1.26) and setting \( \tau_k v(t) = v(t-k) \) for any function \( v \) of time, next we write the system (4.2.9)–(4.2.12) and the estimates (4.2.25) in terms of the time-interpolate functions:
\[
\begin{align*}
(\bar{\theta}_m^\ell, \bar{r}_m^\ell) &\in \partial \varphi(\bar{u}_m^\ell, \bar{\chi}_m^\ell) \text{ a.e. in } Q, \\
\bar{q}_m^\ell &\in -\partial \Phi(\tau_k \bar{\theta}_m^\ell, \nabla \bar{\theta}_m^\ell) \text{ a.e. in } Q, \\
\frac{\partial u_m^\ell}{\partial t} + \nabla \cdot \bar{q}_m^\ell &= \bar{f}_m^\ell \text{ in } V', \\
a \frac{\partial \chi_m^\ell}{\partial t} + \bar{r}_m^\ell &= 0 \text{ a.e. in } Q, \\
\| u_m^\ell \|_{L^\infty(0,T;L^2(\Omega))}, \| \theta_m^\ell \|_{L^2(0,T;V)}, \| \chi_m^\ell \|_{H^1(0,T;L^2(\Omega))} &\leq C_2.
\end{align*}
\]
(4.2.26)

(4.2.27)

(4.2.28)

(4.2.29)

(4.2.30)

By (4.2.27) and by the second inequality of (4.2.5), we then have
\[
\| \bar{q}_m^\ell \|_{L^2(Q)} \leq C_3, \text{ whence } \| \nabla \cdot \bar{q}_m^\ell \|_{L^2(0,T;V')} \leq C_4.
\]
(4.2.31)

By comparing the terms of the approximate equation (4.2.28), we thus get
\[
\| u_m^\ell \|_{H^1(0,T;V')} \leq C_5.
\]
(4.2.32)

(iii) Limit procedure. By the above uniform estimates, there exist \( u, \theta, \chi, r, \bar{q} \) such that, possibly taking \( m \to \infty \) along a subsequence,
\[
\begin{align*}
& u_m \to u \text{ weakly star in } L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;V'), \\
& \chi_m \to \chi \text{ weakly star in } L^\infty(Q) \cap H^1(0,T;L^2(\Omega)), \\
& \theta_m, \bar{\theta}_m \to \theta \text{ weakly in } L^2(0,T;V), \\
& r_m, \bar{r}_m \to r \text{ weakly in } L^2(Q), \\
& \bar{q}_m, \bar{\theta}_m \to \bar{q} \text{ weakly in } L^2(Q)^3.
\end{align*}
\]
(4.2.33)

(4.2.34)

(4.2.35)

(4.2.36)

(4.2.37)

\[100\] Dealing with spaces of (vector-valued) time-dependent functions, the weak (star) convergence of the linear interpolates is equivalent to that of the piecewise-constant interpolates, and the two limits coincide, provided that no time differentiability is involved. The convergences (4.2.33) and (4.2.34) thus entail \( \bar{u}_m \to u \) weakly star in \( L^\infty(0,T;L^2(\Omega)) \) and \( \bar{\chi}_m \to \chi \) weakly star in \( L^\infty(Q) \).
Possibly taking $m \to \infty$ along a further subsequence, we may also assume that
\[
\chi_m(\cdot, T) \rightharpoonup \chi(\cdot, T) \quad \text{weakly star in } L^\infty(\Omega).
\] (4.2.38)
By passing to the limit in (4.2.28) and (4.2.29) we then get (4.1.12) and (4.1.13). Because of (4.2.34) the initial condition of $\chi$ is preserved in the limit. We are thus just left with the proof of (4.1.10) and (4.1.11), that we shall perform in the next steps.

(iv) **Proof of (4.1.10).** By (3.1.33), the convergence (4.2.33) yields
\[
\chi_m \rightharpoonup \chi \quad \text{strongly in } C^0([0, T]; V').
\] (4.2.39)
As $\bar{u}_m - u_m \to 0$ in $L^2(0, T; V')$, by (4.2.35) we then get
\[
\int_Q \bar{u}_m \bar{\theta}_m \, dx \, dt = \int_0^T V' \langle \bar{u}_m - u_m, \bar{\theta}_m \rangle_V \, dt + \int_Q u_m \bar{\theta}_m \, dx \, dt \to \int_Q u \theta \, dx \, dt. \quad (4.2.40)
\]
Moreover by (4.1.13), (4.2.29) and (4.2.38)
\[
\limsup_{m \to \infty} \int_Q \bar{r}_m \bar{\chi}_m \, dx \, dt = -a \liminf_{m \to \infty} \int_Q \frac{\partial \chi_m}{\partial t} \bar{\chi}_m \, dx \, dt
\]
\[
= -a \liminf_{m \to \infty} \int_\Omega \left( |\chi_m(x, T)|^2 - |\chi_0|^2 \right) \, dx
\]
\[
\leq -\frac{a}{2} \int_\Omega \left( |\chi(x, T)|^2 - |\chi_0|^2 \right) \, dx
\]
\[
= -a \int_Q \frac{\partial \chi}{\partial t} \chi \, dx \, dt = \int_Q r \chi \, dx \, dt. \quad (4.2.41)
\]
Note that the inclusion (4.2.9) also reads
\[
\int_Q [\bar{\theta}_m(\bar{u}_m - \bar{u}) + \bar{r}_m(\bar{\chi}_m - \bar{\chi})] \, dx \, dt \geq \int_Q [\varphi(\bar{u}_m, \bar{\chi}_m) - \varphi(\bar{u}, \bar{\chi})] \, dx \, dt
\]
\[
\forall (\bar{u}, \bar{\chi}) \in L^2(Q) \quad \text{such that } (\bar{u}, \bar{\chi}) \in \text{Dom}(\varphi), \ a.e. \ in \ Q. \quad (4.2.42)
\]
By (4.2.40) and (4.2.41), passing to the limit in the latter inequality (more precisely, the superior limit in the left side and the inferior limit in the right side), we get
\[
\int_Q [\theta(u - \bar{u}) + r(\chi - \bar{\chi})] \, dx \, dt \geq \int_Q [\varphi(u, \chi) - \varphi(\bar{u}, \bar{\chi})] \, dx \, dt
\]
\[
\forall (\bar{u}, \bar{\chi}) \in L^2(Q)^2: (\bar{u}, \bar{\chi}) \in \text{Dom}(\varphi), \ a.e. \ in \ Q, \quad (4.2.43)
\]
and this is tantamount to (4.1.10).

(v) **Strong convergence of $\bar{\theta}_m$.** We claim that
\[
\bar{\theta}_m \rightharpoonup \theta \quad \text{strongly in } L^1(Q). \quad (4.2.44)
\]
First note that, by Rockafellar’s Theorem 5.2.3,
\[
\partial \psi_1 + \partial \psi_2(\theta, \chi) = \partial \psi_1(\theta) + \partial \psi_2(\theta, \chi).
\]
The analogous property for $\partial \chi$ is trivial, as $\psi_1$ is independent of $\chi$. Regarding $\psi_1(\theta)$ as a function of $(\theta, \chi)$ that is independent of $\chi$, and defining $\tilde{\partial}$ as in (5.3.17), we thus have $\tilde{\partial}(\psi_1 + \psi_2) = \tilde{\partial}\psi_1 + \tilde{\partial}\psi_2$. The hypothesis (4.2.3) and the inclusion (4.2.9) then yield

$$(\tilde{u}_m, \tilde{r}_m) = \tilde{\partial}\psi_1(\tilde{\theta}_m) + \tilde{\partial}\psi_2(\tilde{\theta}_m, \tilde{\chi}_m) \quad \text{a.e. in } Q, \forall m;$$

thus there exist $\tilde{u}_{1m}, \tilde{u}_{2m} \in L^2(\Omega)$ such that

$$
\begin{cases}
(\tilde{u}_m, \tilde{r}_m) = (\tilde{u}_{1m}, 0) + (\tilde{u}_{2m}, \tilde{r}_m), \\
(\tilde{u}_{1m}, 0) \in \tilde{\partial}\psi_1(\tilde{\theta}_m), \\
(\tilde{u}_{2m}, \tilde{r}_m) \in \tilde{\partial}\psi_2(\tilde{\theta}_m, \tilde{\chi}_m)
\end{cases}
\quad \text{a.e. in } Q, \forall m. \quad (4.2.45)
$$

By Theorem 5.3.4 the operators $\tilde{\partial}\psi_1$ and $\tilde{\partial}\psi_2$ are maximal monotone. We can thus apply Proposition 5.5.4, or rather the Remark (i) that follows that statement, taking $(\tilde{\theta}_m)$ in place of $\theta$, $u^*, \beta_i$, respectively (for $i = 1, 2$). We thus infer that

$$(u_1, 0) \in \tilde{\partial}\psi_1(\theta), \quad \text{namely } u_1 \in \partial\psi_1(\theta) \quad \text{a.e. in } Q. \quad (4.2.46)$$

Let us also recall (4.2.33) and (4.2.35). By the strict convexity of $\psi_1$ (cf. (4.2.3)), by (4.2.40) and by Proposition 5.4.3, the claim (4.2.44) then follows.

(vi) An auxiliary inequality. We claim that

$$
\limsup_{m \to \infty} - \int_0^t d\tau \int_\Omega \tilde{\bar{q}}_m \cdot \nabla \tilde{\bar{\theta}}_m \, dx 
\leq - \int_0^t d\tau \int_\Omega \bar{q} \cdot \nabla \theta \, dx,
$$

for a.e. $t \in [0, T]$. \quad (4.2.47)

In view of proving this inequality, notice that (4.2.23) also reads

$$
\int_\Omega \left[ \psi(u_m, \chi_m)(\cdot, t) - \psi(u_0^m, \chi_0^m) \right] \, dx 
- \int_0^t d\tau \int_\Omega \tilde{\bar{q}}_m \cdot \nabla \tilde{\bar{\theta}}_m \, dx
+ a \int_0^t d\tau \int_\Omega \left| \frac{\partial \chi_m}{\partial t} \right|^2 \, dx
\leq \int_0^t \langle \bar{f}_m, \tilde{\bar{\theta}}_m \rangle \, d\tau
\quad \text{for a.e. } t \in ]0, T[.
$$

Moreover by (4.1.10) and by Proposition 5.2.7

$$
\int_0^t \left\langle \frac{\partial \theta}{\partial \tau}, \theta \right\rangle \, d\tau + \int_0^t d\tau \int_\Omega \frac{\partial \chi}{\partial \tau} \, dx
= \int_\Omega \left[ \psi(u, \chi)(\cdot, t) - \psi(u_0^0, \chi_0^0) \right] \, dx
\quad \text{for a.e. } t \in ]0, T[. \quad (4.2.49)
$$

Note that Eq. (4.1.5) holds in $L^2(0, T; V')$, cf. (4.1.12). Let us next multiply (4.1.5) by $\theta$, multiply (4.1.13) by $\frac{1}{\alpha}\frac{\partial \chi}{\partial t}$, sum these formulas, and integrate in time. By (4.2.49) this yields

$$
\int_\Omega \left[ \psi(u, \chi)(\cdot, t) - \psi(u_0^0, \chi_0^0) \right] \, dx 
- \int_0^t d\tau \int_\Omega \tilde{\bar{q}} \cdot \nabla \theta \, dx
+ a \int_0^t d\tau \int_\Omega \left| \frac{\partial \chi}{\partial t} \right|^2 \, dx
= \int_0^t \langle f, \theta \rangle \, d\tau
\quad \text{for a.e. } t \in ]0, T[. \quad (4.2.50)
$$
Notice that, by the lower semicontinuity of the convex integral functionals,
\begin{equation}
\liminf_{m \to \infty} \left( \int_{\Omega} \phi(u_m, \chi_m)(\cdot, t) \, dx + a \int_{0}^{t} \int_{\Omega} \left| \frac{\partial \chi_m}{\partial t} \right|^2 \, dx \right) \\
\geq \int_{\Omega} \phi(u, \chi)(\cdot, t) \, dx + a \int_{0}^{t} \int_{\Omega} \left| \frac{\partial \chi}{\partial t} \right|^2 \, dx \quad \forall t \in [0, T].
\end{equation}

The formulas (4.2.48), (4.2.50), (4.2.51) yield (4.2.47).

(vii) Proof of (4.1.11). For any \( \tilde{\xi} \in L^2(\Omega)^3 \),
\begin{equation}
\int_{Q} \tilde{q}_m \cdot (\tilde{\xi} - \nabla \tilde{\theta}_m) \, dx \, dt \geq \int_{Q} \left[ \Phi(\tau_k \tilde{\theta}_m, \nabla \tilde{\theta}_m) - \Phi(\tau_k \tilde{\theta}_m, \tilde{\xi}) \right] \, dx \, dt.
\end{equation}

Moreover, by (4.1.2), (4.2.44) and by the lower semicontinuity of the integral functional associated to \( \Phi(\theta, \cdot) \),
\begin{equation}
\liminf_{m \to \infty} \int_{Q} \Phi(\tau_k \tilde{\theta}_m, \nabla \tilde{\theta}_m) \, dx \, dt \geq \int_{Q} \Phi(\theta, \nabla \theta) \, dx \, dt,
\end{equation}

\begin{equation}
\int_{Q} \Phi(\tau_k \tilde{\theta}_m, \tilde{\xi}) \, dx \, dt \to \int_{Q} \Phi(\theta, \tilde{\xi}) \, dx \, dt.
\end{equation}

By (4.2.47), (4.2.53) and (4.2.54), by passing to the superior limit in the left side and to the inferior limit in the right side of (4.2.52), we then get
\begin{equation}
\int_{Q} \tilde{q} \cdot (\tilde{\xi} - \nabla \theta) \, dx \, dt \geq \int_{Q} \left[ \Phi(\theta, \nabla \theta) - \Phi(\theta, \tilde{\xi}) \right] \, dx \, dt
\end{equation}
for any \( \tilde{\xi} \in L^2(\Omega)^3 \), that is (4.1.11).

Modelling remarks. (i) In Problem 4.1.1 we formulated the constitutive law (4.1.3) in terms of the convex potential \( \psi \). This relation is equivalent to
\begin{equation}
u \in \partial_0 \psi(\theta, \chi), \quad \tau \in \partial_\chi(-\psi)(\theta, \chi).
\end{equation}

Next we show that for the two-phase system it is equivalent to construct the functions \( \psi \) and \( \varphi \). Let us assume that the constitutive relation between \( u \) and \( \theta \) is known in each phase; that is,
\begin{equation}
u \in \partial_0 \psi_s(\theta) \quad \text{in the solid,} \quad u \in \partial_0 \psi_\ell(\theta) \quad \text{in the liquid},
\end{equation}
for given convex functions \( \psi_s \) and \( \psi_\ell \). This suggests to set \( \psi(\theta, -1) := \psi_s(\theta), \psi(\theta, 1) := \psi_\ell(\theta) \), and to extend \( \psi(\theta, \chi) \) by linear interpolation:
\begin{equation}
\psi(\theta, \chi) := \psi_s(\theta) \frac{1 - \chi}{2} + \psi_\ell(\theta) \frac{1 + \chi}{2} \quad \forall (\theta, \chi) \in \mathbf{R} \times [-1, 1].
\end{equation}

This function is not globally convex w.r.t. the pair \( (\theta, \chi) \); actually, it is convex in \( \theta \) and linear (hence concave) in \( \chi \). By Theorem 5.3.3, a convex function \( \varphi(u, \chi) \) is then retrieved by partially conjugating \( \psi(\theta, \chi) \) w.r.t. \( \theta \), and in turn \( \psi \) is the partial conjugate of \( \varphi \) w.r.t. \( u \).
\[ \varphi(u, \chi) := \sup_{\theta \in \mathbb{R}} \left[ u \theta - \psi(\theta, \chi) \right] \quad \forall (u, \chi) \in \mathbb{R} \times [-1, 1], \]

\[ \psi(\theta, \chi) := \sup_{u \in \mathbb{R}} \left[ u \theta - \varphi(u, \chi) \right] \quad \forall (\theta, \chi) \in \mathbb{R} \times [-1, 1]. \quad (4.2.59) \]

This is equivalent to the dual interpolation procedure. Setting \( \varphi_s := \psi_s^* \) and \( \varphi_l := \psi_l^* \), the prescribed relations (4.2.57) indeed also read

\[ \theta \in \partial \varphi_s(u) \quad \text{in the solid,} \quad \theta \in \partial \varphi_l(u) \quad \text{in the liquid.} \quad (4.2.60) \]

By interpolation a nonconvex function \( \tilde{\varphi} \) is obtained:

\[ \tilde{\varphi}(u, \chi) := \varphi_s(u) \frac{1 - \chi}{2} + \varphi_l(u) \frac{1 + \chi}{2} \quad \forall (u, \chi) \in \mathbb{R} \times [-1, 1]. \quad (4.2.61) \]

(ii) The equality (4.2.50) accounts for the balance of the function \( \varphi \) (here rescaled by the factor \( \tau_E \)), cf. (2.2.15) and (2.2.17). The first term is the total variation of \( \int_{\Omega} \varphi \, dx \) in the time interval \([0, \ell k]\). The opposite of the second and the third terms represent the (nonnegative) amount of \( \varphi \) that is \emph{dissipated} in that time interval. The second member is the contribution of the heat source (or sink) \( f \). The function \( \varphi \) is the potential of (2.2.15) rescaled by the constant factor \( \tau_E \), and this balance accounts for the consistency of this model with the second principle of thermodynamics, as we saw in Section 2.2.

(iii) If the function \( \psi \) were strictly convex w.r.t. \( \chi \), then \( \psi_s(\theta, \chi) \) would depend on \( \chi \) continuously for \( 0 < \chi < 1 \) for any \( \theta \). This would exclude the occurrence of sharp interfaces between the phases, so that Problem 4.1.1 would represent heat-diffusion with phase transition smoothed out in a temperature interval.\(^{101}\) In this case the above argument might also be simplified, for Corollary 5.4.2 would also entail the strong convergence of \( \chi_m \) in \( L^1(Q) \), without the need of the hypothesis (4.2.3).

One might prove several further results for Problem 4.1.1. For instance, the solution depends weakly-continuously on the data in the sense of Theorem 3.1.2. In presence of a composite material, one might also homogenize this problem along the lines of [458].\(^{102}\) On the other hand the uniqueness of the solution does not seem obvious, because of the multiple nonlinearity of the problem. However, if (4.1.14) is reduced to the linear Fourier law \( \vec{q} = -k \cdot \nabla \theta \), after time integration the uniqueness may be proved along the lines of Section 3.3.

5. Convexity and other analytical tools

The analysis of Stefan-type problems requires several tools of linear and nonlinear functional analysis. In this appendix we briefly review basic notions of convex analysis, maximal monotone and accretive operators, nonlinear semigroups of contractions in Banach spaces, \( \Gamma \)-convergence, and others. We review some definitions, and state few results that are referred to in the remainder of this survey. We just display some of the most simple proofs. For an appropriate treatment we refer to the literature that is quoted in the respective sections.

\(^{101}\) This is physically acceptable for several materials, e.g. of organic origin.

\(^{102}\) The homogenization of phase transitions was also studied e.g. by Ansini, Braides and Chiadò Piat [26], Bossavit and Damlamian [79], Damlamian [159], Rodrigues [393], Visintin [458].
5.1. Convex and lower semicontinuous functions

In this and in the next two sections we outline some notions and properties of convex analysis.103 We assume throughout that $B$ is a real Banach space equipped with the norm $\| \cdot \|$.104 By means of the pairing $\langle \cdot, \cdot \rangle$, we put $B$ in duality with its dual space $B^*$, by equipping these spaces respectively with the weak and the weak star topology. In this way $B$ will play the role of the dual of $B^*$, even if $B$ is not reflexive. For any function $F : B \to \tilde{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ let us set

\begin{align*}
\text{(5.1.1)} & \quad \text{Dom}(F) := \{ v \in B : F(v) < +\infty \} : \text{(effective) domain of } F, \\
\text{(5.1.2)} & \quad \text{epi}(F) := \{ (v, a) \in B \times \mathbb{R} : F(v) \leq a \} : \text{epigraph of } F. 
\end{align*}

Let us also define the indicator function of any set $K \subset B$:

\begin{equation}
\text{(5.1.3)} \quad I_K : B \to \tilde{\mathbb{R}} : v \mapsto \begin{cases} 0 & \text{if } v \in K, \\ +\infty & \text{if } v \notin K. \end{cases}
\end{equation}

This definition allows one to reformulate constrained minimization problems as unconstrained ones, for

\begin{equation}
\text{(5.1.4)} \quad u = \inf_K F \iff u = \inf_B (F + I_K).
\end{equation}

Any set $K \subset B$ is said to be convex if either it is empty or

\begin{equation}
\text{(5.1.5)} \quad \lambda v_1 + (1 - \lambda)v_2 \in K \quad \forall v_1, v_2 \in K, \quad \forall \lambda \in ]0, 1[.
\end{equation}

A function $F : B \to \tilde{\mathbb{R}}$ is said to be convex if

\begin{equation}
\text{(5.1.6)} \quad F(\lambda v_1 + (1 - \lambda)v_2) \leq \lambda F(v_1) + (1 - \lambda)F(v_2) \\
\forall v_1, v_2 \in B, \quad \forall \lambda \in ]0, 1[.
\end{equation}

with obvious conventions for the arithmetical operations in $\tilde{\mathbb{R}}$. If the inequality (5.1.6) is strict for any $v_1 \neq v_2$, the function $F$ is said to be strictly convex. The function $F$ is said to be lower semicontinuous if the set $\{ v \in B : F(v) \leq a \}$ is closed for any $a \in \mathbb{R}$. $F$ is said to be proper if it is not identically equal to $+\infty$.

**Proposition 5.1.1.**

(i) A function $F : B \to \tilde{\mathbb{R}}$ is convex (lower semicontinuous, resp.) if and only if $\text{epi}(F)$ is convex (closed, resp.).

(ii) A set $K \subset B$ is convex (closed, resp.) if and only if $I_K$ is convex (lower semicontinuous, resp.).

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103 See e.g. the monographs quoted in the item (VIII) of the Bibliographical Note in Section 6.

104 This theory might also be developed in the more general framework of topological vector spaces, see e.g. Ekeland and Temam [188], Moreau [339]. One may also deal with complex spaces, just replacing the duality pairing by its real part.
PROPOSITION 5.1.2.
(i) If \( \{ F_i : B \to \tilde{\mathbb{R}} \}_{i \in I} \) is a family of convex (lower semicontinuous, resp.) functions, then their upper hull \( F : v \mapsto \sup_{i \in I} F_i(v) \) is convex (lower semicontinuous, resp.).
(ii) If \( \{ K_i \}_{i \in I} \) is a family of convex (closed, resp.) subsets of \( B \), then their intersection \( \bigcap_{i \in I} K_i \) is convex (closed, resp.).

Let us denote by \( \Gamma(B) \) the class of functions \( F : B \to \tilde{\mathbb{R}} \) that are the upper hull of a family of continuous and affine functions \( B \to \mathbb{R} \). This consists of the class \( \Gamma_0(B) \) of proper, convex, lower semicontinuous functions, and of the function identically equal to \( +\infty \).

By part (ii) of Proposition 5.1.2, for any set \( K \subset B \) the intersection of the convex and closed subsets of \( B \) that contain \( K \) is convex and closed; this is named the \emph{closed convex hull} of \( K \), and is denoted by \( \text{co}(K) \). Similarly, let us consider any function \( F : B \to (-\infty, +\infty] \) that has a convex lower bound. By part (i) of Proposition 5.1.2, the upper hull of all affine lower bounds of \( F \) is convex and lower semicontinuous; this is the largest lower bound of \( F \) in \( \Gamma(B) \), and is named the \( \Gamma \)-regularized function of \( F \). Its epigraph coincides with the closed convex hull of the epigraph of \( F \).

5.2. Legendre–Fenchel transformation and subdifferential

Let \( F : B \to \tilde{\mathbb{R}} \) be a proper function. The function
\[
F^* : B^* \to \tilde{\mathbb{R}} : u^* \mapsto \sup_{u \in B} \{ \langle u^*, u \rangle - F(u) \}
\]
(5.2.1)
is called the (convex) \emph{conjugate function} of \( F \). If \( F^* \) is proper, its conjugate function
\[
F^{**} : B \to \tilde{\mathbb{R}} : u \mapsto \sup_{u^* \in B^*} \{ \langle u^*, u \rangle - F^*(u^*) \}
\]
(5.2.2)
is called the \emph{biconjugate function} of \( F \). (Notice that we defined \( F^{**} \) on \( B \) rather than the bidual space \( B^{**} \).) If the function \( F \) depends on two (or more) variables, one may also introduce the \emph{partial conjugate} function w.r.t. any of these variables. For instance, if \( F : B^2 \to \tilde{\mathbb{R}} \), then its partial conjugate w.r.t. the first variable reads
\[
G : B^* \times B \to \tilde{\mathbb{R}} : (u^*, w) \mapsto \sup_{u \in B} \{ \langle u^*, u \rangle - F(u, w) \}.
\]
(5.2.3)

THEOREM 5.2.1. For any proper function \( F : B \to \tilde{\mathbb{R}} \) such that \( F^* \) is also proper,
\[
F^* \in \Gamma(B^*); \quad F^{**} \leq F;
\]
(5.2.4)
\[
F^{**} = F \iff F \in \Gamma(B); \quad (F^*)^{**} = F^*.
\]
Moreover, \( F^{**} \) coincides with the \( \Gamma \)-regularized function of \( F \) (Fenchel–Moreau theorem).

The conjugacy transformation \( F \mapsto F^* \) is a bijection between \( \Gamma_0(B) \) and \( \Gamma_0(B^*) \).

We define the \emph{subdifferential} \( \partial F : \text{Dom}(F) \subset B \to 2^{B^*} \) (the power set) of any proper function \( F : B \to \tilde{\mathbb{R}} \) as follows:
Fig. 10. The drawn subtangent straight-line is characterized by the equation \( z = \langle u^*, v - u \rangle + F(u) \), or equivalently \( z = \langle u^*, v \rangle - F^*(u^*) \), for any \( u^* \in \partial F(u) \).

\[
\partial F(u) := \left\{ u^* \in B^* : \langle u^*, u - v \rangle \geq F(u) - F(v), \forall v \in B \right\}
\]  
\( \forall u \in \text{Dom}(F), \)  
\( (5.2.5) \)

cf. Figure 10. Dually, we define \( \partial F^*(u^*) : \text{Dom}(F^*) \subset B^* \rightarrow 2^{B^*} : \)

\[
\partial F^*(u^*) := \left\{ u \in B : \langle u, u^* - v^* \rangle \geq F^*(u^*) - F^*(v^*), \forall v^* \in B^* \right\}
\]  
\( \forall u^* \in \text{Dom}(F^*). \)  
\( (5.2.6) \)

Note that \( \partial F(u) = \emptyset \) is not excluded, so that one may also take the subdifferential of an either nonconvex or nonlower-semicontinuous function at any point of its domain. We also set

\[ \partial F(u) := \emptyset \quad \forall u \in B \setminus \text{Dom}(F), \quad \partial F^*(u^*) := \emptyset \quad \forall u^* \in B^* \setminus \text{Dom}(F^*). \]

If the function \( F \) depends on two or more variables, one may also introduce the partial subdifferential w.r.t. one of its arguments, extending the notion of partial derivative. For instance, if \( F : B^2 \rightarrow \tilde{\mathbb{R}} \), then the partial subdifferential \( \partial_u F(u, w) \) is defined as in (5.2.5), by freezing the dependence on the argument \( w \).

**Proposition 5.2.2.** Let \( F : B \rightarrow \tilde{\mathbb{R}} \). Then for any \( u \in B \) and any \( u^* \in B^* \):

\[
F(u) + F^*(u^*) \geq \langle u^*, u \rangle,
\]  
\( (5.2.7) \)

\[
u^* \in \partial F(u) \quad \Leftrightarrow \quad F(u) + F^*(u^*) = \langle u^*, u \rangle,
\]  
\( (5.2.8) \)

\[
u^* \in \partial F(u) \quad \Rightarrow \quad u \in \partial F^*(u^*),
\]  
\( (5.2.9) \)

\[
\left[ F(u) = F^{**}(u), \ u \in \partial F^*(u^*) \right] \quad \Rightarrow \quad u^* \in \partial F(u),
\]  
\( (5.2.10) \)

\[
F \in \Gamma_0^1(B) \quad \Rightarrow \quad \partial F^* = (\partial F)^{-1}.
\]  
\( (5.2.11) \)

(5.2.7) follows from the definition of \( F^* \). By taking the supremum over all test functions \( v \) in (5.2.5), we get \( F(u) + F^*(u^*) \leq \langle u^*, u \rangle \) whenever \( u^* \in \partial F(u) \); (5.2.7) then entails the equality. The opposite implication directly follows from the definition of \( F^* \). (5.2.8) is thus established. The statements (5.2.9)–(5.2.11) are easily proved by means of (5.2.8).
THEOREM 5.2.3 (Rockafellar). Let $F_1, F_2 : B \to \tilde{\mathbb{R}}$. Then
\[
\partial F_1(u) + \partial F_2(u) \subset \partial (F_1 + F_2)(u) \quad \forall u \in \text{Dom}(F_1) \cap \text{Dom}(F_2).
\]
The opposite inclusion holds whenever $F_1$ and $F_2$ are both convex and lower semicontinuous, and either $F_1$ or $F_2$ is continuous at some point $u_0 \in \text{Dom}(F_1) \cap \text{Dom}(F_2)$.

PROPOSITION 5.2.4. Let $F : B \to \tilde{\mathbb{R}}$ be convex and proper. Then $F$ is locally Lipschitz-continuous at the interior of $\text{Dom}(F)$, and there $\partial F \neq \emptyset$.

EXAMPLE. Let $B$ be a real Hilbert space, denote it by $H$, and identify it with its dual space, so that the duality pairing coincides with the scalar product: $\langle u, v \rangle = (u, v)$ for any $u, v \in H$. Let also $1 \leq p < +\infty$ and set $p' = p/(p - 1)$ if $p > 1$, $p' = \infty$ if $p = 1$. Let us consider the functional $F_p(u) = \|u\|^p/p$ for any $u \in H$. If $p > 1$, then
\[
\partial F_p(u) = \|u\|^{p-2}u \quad \forall u \in H, \quad F_p^*(v) = \frac{1}{p'}\|v\|^{p'} \quad \forall v \in H.
\]
On the other hand for $p = 1$,
\[
\partial F_1(u) = \{\|u\|^{-1}u\} \quad \forall u \in H \setminus \{0\}, \quad \partial F_1(0) = \{v \in H : \|v\| \leq 1\}, \quad F_1^*(v) = 0 \text{ if } \|u\| \leq 1, \quad F_1^*(v) = +\infty \text{ otherwise}.
\]
In particular, if $H = \mathbb{R}$ then $\partial F_1$ coincides with the multi-valued sign function:
\[
\text{sign}(u) := \{-1\} \quad \text{if } u < 0, \quad \text{sign}(0) := [-1, 1], \quad \text{sign}(u) := \{1\} \quad \text{if } u > 0.
\]
Here (5.2.7) and (5.2.8) read
\[
\frac{1}{p}\|u\|^p + \frac{1}{p'}\|v\|^{p'} \geq (u, v),
\]
\[
v = \|u\|^{p-2}u \iff \frac{1}{p}\|u\|^p + \frac{1}{p'}\|v\|^{p'} = (u, v) \quad \forall u, v \in H.
\]
(For $H = \mathbb{R}$, the former inequality is the classical Young inequality.) A similar example applies if $B$ is a Banach space, but this requires the introduction of the notion of duality mapping.

PROPOSITION 5.2.5. For any proper, convex, lower semicontinuous function $F : B \to \tilde{\mathbb{R}}$, any $u \in B$ and any $u^* \in B^*$, the following statements are mutually equivalent:
\[
u^* \in \partial F(u),
\]
\[
u \in \partial F^*(u^*),
\]
\[
u \in \text{Dom}(F), \quad \langle u^*, u - v \rangle \geq F(u) - F(v) \quad \forall v \in B,
\]
\[
u^* \in \text{Dom}(F^*), \quad \langle u^*, u^* - v^* \rangle \geq F^*(u^*) - F^*(v^*) \quad \forall v^* \in B^*,
\]
\[
\langle u, u^* \rangle \geq F(u) + F^*(u^*),
\]
\[
\langle u, u^* \rangle = F(u) + F^*(u^*).
\]
The equivalence between (5.2.17) and (5.2.18) follows from (5.2.10). The inclusions (5.2.17) and (5.2.18) are respectively equivalent to the variational inequalities (5.2.19) and (5.2.20), by the definitions of $\partial F$ and $\partial F^*$. The inequality (5.2.21) is equivalent to (5.2.19) by the definition of $F^*$. Finally, (5.2.21) is equivalent to (5.2.22) because of (5.2.7).

The next statement is just a particular case of the latter proposition.

**COROLLARY 5.2.6.** For any (nonempty) closed convex set $K \subset B$, any $u \in B$ and any $u^* \in B^*$, the following statements are mutually equivalent:

\begin{align*}
    u^* & \in \partial I_K(u), \quad (5.2.23) \\
    u & \in \partial I_K^*(u^*), \quad (5.2.24) \\
    u & \in K, \quad \langle u^*, u - v \rangle \geq 0 \quad \forall v \in K, \quad (5.2.25) \\
    \langle u, u^* - v^* \rangle & \geq I_K^*(u^*) - I_K^*(v^*) \quad \forall v^* \in B^*, \quad (5.2.26) \\
    u & \in K, \quad \langle u, u^* \rangle \geq I_K^*(u^*), \quad (5.2.27) \\
    u & \in K, \quad \langle u, u^* \rangle = I_K^*(u^*). \quad (5.2.28)
\end{align*}

The next statement is often applied.

**PROPOSITION 5.2.7.** Let $B^*$ be a Banach space, $F : B \to \tilde{\mathbb{R}}$ be convex and lower semi-continuous, and $p \in [1, +\infty]$. If

\begin{align*}
    u & \in W^{1,p}(0, T; B), \quad w \in L^{p'}(0, T; B^*), \quad (5.2.29) \\
    w & \in \partial F(u) \quad \text{a.e. in } ]0, T[, \quad (5.2.30)
\end{align*}

then

\begin{equation}
    F(u) \in W^{1,1}(0, T), \quad \frac{d}{dt} F(u) = \left\langle w, \frac{du}{dt} \right\rangle \quad \text{a.e. in } ]0, T[. \quad (5.2.31)
\end{equation}

### 5.3. Saddle points

Let $U$, $V$ be nonempty subsets of two real topological vector spaces $X_1$, $X_2$ (resp.), and $L : U \times V \to \mathbb{R}$. (We assume that this function is finite in order to simplify the presentation.) Note that

\begin{equation}
    \inf_{u \in U} L(u, \bar{v}) \leq \sup_{v \in V} L(\bar{u}, v) \quad \forall (\bar{u}, \bar{v}) \in U \times V. \quad (5.3.1)
\end{equation}

A point $(\bar{u}, \bar{v}) \in U \times V$ is called a saddle point of $L$ whenever the opposite inequality is fulfilled, or equivalently

\begin{equation}
    L(u, v) \leq L(\bar{u}, \bar{v}) \quad \forall (u, v) \in U \times V. \quad (5.3.2)
\end{equation}

This is also equivalent to the so-called min-max equality:

\begin{equation}
    \min_{u \in U} \sup_{v \in V} L(u, v) = \max_{v \in V} \inf_{u \in U} L(u, v). \quad (5.3.3)
\end{equation}
In view of the next two statements, we shall say that a function \( f : X \to \mathbb{R} \) is quasi-convex if for any \( a \in \mathbb{R} \) the sublevel set \( \{ v \in X : f(v) \leq a \} \) is convex, and that it is quasi-concave if \( -f \) is quasi-convex. Obviously, any convex (concave, resp.) function is quasi-convex (quasi-concave, resp.), but the converse may fail: for instance, any nondecreasing real function is quasi-convex.

The next statement is often used in the study of saddle points.

**Theorem 5.3.1 (Fan inequality).** Let \( K \) be a compact convex subset of a real topological vector space \( X \), and \( \varphi : K^2 \to \mathbb{R} \) be such that

\[
\varphi(\cdot, y) \text{ is lower semicontinuous } \forall y \in K, \\
\varphi(x, \cdot) \text{ is quasi-concave } \forall x \in K.
\]

Then

\[
\min_{x \in K} \sup_{y \in K} \varphi(x, y) \leq \sup_{y \in K} \varphi(y, y). \tag{5.3.6}
\]

**Theorem 5.3.2 (Existence of a saddle point – Von Neumann – Sion).** Let \( U \) and \( V \) be nonempty compact convex subsets of two real topological vector spaces \( X_1 \) and \( X_2 \) (resp.), and \( L : U \times V \to \mathbb{R} \) be such that

\[
L(\cdot, v) \text{ is quasi-convex and lower semicontinuous } \forall v \in V, \\
L(u, \cdot) \text{ is quasi-concave and upper semicontinuous } \forall u \in U. \tag{5.3.7}
\]

Then \( L \) has a saddle point, and more precisely

\[
\min_{u \in U} \max_{v \in V} L(u, v) = \max_{v \in V} \min_{u \in U} L(u, v). \tag{5.3.8}
\]

We just show that, under the strengthened hypotheses that \( L(\cdot, v) \) is convex and \( L(u, \cdot) \) is concave, this statement follows from Theorem 5.3.1. Let us first set

\[
\varphi((\bar{u}, \bar{v}), (u, v)) := L(\bar{u}, v) - L(u, \bar{v}) \quad \forall (\bar{u}, \bar{v}), (u, v) \in K := U \times V,
\]

and notice that the hypotheses of Fan’s theorem are fulfilled. There exists then \((\bar{u}, \bar{v}) \in K\) such that

\[
\varphi((\bar{u}, \bar{v}), (u, v)) = \min_{(\bar{u}, \bar{v}) \in K} \sup_{(u, v) \in K} \varphi((\bar{u}, \bar{v}), (u, v)) \\
\leq \sup_{(u, v) \in K} \varphi((u, v), (u, v)) = 0. \tag{5.3.9}
\]

Thus \((\bar{u}, \bar{v})\) fulfills (5.3.3), namely it is a saddle point of \( L \). Actually, by the compactness of \( U \) and \( V \), in this case the min-max equality has the more precise form (5.3.8).

The function \( L \) may be allowed to attain the values \( \pm \infty \), but then some care must be paid in defining lower and upper semicontinuity.\(^{105}\)

\(^{105}\) See e.g. Rockafellar [388], [389, Section 33].
THEOREM 5.3.3. Let $B_1$, $B_2$ be two real Banach spaces, $F \in \Gamma_0(B_1 \times B_2)$, and $L$ be the partial conjugate of $F$ w.r.t. $u$:

$$L(u^*, w) = \sup_{u \in B_1} \left\{ (u^*, u) - F(u, w) \right\} \quad \forall (u^*, w) \in B_1^* \times B_2. \quad (5.3.10)$$

Then: (i)

$$L(\cdot, w) \text{ is convex and lower semicontinuous } \forall w \in B_2, \quad (5.3.11)$$

$$L(u^*, \cdot) \text{ is concave } \forall u^* \in B_1^*. \quad (5.3.12)$$

(In general $L(u^*, \cdot)$ need not be upper semicontinuous.)

(ii) Moreover, let $B_1$ be reflexive, and $F$ be coercive with respect to $u$ locally uniformly with respect to $w$, in the sense that

$$\forall \text{ bounded } S \subset B_2, \forall M > 0,$$

$$\left\{ u \in B_1 : F(u, w) \leq M, \forall w \in S \right\} \text{ is bounded.} \quad (5.3.13)$$

Then $L(u^*, \cdot)$ is upper semicontinuous for any $u^* \in B_1^*$.

(iii) Under the above hypotheses, for any $(u, w) \in B_1 \times B_2$ and any $(u^*, w^*) \in B_1^* \times B_2^*$,

$$\begin{cases}
    u^* \in \partial_u F(u, w), \\
    w^* \in \partial_w F(u, w)
\end{cases} \iff \begin{cases}
    u \in \partial_u^* L(u^*, w), \\
    w^* \in \partial_w^* (-L)(u^*, w). \quad (5.3.14)
\end{cases}$$

(iv) Conversely, if $L : B_1^* \times B_2 \to \mathbb{R}$ fulfills (5.3.11) and (5.3.12), then the mapping

$$G(u, w) := \sup_{u^* \in B_1^*} \left\{ (u^*, u) - L(u^*, w) \right\} \quad \forall (u, w) \in B_1 \times B_2 \quad (5.3.15)$$

is convex and lower semicontinuous. Moreover, if $L$ fulfills (5.3.10) then $F = G$.

PARTIAL PROOF. (i) The statement (5.3.11) directly follows from part (i) of Proposition 5.1.2.

In view of proving (5.3.12), let us fix any $u^* \in B_1^*$, any $u', u'' \in B_1$, any $w', w'' \in B_2$ and any $\lambda \in [0, 1]$. By the convexity of $F$ we have

$$L(u^*, \lambda w' + (1 - \lambda) w'')$$

$$\geq \langle u^*, \lambda u' + (1 - \lambda) u'' \rangle - F(\lambda u' + (1 - \lambda) u'', \lambda w' + (1 - \lambda) w'')$$

$$\geq \lambda (\langle u^*, u' \rangle - F(u', w')) + (1 - \lambda) (\langle u^*, u'' \rangle - F(u'', w'')).$$

By taking the supremum with respect to $u'$ and $u''$, we then get

$$L(u^*, \lambda w' + (1 - \lambda) w'') \geq \lambda L(u^*, w') + (1 - \lambda) L(u^*, w'').$$

The property (5.3.12) has thus been proved.

(ii) Let us fix any sequence $\{w_n\}$ in $B_2$ that weakly converges to some $w \in B_2$; $\{w_n\}$ is necessarily bounded. If $M := \limsup_{n \to \infty} L(u^*, w_n) = -\infty$ then trivially $L(u^*, w) \geq M$. If instead $M > -\infty$ then there exists a sequence $\{u_n\}$ in $B_1$ such that for $n$ large enough

$$\langle u^*, u_n \rangle - F(u_n, w_n) \geq L(u^*, w_n) + 1/n \quad \forall n \in \mathbb{N} \quad \text{if } M < +\infty. \quad (5.3.16)$$
\(\langle u^*, u_n \rangle - F(u_n, w_n) \geq n \quad \forall n \in \mathbb{N} \) if \(M = +\infty\).

By (5.3.13) the sequence \(\{u_n\}\) is confined to a bounded subset of the reflexive space \(B_1\). Hence there exists \(u \in B_1\) such that, as \(n\) diverges along a further subsequence (not relabelled), \(u_n \to u\) weakly in \(B_1\). Passing to the limit in (5.3.16) on this subsequence, by the lower semicontinuity of \(F\) we then get
\[
\langle u^*, u \rangle - F(u, w) \geq \limsup_{n \to \infty} (\langle u^*, u_n \rangle - F(u_n, w_n)) \geq M.
\]
Thus \(L(u^*, w) \geq M\).

(iii) For the proof of (5.3.14) see e.g. Barbu and Precupanu [49, p. 135], Rockafellar [388], [389, p. 395].

(iv) The convexity and lower semicontinuity of \(G\) follow from Proposition 5.1.2. As \(G(\cdot, w)\) is the biconjugate function of \((\cdot, w)\), by Theorem 5.2.1 we conclude that \(F = G\). \(\square\)

For any function \(L : U \times V \to \mathbb{R}\), let us define the double subdifferential
\[
\tilde{\partial}L : U \times V \to 2^{B_1^*} \times 2^{B_2^*} : (u, v) \mapsto \left(\partial_u L(u, v), \partial_v [-L(u, v)]\right).
\]
This definition is especially convenient if \(L\) is convex-concave, cf. (5.3.18) below.

Incidentally, note that any \((u, v)\) is a saddle point of \(L\) if and only if \((0, 0) \in \tilde{\partial}L(u, v)\).

**Theorem 5.3.4.** Let \(U\) and \(V\) be nonempty, closed, convex subsets of two real Banach spaces \(B_1\) and \(B_2\) (respect.), with at least one of them reflexive. Let \(L : U \times V \to \mathbb{R}\) be such that
\[
L(\cdot, v) \text{ is convex and lower semicontinuous } \forall v \in V, \\
L(u, \cdot) \text{ is concave and upper semicontinuous } \forall u \in U.
\]
The operator \(\tilde{\partial}L\) is then maximal monotone. \(\square\)

### 5.4. Compactness by strict convexity

Let \(K\) be a closed subset of \(\mathbb{R}^N\). A point \(\xi \in K\) is said **extremal** for \(K\) if
\[
\xi = \lambda \xi' + (1 - \lambda) \xi'' \in K, \quad \xi', \xi'' \in K, \quad 0 < \lambda < 1 \quad \Rightarrow \quad \xi = \xi' = \xi''.
\]

Let \(\Omega\) be a domain of \(\mathbb{R}^N\). A multi-valued mapping \(K : \Omega \to 2^{\mathbb{R}^M}\) is said to be measurable if there exists a sequence of measurable single-valued functions \(\{k_m : \Omega \to \mathbb{R}^M\}\) such that \(\bigcup_{m \in \mathbb{N}} k_m(x)\) is dense in \(K(x)\) for a.a. \(x \in \Omega\). \(\square\)

---

106 See e.g. Barbu and Precupanu [49, p. 137], Rockafellar [388], [389, p. 396], [390].
107 See Section 5.6 for the definition of the latter notion.
108 See e.g. Castaing and Valadier [119, Section III.2], Ioffe and Tihomirov [278, Section 8.1].
THEOREM 5.4.1. \( K : \Omega \rightarrow 2^{\mathbb{R}^M} \) be measurable, and \( K(x) \) be closed and convex for a.a. \( x \in \Omega \). If

\[
\begin{align*}
  v_n &\rightarrow v \quad \text{weakly in } L^1(\Omega)^M, \\
v_n(x) &\in K(x) \quad \text{for a.a. } x \in \Omega, \quad \forall n, \\
v(x) &\text{ is an extremal point of } K(x) \text{ for a.a. } x \in \Omega,
\end{align*}
\]

then

\[
v_n \rightarrow v \quad \text{strongly in } L^1(\Omega)^M.
\]

By and large, the rational behind this result is that if a sequence of function converges weakly in \( L^1(\Omega)^M \) without oscillating around the limit value, then it also converges strongly. For instance, this applies to any \( L^1 \)-weakly vanishing sequence of nonnegative scalar functions. (On the other hand, it fails in \( L^p \)-spaces with \( p > 1 \).)

COROLLARY 5.4.2. If \( \varphi : \mathbb{R}^M \rightarrow \tilde{\mathbb{R}} \) is strictly convex, lower semicontinuous, and

\[
\begin{align*}
  u_n &\rightarrow u \quad \text{weakly in } L^1(\Omega)^M, \\
  \int_\Omega \varphi(u_n) \, dx &\rightarrow \int_\Omega \varphi(u) \, dx \quad (< +\infty),
\end{align*}
\]

then

\[
\begin{align*}
  u_n &\rightarrow u \quad \text{strongly in } L^1(\Omega)^M, \\
  \varphi(u_n) &\rightarrow \varphi(u) \quad \text{strongly in } L^1(\Omega).
\end{align*}
\]

OUTLINE OF THE PROOF. By (5.4.7) and by the convexity of \( \varphi \), it is not difficult to see that \( \varphi(u_n) \rightarrow \varphi(u) \) weakly in \( L^1(\Omega)^M \). Note that \((u, \varphi(u))\) is an extremal point of \( K := \text{epi}(\varphi) \subset \mathbb{R}^{M+1} \) a.e. in \( \Omega \). It then suffices to apply Theorem 5.4.1 taking \( v_n = (u_n, \varphi(u_n)) \). □

PROPOSITION 5.4.3. In Corollary 5.4.2 the assumption (5.4.7) holds whenever

\[
\begin{align*}
  w_n := \varphi(u_n) &\rightarrow w := \varphi(u) \quad \text{weakly in } L^1(\Omega)^M, \\
  \int_\Omega u_n \cdot w_n \, dx &\rightarrow \int_\Omega u \cdot w \, dx.
\end{align*}
\]

(\( \varphi \) is single-valued, because of the hypothesis of strict convexity.)

PROOF. The Fenchel property yields

\[
\begin{align*}
  \int_\Omega \varphi(u_n) \, dx + \int_\Omega \varphi^*(w_n) \, dx &= \int_\Omega u_n \cdot w_n \, dx, \\
  \int_\Omega \varphi(u) \, dx + \int_\Omega \varphi^*(w) \, dx &= \int_\Omega u \cdot w \, dx.
\end{align*}
\]

\[\text{See Visintin [445], [453, Section X.1].}\]
By the lower semicontinuity of these integral functionals and by (5.4.11), we then infer (5.4.7) (and \( \int_{\Omega} \varphi^*(w_n) \, dx \rightarrow \int_{\Omega} \varphi^*(w) \, dx \)). □

Dually, if the conjugate function \( \varphi^* \) is strictly convex, then (5.4.6), (5.4.10) and (5.4.11) entail the strong \( L^1 \)-convergence of the sequence \( \{w_n\} \).

### 5.5. Maximal monotone operators

In this section we briefly illustrate the notion of maximal monotone operator in a real Banach space \( B \).\(^{110}\)

An operator \( A : B \rightarrow 2^{B^*} \) is said monotone if, setting \( \text{graph}(A) := \{(u, u^*) : u^* \in A(u)\} \),

\[
\langle u^* - v^*, u - v \rangle \geq 0 \quad \forall (u, u^*), (v, v^*) \in \text{graph}(A).
\] (5.5.1)

This operator is said maximal monotone if it is monotone and its graph is not properly included in that of any other monotone operator \( B \rightarrow 2^{B^*} \). It is said cyclically monotone if

\[
\sum_{i=1}^{m} \langle u_i^*, u_i - u_{i-1} \rangle \geq 0 \quad (\text{setting } u_0 = u_m).
\] (5.5.2)

(For \( m = 2 \) the inequality (5.5.1) is obviously retrieved.) By the Hausdorff maximal chain theorem (a consequence of the Zorn lemma), it is easy to see that any monotone operator \( A : B \rightarrow 2^{B^*} \) can be extended to a maximal monotone operator. The inverse of a monotone operator is defined as the operator that has the inverse graph: \( u \in A^{-1}(u^*) \) if and only if \( u^* \in A(u) \). If \( A \) is maximal monotone, then the inverse operator \( A^{-1} \) is also maximal monotone.

**Theorem 5.5.1.** Let \( B \) be reflexive and \( A : B \rightarrow 2^{B^*} \) be maximal monotone. If

\[
\frac{\langle w, v \rangle}{\|v\|} \rightarrow +\infty \quad \text{as} \quad w \in A(v), \|v\| \rightarrow \infty,
\] (5.5.3)

then for any \( f \in B^* \) there exists \( u \in B \) such that \( A(u) \ni f \).

The latter theorem extends the next result to Banach spaces.

**Theorem 5.5.2 (Minty and Browder).** Let \( H \) be a real Hilbert space. A monotone operator \( A : H \rightarrow 2^H \) is maximal monotone if and only if the mapping \( A + \lambda I \) is surjective for some (equivalently, for any) \( \lambda > 0 \).

**Theorem 5.5.3 (Rockafellar).** For any \( F \in \Gamma_0(B) \) the operator \( \partial F \) is maximal monotone.

An operator \( A : B \rightarrow 2^{B^*} \) is maximal monotone and cyclically monotone if and only if \( A = \partial F \) for some proper lower semicontinuous convex function \( F : B \rightarrow \tilde{\mathbb{R}} \).

\(^{110}\) See e.g. the monographs quoted in the item (IX) of the Bibliographical Note in Section 6.
**Proposition 5.5.4.** Let $\alpha_1, \alpha_2 : B \to 2^{B^*}$ be two maximal monotone operators, $\{u_{1n}\}$, $\{u_{2n}\}$ be two sequences in $B$, and $\{u_n^*\}$ be a sequence in $B^*$. If

\begin{align}
\quad u_n^* &\in \alpha_1(u_{1n}) \cap \alpha_2(u_{2n}) \quad \forall n \in \mathbb{N}, \\
u_{1n} &\to u_i \text{ weakly in } B, \text{ for } i = 1, 2, \\
u_n^* &\to u^* \text{ weakly star in } B^*, \\
\lim_{n \to \infty} \langle u_n^*, u_{1n} + u_{2n} \rangle &\leq \langle u^*, u_1 + u_2 \rangle,
\end{align}

then $u^* \in \alpha_1(u_1) \cap \alpha_2(u_2)$.

**Proof.** By (5.5.4),

\[ \langle u_n^* - v_i^*, u_{1n} - v_i \rangle \geq 0 \quad \forall (v_i, v_i^*) \in \text{graph}(\alpha_i), \text{ for } i = 1, 2; \]

by adding these two inequalities we get

\[ \langle u_n^*, u_{1n} + u_{2n} \rangle - \sum_{i=1,2} \left( \langle u_n^*, v_i \rangle + \langle v_i^*, u_{1n} - v_i \rangle \right) \geq 0. \]

By passing to the inferior limit as $n \to \infty$, the hypotheses (5.5.5)–(5.5.7) yield

\[ \langle u^*, u_1 + u_2 \rangle - \sum_{i=1,2} \left( \langle u^*, v_i \rangle + \langle v_i^*, u_i - v_i \rangle \right) \geq 0. \]

By selecting either $v_1 = u_1$ or $v_2 = u_2$, we then obtain

\[ \langle u^* - v_i^*, u_i - v_i \rangle \geq 0 \quad \forall (v_i, v_i^*) \in \text{graph}(\alpha_i), \text{ for } i = 1, 2, \]

namely $u^* \in \alpha_1(u_1) \cap \alpha_2(u_2)$. \hfill \square

**Remarks.**

(i) The latter statement also admits a dual formulation. In fact, denoting by $\beta_i$ the inverse of the operator $\alpha_i (i = 1, 2)$, the hypothesis (5.5.4) and the thesis respectively also read

\begin{align}
\quad u_{1n} &\in \beta_1(u_n^*), \quad u_{2n} \in \beta_2(u_n^*) \quad \forall n \in \mathbb{N}, \\
u_1 &\in \beta_1(u^*), \quad u_2 \in \beta_2(u^*). \label{5.5.8}
\end{align}

(ii) Although we stated Proposition 5.5.4 for two operators, the further extension to an either finite or even countable family of maximal monotone operators is straightforward. On the other hand for $\alpha_1 = \alpha_2$ we get the next statement, that is often applied in the analysis of nonlinear problems.

**Corollary 5.5.5.** Let $\alpha : B \to 2^{B^*}$ be a maximal monotone operator and $\{(u_n, u_n^*)\}$ be a sequence in $B \times B^*$. If

\begin{align}
\quad u_n^* &\in \alpha(u_n) \quad \forall n \in \mathbb{N}, \label{5.5.10} \\
u_n &\to u \text{ weakly in } B, \label{5.5.11}
\end{align}

then $u \in \alpha(u)$. \hfill \square
\[ u_n^* \to u^* \quad \text{weakly star in} \quad B^*, \quad (5.5.12) \]
\[ \liminf_{n \to \infty} \langle u_n^*, u_n \rangle \leq \langle u^*, u \rangle, \quad (5.5.13) \]
then \( u^* \in \alpha(u) \).

5.6. \textbf{M-accretive operators and semigroups}

In this section we illustrate nonlinear semigroups of contractions in Banach spaces.\textsuperscript{111}

After dealing with (multi-valued) operators \( B \to B^* \), next we consider a different notion of monotonicity for operators that map \( B \) to itself. Of course this distinction makes sense only if \( B \) is not a Hilbert space. An operator \( A : B \to 2^B \) is said \textit{accretive} if
\[ \forall u_1, u_2 \in \text{Dom}(A), \quad \forall \lambda > 0, \quad \| u_1 - u_2 \| \leq \| u_1 - u_2 + \lambda (v_1 - v_2) \|. \quad (5.6.1) \]

\( A \) is said \textit{m-accretive} if it is accretive and \( I + \lambda A \) is surjective for some \( \lambda > 0 \) (equivalently, for any \( \lambda > 0 \)). By the Minty and Browder Theorem 5.5.2, an operator that acts on a Hilbert space is m-accretive if and only if it is maximal monotone.

\textbf{Cauchy problem.} Let \( A : B \to 2^B, \ T > 0, \ f \in L^1(0, T; B), \ u^0 \in \text{Dom}(A) \) (the strong closure of the domain of \( A \)), and consider the equation
\[ \frac{du}{dt} + A(u) \ni f \quad \text{in} \quad ]0, T[. \quad (5.6.2) \]

A function \( u : ]0, T[ \to B \) is called a \textit{strong solution} of this equation if
(i) \( u \) is absolutely continuous on any interval \([a, b] \subset ]0, T[\), and strongly differentiable a.e. in \([0, T]\).
(ii) \( u \in \text{Dom}(A) \) a.e. in \([0, T]\), and
(iii) Eq. (5.6.2) is fulfilled a.e. in \([0, T]\).

On the other hand, \( u : ]0, T[ \to B \) is called a \textit{mild solution} of (5.6.2) if there exists a sequence \( \{(u_n, f_n)\} \) such that \( u_n \) is a strong solution of the same equation with \( f_n \) in place of \( f \) for any \( n \), and
\[ u_n \to u \quad \text{in} \quad B, \quad \text{locally uniformly in} \quad ]0, T[, \]
\[ f_n \to f \quad \text{strongly in} \quad L^1(0, T; B). \quad (5.6.3) \]

These notions are easily extended to the Cauchy and periodic problems associated with (5.6.2):

(CP) \[
\begin{aligned}
\frac{du}{dt} + A(u) &\ni f \quad \text{in} \quad ]0, T[,

u(0) &= u^0,
\end{aligned}
\]

(PP) \[
\begin{aligned}
\frac{du}{dt} + A(u) &\ni f \quad \text{in} \quad ]0, T[,

u(0) &= u(T).
\end{aligned}
\]

Here \( u \) is also assumed to be continuous at \( t = 0 \) for (CP), at \( t = 0, T \) for (PP).

\textsuperscript{111} See e.g. see e.g. the monographs quoted in the item (X) of the Bibliographical Note in Section 6.
THEOREM 5.6.1. Let $B$ be a Banach space and $A : B \to 2^B$ be an $m$-accretive operator. Then:

(i) If $f \in L^1(0, T; B)$ and $u^0 \in \overline{\text{Dom}}(A)$, then the Cauchy problem (CP) has a mild solution.

(ii) If $f_i \in L^1(0, T; B)$, $u^0_i \in \overline{\text{Dom}}(A)$, and $u_i$ is a corresponding mild solution of (CP) for $i = 1, 2$, then

$$
\|u_1(t) - u_2(t)\| \\
\leq \|u^0_1 - u^0_2\| + \int_0^t \|f_1(s) - f_2(s)\| \, ds \quad \forall t \in [0, T].
$$

(The mild solution is thus unique.)

(iii) If the mild solution of (CP) is absolutely continuous on any interval $[a, b] \subset ]0, T[\setminus\{0, T\}$ and is strongly differentiable a.e. in $]0, T[$, then it is a strong solution.

(iv) If $f : ]0, T[ \to B$ has bounded variation and $u^0 \in \text{Dom}(A)$, then the mild solution of (CP) is Lipschitz-continuous in $]0, T[$.

(v) If the operator $A - aI$ is accretive for some constant $a > 0$, then the periodic problem (PP) has one and only one mild solution.

A Banach space $B$ is said to have the Radon–Nikodým property if any Lipschitz-continuous mapping $[0, 1] \to B$ is strongly differentiable a.e. in $]0, T[$. This holds if either $B$ is reflexive or it is separable and has a pre-dual. For instance, this applies to all reflexive Banach spaces and to $l^1$, but neither to $L^1(\Omega)$ nor to $L^\infty(\Omega)$.

By parts (iii) and (iv) of Theorem 5.6.1, the mild solution of the Cauchy problem (CP) is a strong solution whenever $B$ has the Radon–Nikodým property, $f : ]0, T[ \to B$ has bounded variation and $u^0 \in \text{Dom}(A)$.

Let us now assume that $f \equiv 0$. For any $u^0 \in \overline{\text{Dom}}(A)$ let $u$ be the mild solution of (CP), and set $S(t)u^0 := u(t)$ for any $t \geq 0$. The mapping $t \mapsto S(t)$ is then a continuous semigroup of contractions, for it is a continuous semigroup and

$$
\|S(t)u^0_1 - S(t)u^0_2\| \leq \|u^0_1 - u^0_2\| \quad \forall t \geq 0, \forall u^0_1, u^0_2 \in \overline{\text{Dom}}(A).
$$

(5.6.7)

For any Lipschitz-continuous operator $F : B \to B$, the above results are easily extended to the operator $\tilde{A} := A + F$. In this case, denoting by $\omega$ the Lipschitz constant of $F$, $t \mapsto S(t)$ is a continuous semigroup of $\omega$-contractions, for (5.6.7) is replaced by

$$
\|S(t)u^0_1 - S(t)u^0_2\| \leq e^{\omega t} \|u^0_1 - u^0_2\| \quad \forall t \geq 0, \forall u^0_1, u^0_2 \in \overline{\text{Dom}}(A).
$$

(5.6.8)

$T$-accretiveness. A Banach space $B$ is called a Banach lattice if it is an ordered set such that any finite nonempty subset admits infimum and supremum, and, setting $|u| := \sup\{u, -u\}$ and $u \leq v$ if $u = \inf\{u, v\}$, it satisfies the following conditions, for any $u, v, w \in B$:

(i) if $u \leq v$ then $u + w \leq v + w$,

(ii) if $u \leq v$ and $\alpha > 0$, then $\alpha u \leq \alpha v$,

(iii) if $u \leq v$ then $-v \leq -u$.

112 See, e.g., Bénilan [54], Diestel and Uhl [181, Chapter III], Kufner, John and Fučík [298, Section 2.22.5].
(iv) if $|u| \leq |v|$ then $\|u\| \leq \|v\|$.

For any $u \in B$, let us set

$$u^+ := \sup\{u, 0\} \quad \text{and} \quad u^- := \sup\{-u, 0\}.$$

An operator $A : B \to 2^B$ is then said to be $T$-accretive if

$$\forall u_i \in \text{Dom}(A), \forall v_i \in A(u_i) \ (i = 1, 2), \ \forall \lambda > 0,$$

$$\| (u_1 - u_2)^+ \| \leq \| [u_1 - u_2 + \lambda (v_1 - v_2)]^+ \|. \quad (5.6.9)$$

Any $T$-accretive operator in $B$ is also accretive whenever

$$\|u^+\| \leq \|v^+\|, \quad \|u^-\| \leq \|v^-\| \quad \Rightarrow \quad \|u\| \leq \|v\| \ \forall u, v \in B. \quad (5.6.10)$$

**Theorem 5.6.2.** If $B$ is a Banach lattice and $A$ is $m$- and $T$-accretive, then the mild solution of the Cauchy problem (CP) depends monotonically on the data. That is, if $u_i$ is the mild solution corresponding to $u_0^i, f_i \ (i = 1, 2)$ and $u_1^0 \leq u_2^0, f_1 \leq f_2$, then $u_1 \leq u_2$.

### 5.7. Perimeter and curvature

In this section we state a result about sets of finite perimeter in the sense of Caccioppoli.

For any measurable function $v : \Omega \to \mathbb{R}$, let us first define the total variation functional

$$\int_{\Omega} |\nabla v| := \sup \left\{ \int_{\Omega} v \nabla \cdot \vec{\eta} : \vec{\eta} \in C_0^1(\Omega)^N, \ |\vec{\eta}| \leq 1 \ \text{in} \ \Omega \right\}. \quad (5.7.1)$$

The domain of this operator in $L^1(\Omega)$ is thus the space $BV(\Omega)$. Let us also set

$$P(v) := \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla v| \ (\leq +\infty) \quad & \text{if } |v| = 1 \ \text{a.e. in } \Omega, \\ +\infty \quad & \text{otherwise.} \end{cases} \quad (5.7.2)$$

If $v \in \text{Dom}(P)$ then $P(v)$ is the perimeter in $\Omega$ in the sense of Caccioppoli of the set $\Omega^+ = \{x \in \Omega : v(x) = 1\}$. Whenever $\Omega^+$ is of Lipschitz class, this perimeter coincides with the bidimensional Hausdorff measure of $\partial \Omega^+$.

Let us now fix any $g \in L^1(\Omega)$, any constants $a > 0$ and $b$, and set

$$\Phi(v) := \begin{cases} a \int_{\Omega} |\nabla v| + b \int_{\Gamma} \gamma_0 v \, d\sigma + \int_{\Omega} g \, v \, dx \quad & \forall v \in \text{Dom}(P), \\ +\infty \quad & \forall v \in L^1(\Omega) \setminus \text{Dom}(P). \end{cases} \quad (5.7.3)$$

This operator is well-defined, for the trace operator $\gamma_0$ maps $BV(\Omega)$ to $L^1(\Gamma)$ (and is continuous).

**Proposition 5.7.1.** Under the above assumptions, the functional $\Phi$ is lower semicontinuous with respect to the strong topology of $L^1(\Omega)$ if and only if $|b| \leq a$. In that case $\Phi$ has an (in general nonunique) absolute minimizer.

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113 See e.g. the monographs quoted in the item (XV) of the Bibliographical Note in Section 6.

114 See e.g. Massari and Pepe [322].
The existence of a minimizer of $\Phi$ then follows by the direct method of the calculus of variations.\textsuperscript{115}

**Theorem 5.7.2** (Gibbs–Thomson law and contact angle condition). Let $|b| \leq a$, $g \in W^{1,p}(\Omega)$ with $p > 3$, $u \in L^1(\Omega)$ and

$$\liminf_{v \to u} \frac{\Phi(v) - \Phi(u)}{\|v - u\|_{L^1(\Omega)}} \geq 0 \quad \text{as } v \to u \text{ in } L^1(\Omega).$$

(5.7.4)

Then:

(i) The boundary $S$ in $\Omega$ of the set $\Omega^+ := \{ x \in \Omega : u(x) = 1 \}$ is a surface of class $C^{1,(p-N)/2p}$.

(ii) Denoting by $\bar{n}$ the unit normal vector to the surface $S$ oriented towards $\Omega^+$, $\kappa := \frac{1}{2} \nabla S \cdot \bar{n} \in L^p(S)$.\textsuperscript{116} Moreover, equipping $S$ with the two-dimensional Hausdorff measure and denoting by $\gamma_0$ the trace operator $W^{1,p}(\Omega) \to W^{1-1/p,p}(S)$,

$$\kappa = \gamma_0 g \quad \text{a.e. on } S.$$  

(5.7.5)

(iii) If $\Gamma$ is of class $C^1$, then, denoting by $\omega$ the angle between $\bar{n}$ and the outward normal vector to $\Gamma$, and equipping $S \cap \Gamma$ with the one-dimensional Hausdorff measure,

$$\cos \omega = b/a \quad \text{a.e. on } S \cap \Gamma.$$  

(5.7.6)

Part (i) follows from a classic result of Almgren [14]. Parts (ii) and (iii) may be proved by representing $S$ locally in Cartesian form, and then letting the first variation of $\Phi$ vanish for any local Cartesian perturbation of the interface.\textsuperscript{117}

### 5.8. $\Gamma$-convergence

In this section we state De Giorgi’s notion of $\Gamma$-convergence, and some basic results of this theory.\textsuperscript{118}

Let $(X, d)$ be a metric space, $f_n$ ($n \in \mathbb{N}$) and $f$ be functions $X \to \mathbb{R} \cup \{\pm \infty\}$. If for some $u \in X$

(i) for any sequence $\{u_n\}$ in $X$,

if $u_n \to u$ then $\liminf_{n \to \infty} f_n(u_n) \geq f(u)$, \hspace{1cm} (5.8.1)

(ii) there exists a sequence $\{u_n\}$ in $X$ such that $u_n \to u$ and $\limsup_{n \to \infty} f_n(u_n) \leq f(u)$, \hspace{1cm} (5.8.2)

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\textsuperscript{115} See e.g. Braides and Defranceschi [82], Buttazzo, Giaquinta and Hildebrandt [94], Carbone and De Arcangelis [117], Dal Maso [155], Evans and Gariepy [201], Giusti [245].

\textsuperscript{116} $\kappa$ is the tangential divergence of $\bar{n}/2$ over $S$, in the sense of $H^{-1}(S)$, say. Thus $\kappa$ equals the mean curvature of $S$.

\textsuperscript{117} See e.g. Visintin [453, Section VI.4].

\textsuperscript{118} See e.g. the pioneering papers De Giorgi [168], De Giorgi and Franzoni [169], and the monographs quoted in the item (XIV) of the Bibliographical Note in Section 6.
then we say that $f_n \Gamma$-converges to $f$ (in $(X, d)$) at $u$, and write $f_n(u) \xrightarrow{\Gamma} f(u)$. If this occurs for any $u \in X$, then we say that $f_n \Gamma$-converges to $f$, and write $f_n \xrightarrow{\Gamma} f$. The function $f$ is then lower semicontinuous.

**Proposition 5.8.1 (Compactness).** Let $(X, d)$ be a separable metric space and $\{f_n\}$ be a sequence of functions $X \to \mathbb{R}$. A subsequence of $\{f_n\}$ then $\Gamma$-converges to some function $f : X \to \mathbb{R}$.

**Proposition 5.8.2 (Minimization).** Let $(X, d)$ be a metric space, and $\{f_n\}$ is a sequence of functions $X \to \mathbb{R}$ such that $f_n \Gamma$-converges to $f$. If a sequence $\{u_n\} \subset X$ and $u \in X$ are such that

$$f_n(u_n) \leq \inf f_n + \frac{1}{n} \quad \forall n, \quad u_n \to u \quad \text{in} \ X, \quad (5.8.3)$$

then $f_n(u_n) \to f(u)$ and $f(u) = \inf f$. (Thus $\inf f_n \to \inf f$.)

**Proof.** By (5.8.1), $f(u) \leq \liminf_{n \to \infty} f_n(u_n)$. Moreover, for any $v \in X$, there exists a sequence $\{v_n\} \subset X$ such that $v_n \to v$ in $X$ and $f_n(v_n) \to f(v)$. As by (5.8.3), $f_n(u_n) \leq f(v_n) + \frac{1}{n}$ for any $n$, we get

$$f(u) \leq \liminf_{n \to \infty} f_n(u_n) \leq \lim_{n \to \infty} f_n(v_n) = f(v) \quad \forall v \in X.$$ 

Thus $f(u) = \inf f$. If $\{\tilde{u}_n\}$ is the sequence prescribed by (5.8.2), then

$$f(u) \geq \limsup_{n \to \infty} f_n(\tilde{u}_n) \geq \limsup_{n \to \infty} (\inf f_n) \geq (\text{by (5.8.3)}) \limsup_{n \to \infty} f_n(u_n).$$

Thus $f_n(u_n) \to f(u)$.

The two latter propositions entail the next statement, that shows that the notion $\Gamma$-convergence is especially appropriate for the study of the limit behaviour of minimization problems.

**Corollary 5.8.3.** Let $(X, d)$ be a separable metric space, $\{f_n\}$ be a sequence of functions $X \to \mathbb{R}$, and $\{u_n\}$ be a compact sequence of $X$ such that $f_n(u_n) = \inf f_n$ for any $n$. Then there exist $f$ and $u$ such that, as $n \to \infty$ along a suitable sequence (not relabelled),

$$f_n \xrightarrow{\Gamma} f, \quad u_n \to u \quad \text{in} \ X, \quad f_n(u_n) \to f(u) = \inf f. \quad (5.8.4)$$

The next statement has been applied to several models of multi-phase systems. It may be noticed that in this case the intersection between the domain of the sequence of functionals, $H^1(\Omega) \cap L^4(\Omega)$, and that of the $\Gamma$-limit, $\{v \in BV(\Omega) : |v| = 1 \text{ a.e. in } \Omega\}$, is reduced to the two constant functions $v \equiv \pm 1$.

\textsuperscript{119} I.e., a space equipped with a countable basis of open sets, in the topology induced by the metric.
THEOREM 5.8.4. Let $\Omega$ be a Lipschitz domain of $\mathbb{R}^N$, and for any $v \in L^1(\Omega)$ set (cf. (5.7.1))

\[
f_n(v) := \begin{cases} 
\int_{\Omega} \left( \frac{1}{n} |\nabla v|^2 + n(v^2 - 1)^2 \right) \, dx & \text{if } v \in H^1(\Omega) \cap L^4(\Omega), \\
+\infty & \text{otherwise},
\end{cases}
\]

(5.8.5)

\[
f(v) := \begin{cases} 
\frac{4}{3} \int_{\Omega} |\nabla v| & \text{if } |v| = 1 \text{ a.e. in } \Omega, \\
+\infty & \text{otherwise}.
\end{cases}
\]

(5.8.6)

The sequence $f_n$ then $\Gamma$-converges to $f$ in $L^1(\Omega)$. (Note that $f = \frac{3}{8} P$, cf. (5.7.2)).

Half of this proof is not difficult, and allows us to justify the occurrence of the constant $4/3$ in (5.8.6). Setting $a(y) := 2y^3/3 - 2y$ for any $y \in \mathbb{R}$, by the obvious inequality $b^2 + c^2 \geq 2bc$, for any sequence $\{u_n\}$ in $H^1(\Omega)$ we have

\[
f_n(u_n) = \int_{\Omega} \left( \frac{1}{n} |\nabla u_n|^2 + n(u_n^2 - 1)^2 \right) \, dx \\
\geq 2 \int_{\Omega} |\nabla u_n||u_n^2 - 1| \, dx = \int_{\Omega} |\nabla a(u_n)| \, dx.
\]

Notice also that, if $u_n \to u$ in $L^1(\Omega)$ and $\liminf_{n \to \infty} f_n(u_n) < +\infty$, then $|u| = 1$ a.e. in $\Omega$. Hence

\[
\liminf_{n \to \infty} f_n(u_n) \geq \begin{cases} 
\int_{\Omega} |\nabla a(u)| & \text{if } |u| = 1 \text{ a.e. in } \Omega, \\
+\infty & \text{otherwise}.
\end{cases}
\]

Moreover, as $|a(\pm 1)| = 4/3$,

\[
\int_{\Omega} |\nabla a(u)| = |a(\pm 1)| \int_{\Omega} |\nabla u| = \frac{4}{3} \int_{\Omega} |\nabla u| \quad \text{if } |u| = 1 \text{ a.e. in } \Omega.
\]

We thus checked (5.8.1). The construction of a recovery sequence fulfilling (5.8.2) is less obvious and is here omitted.

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\[\text{See Modica and Mortola [337].}\]
6. Bibliography

*Bibliographical note.* A large mathematical literature has been devoted to the analysis of the Stefan problem and of its generalizations. Below we loosely gather a number of monographs and proceedings under few headings, mainly for convenience of reference within this survey. These references concern Stefan-type and free boundary problems, as well as some topics of the analysis of nonlinear PDEs and calculus of variations.\(^ {121}\)

(I) Monographs on the Stefan problem and related models:

Alexiades and Solomon [8], Avdonin [43], Brokate and Sprekels [91], Datzef [166], Elliott and Ockendon [193], Fasano [206], Gupta [249], Gurtin [256], Hill [267], Meirmanov [331], Romano [404], Rubinstein [410], Visintin [453], Yamaguchi and Nogi [474].

(II) Surveys on the Stefan problem and related models:

Andreucci, Herrero and Velázquez [25], Damlamian [160], Danilyuk [164], Fasano [205], Fasano and Primicerio [213], Magenes [318,320], Niezgódka [355], Oleinik, Primicerio and Radkevich [365], Primicerio [377,378], Rodrigues [395,397], Rubinstein [409], Tarzia [434], Visintin [455,457].

(III) Monographs on free boundary problems:

Baiocchi and Capelo [45], Caffarelli and Salsa [100], Chipot [129], Crank [149], Diaz [172], Duvaut and Lions [186], Friedman [234], Monakhov [338], Kinderlehrer and Stampacchia [292], Monakhov [338], Naumann [348], Nečas and Hlaváček [350], Pukhnachev [381], Radkevich and Melikulov [384], Rubinstein and Martuzans [412], Steinbach [431].

(IV) Collections of references on free boundary problems:

Cannon [112], Cryer [153], Tarzia [435,436], Wilson, Solomon and Trent [467].

(V) Proceedings and collective books on free boundary problems and applications:

Albrecht, Collatz and Hoffmann [6], Antontsev, Diaz and Shmarev [28], Antontsev, Hoffmann and Khludnev [29], Argoul, Frémond and Nguyen [30], Athanasopoulos, Makrakis and Rodrigues [37], Bossavit, Damlamian and Frémond [80], Brown and Davis [93], Chadam and Rasmussen [121–123], Colli, Kenmochi and Sprekels [138], Colli, Verdi and Visintin [144], Diaz, Herrero, Liñán and Vázquez [173], Fasano and Primicerio [212], Figueiredo, Rodrigues and Santos [218], Friedman and Spruck [237], Gurtin and McFadden [260], Hoffmann and Sprekels [271,272], Kenmochi [285], Magenes [319], Miranville [332], Miranville, Yin and Showalter [333], Neittanmäki [354], Niezgódka and Pawlow [356], Niezgódka and Strzelecki [357], Ockendon and Hodgkins [362], Rodrigues [396], Wilson, Solomon and Boggs [466], Wrobel and Brebbia [470,471], Wrobel, Brebbia and Sarler [472].

(VI) Monographs on physical and engineering aspects of phase transitions:

Abraham [2], Abeyaratne and Knowles [1], Brice [90], Chalmers [124], Christian [130], Doremus [183], Flemings [222], Kassner [284], Kurz and Fisher [299], Pamplin [366], Papon, Leblond and Meijer [369], Skripov [424], Turnbull [441], Ubbelohde [442], Woodruff [468].

\(^ {121}\) An exercise like this exposes the author to the risk of omissions, even if he has no aim of completeness. Facing so large a literature, more than a risk this is a certainty.
(VII) Monographs on nonequilibrium thermodynamics:
Astarita [31], Callen [111], De Groot [170], De Groot and Mazur [171], Frémond [224], Glansdorff and Prigogine [246], Kondepudi and Prigogine [296], Lavenda [310], Müller [344], Müller and Weiss [345], Prigogine [376], Woods [469].

(VIII) Some monographs on convex analysis:
Attouch [39], Aubin [41], Aubin and Ekeland [42], Barbu and Precupanu [49], Borwein and Lewis [74], Castaing and Valadier [119], Ekeland and Temam [188], Hiriart-Urruty and Lemarechal [269,270], Hörmander [276], Ioffe and Tihomirov [278], Kusraev and Kutateladze [300], Moreau [339], Rockafellar [389,391], Rockafellar and Wets [392], van Tiel [440], Willem [463].

(IX) Some monographs on maximal monotone operators:
Barbu [46,47], Barbu and Precupanu [49], Brezis [86], Browder [92], Lions [311], Pascalei and Sburlan [370], Showalter [421].

(X) Some monographs on nonlinear semigroups of contractions:
Barbu [46,47], Bénilan [54], Bénilan, Crandall and Pazy [58], Brezis [86], Da Prato [165], Miyadera [334], Morosanu [341], Pavel [371].

(XI) Some monographs on variational inequalities and applications to PDEs:
Baiocchi and Capelo [45], Barbu and Precupanu [49], Brezis [84–86], Duvaut and Lions [186], Ekeland and Temam [188], Friedman [234], Gajewski, Gröger and Zacharias [240], Kinderlehrer and Stampacchia [292], Kluge [293], Lions [311], Naumann [348], Nečas and Hlaváček [350], Panagiotopoulos [367,368], Pascali and Sburlan [370], Rodrigues [394], Showalter [421], Vainberg [443], Zeidler [475].

(XII) Some monographs on Sobolev spaces:
Adams [3], Attouch, Buttazzo and Michaille [40], Baiocchi and Capelo [45], Brezis [87], Dautray and Lions [167], Evans [200], Evans and Gariepy [201], Kufner, John and Fučík [298], Lions and Magenes [312], Maz’ja [324], Nečas [349], Tartar [433], Ziemer [478].

(XIII) Some monographs dealing with quasilinear parabolic equations in Sobolev spaces:
Barbu [46,47], Brezis [85,86], Friedman [230], Ladyženskaja, Solonnikov and Ural’ceva [305], Lions [311], Naumann [348], Roubíček [405], Showalter [421], Zeidler [475].

(XIV) Some monographs on $\Gamma$-convergence:
Attouch [39], Braides [81], Braides and Defranceschi [82], Carbone and De Arcangelis [117], Dal Maso [155].

(XV) Some monographs on sets of finite perimeter:
Almgren [14], Colombini, De Giorgi and Piccinini [147], Evans and Gariepy [201], Federer [216], Giusti [244], Morgan [340], Simon [423].

References


Introduction to Stefan-type problems


Introduction to Stefan-type problems

Introduction to Stefan-type problems


Introduction to Stefan-type problems


Introduction to Stefan-type problems


