Boundary-Value Problems for P.D.E.s

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1 P.D.E.s and Boundary-Value Problems

For linear P.D.E.s a classification by *types* is well-established, at least for the second order equations that are most frequently encountered in applications. At first, one distinguishes the *principal part* of the differential operator, which consists of the terms that contain the *highest order* derivatives. One then speaks of elliptic, parabolic, hyperbolic operators (and equations), according to the form of the principal part.

As it is known to the reader, typical examples of elliptic equations are the Poisson equation $-\Delta u = f$ and the equation $\lambda u - \Delta u = f$; here $\Delta := \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$, $\lambda \in \mathbf{R}$, and f is a prescribed function. These equations represent several stationary phenomena. The heat equation $\frac{\partial u}{\partial t} - \Delta u = f$ is parabolic equation, whereas the wave equation $\frac{\partial^2 u}{\partial t^2} - \Delta u = f$ is hyperbolic.

This classification reflects the fact that several basic features of the qualitative behaviour of solutions, including the well-posedness of corresponding boundary- and/or initial-value problems, are determined by the form of the principal part, and solutions are stable with respect to large classes of perturbations which do not involve the principal part.

In many cases the classes of boundary conditions that make a P.D.E. well-posed are also determined by the type of the equation. This, however, does not apply to parabolic equations, which indeed represent a sort of degenerate case, as the principal part corresponds to a singolar matrix. For instance, the principal part of the heat operator $a\partial/\partial t - \Delta$ (with $a \in \mathbf{R}$) is $-\Delta$; but, e.g., the well-posedness of the corresponding initial-value problem depends on the sign of a. [1]

In this section we discuss some examples of boundary-value problems for linear P.D.E.s of second order of elliptic, parabolic and hyperbolic type.

Well Posedness. Following Hadamard, we say that a problem is well-posed whenever for any set of admissible data it has one and only one solution, and this depends continuously on the data. Obviously, this requires that data and solutions range in appropriate topological spaces. Here we briefly discuss some examples.

(i) For $N = 2$, let us consider the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 & \text{in } \Omega := \mathbf{R} \times]0, +\infty[, \\ u(x,0) = g(x), \quad \dfrac{\partial u}{\partial y}(x,0) = 0 & \forall x \in \mathbf{R}, \end{cases} \tag{1.1}$$

with g continuous and bounded. If y is interpreted as a time variable, the two latter conditions may be regarded as initial conditions, and are also named *Cauchy conditions.* Accordingly, one speaks of a *Cauchy problem* for the Laplace equation. For any $n \in \mathbf{N}$, if $g_n(x) := \frac{1}{n}\sin(nx)$ for any $x \in \mathbf{R}$, then by separation of variables one easily finds the solution

$$u_n(x,y) = \frac{1}{n}\sin(nx)\cosh(ny) \qquad \forall (x,y) \in \mathbf{R} \times \mathbf{R}^+. \tag{1.2}$$

[1] To this purpose one might then consider an alternative notion, that besides the principal includes any derivative D^α such that no term of the form D^β with $\beta > \alpha$ occurs. With this convention the heat operator would coincide with its principal part.

As $n \to \infty$, $g_n \to 0$ uniformly in $\mathbf{R}$, and of course $u \equiv 0$ solves the problem for $g \equiv 0$. But, as $n \to \infty$, u_n does not vanish uniformly in $\mathbf{R} \times \mathbf{R}^+$ (actually, not even in any neighbourhood of the straight line $y = 0$). Therefore this Cauchy problem is ill-posed w.r.t. the uniform topology.

On the other hand, if in (1.1) the Laplace equation is replaced by the wave equation, then the Cauchy problem is well-posed w.r.t. the uniform topology. For instance, if g_n is as above then the solution $u_n(x, y) = \frac{1}{n} \sin(nx) \cos(ny)$ vanishes uniformly as $n \to \infty$. The same applies for the heat equation (in this case a single initial condition must be prescribed, as the operator is of first order in time).

Let us now consider the *boundary-value problem*

$$
\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \qquad \text{in } \Omega := \mathbf{R}\times]0, 1[, \\ u(x, 0) = g(x), \quad u(x, 1) = 0 \qquad \forall x \in \mathbf{R}. \end{cases} \tag{1.3}
$$

with g continuous and bounded. This problem is well-posed w.r.t. the uniform topology. [Ex] For instance, if $g_n(x) := \frac{1}{n} \sin(nx)$ for any $x \in \mathbf{R}$ then the solution

$$
u_n(x, y) = \frac{1}{n(1 - e^{2n})} \sin(nx)\big(e^{ny} - e^{n(2-y)}\big)
$$

vanishes uniformly as $n \to \infty$.

These conclusions do not depend on the unboundedness of the domain. The above picture is essentially unchanged in $\Omega :=]0, \pi[^2$, if the conditions $(1.3)_2$ and $u(0, \cdot) = u(\pi, \cdot) = 0$ are appended. Similar conclusions apply for any $N \geq 2$, and if the Laplace, wave and heat equations are respectively replaced by general second order equations of the same type.

Boundary-Value Problems for Elliptic Equations. We introduce some boundary-value problems associated with the equation $-\Delta u + u = f$, which are well-posed in several classes of function spaces. This discussion holds almost unchanged for the Poisson equation, and may be extended to more general elliptic operators. This will also provide a basis for the formulation of initial- and boundary-value problems for evolution equations. Here we shall just introduce these problems *formally,* i.e., without specifying the functional environment and the precise meaning of the equations. Regularity hypotheses and the actual well-posedness of several of these problems will be illustrated in the next sections.

Dirichlet Condition. Let Ω be a bounded domain of $\mathbf{R}^N$, its boundary Γ be of class $C^{0,1}$, and $f : \Omega \to \mathbf{R}$ and $g : \Gamma \to \mathbf{R}$ be prescribed functions. We search for $u : \bar{\Omega} \to \mathbf{R}$ such that

$$
\begin{cases} -\Delta u + \lambda u = f \qquad \text{in } \Omega, \\ u = g \qquad \text{on } \Gamma. \end{cases} \tag{1.4}
$$

This boundary condition is named after Dirichlet, and is said of homogeneous type if g identically vanishes. (1.4) is then called a *Dirichlet problem* for the operator $-\Delta + \lambda I$ ($I :=$ identity operator) for any $\lambda \geq 0$. If the function g is extended to the whole $\bar{\Omega}$, by setting $\tilde{u} := u - g$ and $\tilde{f} := f + \Delta g - \lambda g$, the nonhomogeneous boundary condition is reduced to a homogeneous condition:

$$
\begin{cases} -\Delta \tilde{u} + \lambda \tilde{u} = \tilde{f} \qquad \text{in } \Omega, \\ \tilde{u} = 0 \qquad \text{on } \Gamma. \end{cases} \tag{1.5}
$$

Similarly, if $h : \Omega \to \mathbf{R}$ is any function such that $-\Delta h + \lambda h = f$ in Ω, then, by setting $\tilde{u} := u - h$ in Ω and $\tilde{g} := g - h$ on Γ, we get a homogeneous equation coupled with a nonhomogeneous boundary condition:

$$
\begin{cases} -\Delta \tilde{u} + \lambda \tilde{u} = 0 \qquad \text{in } \Omega, \\ \tilde{u} = \tilde{g} \qquad \text{on } \Gamma. \end{cases} \tag{1.6}
$$

Neumann Condition. Let Ω, Γ, f, g be as above, and let $\vec{\nu}$ be the exterior normal unit vector on Γ. We search for $u : \bar{\Omega} \to \mathbf{R}$ such that

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \vec{\nu}} = g & \text{on } \Gamma. \end{cases} \tag{1.7}$$

This boundary condition is named after Neumann, and is said homogeneous if g identically vanishes. (1.7) is then called a *Neumann problem* for the operator $-\Delta + \lambda I$.

This problem is well-posed for any $\lambda > 0$, but not for $\lambda = 0$. Indeed, under suitable regularity conditions, for $\lambda = 0$ if it has a solution then $\int_\Omega \Delta u \, dx = \int_\Gamma \frac{\partial u}{\partial \vec{\nu}} \, d\sigma$, by the Gauss-Green theorem. This yields the compatibility condition $\int_\Omega f \, dx + \int_\Gamma g \, d\sigma = 0$ on the data. Moreover, still for $\lambda = 0$, if u is a solution then $u + C$ is also a solution for any constant C. Thus for $\lambda = 0$ the problem (1.7) has a solution only if it has an infinity of solutions. This is an example of a more general result, known as the *Fredholm alternative,* which we shall address ahead.

Robin Condition. Let Ω, Γ, $\vec{\nu}$, f, g be as above, and $a : \Gamma \to \mathbf{R}^+$. We search for $u : \bar{\Omega} \to \mathbf{R}$ such that

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \vec{\nu}} + au = g & \text{on } \Gamma. \end{cases} \tag{1.8}$$

This boundary condition is named after Robin, or after Newton, since this includes Newton's cooling law $\frac{\partial u}{\partial \vec{\nu}} + a(u - u_*) = 0$; here u represents the temperature and u_* is an exterior value. This is also labelled as a *condition of the third type,* as sometimes the Dirichlet and the Neumann conditions are respectively named conditions of the first and second type.

Periodicity Condition. If the domain is a product of intervals:

$$\Omega :=]a_1, b_1[\times \ldots \times]a_N, b_N[\qquad (a_i < b_i, \ \forall i),$$

we may search for $u : \bar{\Omega} \to \mathbf{R}$ such that

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega, \\ u|_{x_i = a_i} = u|_{x_i = b_i}, \quad \dfrac{\partial u}{\partial x_i}\Big|_{x_i = a_i} = \dfrac{\partial u}{\partial x_i}\Big|_{x_i = b_i} & (i = 1, \ldots, N). \end{cases} \tag{1.9}$$

This is equivalent to prescribing the equation on an N-dimensional torus.

Mixed Boundary Conditions. Let Ω, Γ, $\vec{\nu}, \lambda$ be as above, and Γ_0, Γ_1 be two nonempty open subsets of Γ such that $\Gamma = \Gamma_0 \cup \Gamma_1$. Let $f : \Omega \to \mathbf{R}$, $g : \Gamma_0 \to \mathbf{R}$, $h : \Gamma_1 \to \mathbf{R}$. The problem of finding $u : \bar{\Omega} \to \mathbf{R}$ such that

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial \vec{\nu}} = h & \text{on } \Gamma_1. \end{cases} \tag{1.10}$$

is named a *mixed Dirichlet-Neumann problem.* One may also consider a *mixed Dirichlet-Robin problem,* or a *mixed Neumann-Robin problem.*

Free Space. One may also deal with the equation $(1.10)_1$ in $\Omega := \mathbf{R}^N$, or in other unbounded domains. In particular, if Ω is the complement of a compact subset of $\mathbf{R}^N$, one speaks of an *exterior problem.* Which natural boundary conditions may be prescribed in this case? In the following sections we shall see that integrability plays an important role in the analysis of P.D.E.s, as several methods assume that data and certain derivatives of solutions vary in L^p-spaces $(1 \le p < +\infty)$. Of course, integrability in an unbounded domain suggests some restrictions on the asymptotic behaviour (although, this does not force asymptotic vanishing!). In this case one may then regard the boundary conditions as implicit in the membership of the solution in some Banach spaces.

For instance, one may deal with the Dirichlet-type problem

$$\begin{cases} -\varDelta u+\lambda u=f & \text{in } \varOmega:=\mathbf{R}^{N-1}\times]0,+\infty[, \\ u=g & \text{on } \mathbf{R}^{N-1}\times\{0\}. \end{cases} \tag{1.11}$$

The corresponding problem for the Poisson equation is not well-posed, as it is clear in the one-dimensional case.

Boundary-Value Problems for Hyperbolic and Parabolic Equations. We introduce some evolution problems which are well-posed in several classes of function spaces. This discussion partly extends that of the stationary equations, as the evolution operators that we consider reduce to elliptic operators under stationary conditions. We deal with the heat and wave equations, as representative of parabolic and hyperbolic equations. More general evolution operators may then be obtained just by replacing the Laplace operator by a more general elliptic operator, and the following discussion partly extends to that setting.

Free Space. Let $\varOmega:=\mathbf{R}^N$, fix any $T>0$, and set $Q:=\varOmega\times]0,T[$. The Cauchy problems for the wave and heat equations respectively read

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}-\varDelta u=f & \text{in } Q, \\ u(x,0)=u^0(x), \quad \dfrac{\partial u}{\partial t}(x,0)=w^0(x) & \text{in } \varOmega, \end{cases} \tag{1.12}$$

$$\begin{cases} \dfrac{\partial u}{\partial t}-\varDelta u=f & \text{in } Q, \\ u(x,0)=u^0(x) & \text{in } \varOmega, \end{cases} \tag{1.13}$$

for prescribed functions f,u^0,w^0.

Cauchy-Dirichlet Problem. We assume that $\varOmega$ is a bounded domain of $\mathbf{R}^N$, and $\varGamma$ is its boundary. We then fix any $T>0$, and set $Q:=\varOmega\times]0,T[$, $\varSigma:=\varGamma\times]0,T[$. Here suitable boundary conditions must be coupled with the initial condition(s). For instance, the Cauchy-Dirichlet problem for the heat equation reads

$$\begin{cases} \dfrac{\partial u}{\partial t}-\varDelta u=f & \text{in } Q, \\ u(x,t)=g(x,t) & \text{on } \varSigma, \\ u(x,0)=u^0(x) & \text{in } \varOmega. \end{cases} \tag{1.14}$$

One may also couple the Cauchy condition with a Neumann condition (Cauchy-Neumann problem), or with another boundary condition. For the wave equation, a second initial condition is in order.

The set $\varGamma\times\{0\}$ is the boundary of $\varOmega\times\{0\}$ and is also a part of the boundary of $\varSigma$. We may find so regular a solution that the double trace on $\varGamma\times\{0\}$ is meaningful only if the data fulfil the compatibility condition

$$g(x,t)=u^0(x) \qquad \text{on } \varGamma\times\{0\}. \tag{1.15}$$

The same applies to other compatibility conditions.

Functional Framework. Different formulations may be attached to the same problem, corresponding to different regularity hypotheses on data and solution. We outline this issue on the Dirichlet problem for the equation $-\varDelta u+\lambda u=f$, for any $\lambda\geq 0$.

Classical Formulation. This setting refers to spaces of either continuous or Hölder-continuous functions. Here f and g are assumed to be (at least) continuous, u is required to belong to $C^2(\Omega) \cap C^0(\bar{\Omega})$; the equation and the boundary condition are then assumed to hold at all points.

Strong Formulation. Here we move to Sobolev spaces. We fix any $p \in [1, +\infty[$, and assume that Ω is at least of class $C^{0,1}$, so that $\gamma_0 : W^{1,p}(\Omega) \to W^{1-1/p,p}(\Gamma)$. For any $f \in L^p(\Omega)$ and any $g \in W^{1-1/p,p}(\Gamma)$, we search for $u \in W^{1,p}(\Omega)$ such that $\Delta u \in L^p(\Omega)$ and

$$\begin{cases} -\Delta u + \lambda u = f & \text{a.e. in } \Omega, \\ \gamma_0 u = g & \text{a.e. on } \Gamma. \end{cases} \tag{1.16}$$

Weak Formulation. The restriction "$\Delta u \in L^p(\Omega)$" is here removed by interpreting the equation in the sense of distributions. We assume that $f \in W^{-1,p}(\Omega)$, $g \in W^{1-1/p,p}(\Gamma)$, and search for $u \in W^{1,p}(\Omega)$ such that

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } \mathcal{D}'(\Omega), \\ \gamma_0 u = g & \text{a.e. on } \Gamma. \end{cases} \tag{1.17}$$

In the analysis of these problems, usually one first deals with the weak formulation. Proving existence of a solution is the first task; one then tries to derive its uniqueness and qualitative properties. Under stronger assumptions on the data, one also tries to establish regularity properties of the weak solution, aiming to show that this is a strong solution, or even a classical one.

2 Elliptic Operators in Nondivergence Form

Some Classical Results for the Laplace Operator. Any function $\Omega \to \mathbf{R}$ such that $\Delta u = 0$ ($\Delta u \geq 0$, $\Delta u \leq 0$, resp.) in $\mathcal{D}'(\Omega)$ is called *harmonic (subharmonic, superharmonic,* resp.), for reasons that will appear clear by the results of this section. Let us denote by ω_N the measure of the unit ball of $\mathbf{R}^N$, so that the $(N-1)$-dimensional measure of the corresponding unit sphere is $N\omega_N$.

Theorem 2.1 *(Mean Value Principle) Let Ω be a domain of $\mathbf{R}^N$ $(N \geq 1)$ and $u \in C^2(\Omega)$. Then $\Delta u \geq 0$ in Ω iff*

$$u(x) \leq \frac{1}{N\omega_N R^{N-1}} \int_{\partial B(x,R)} u(y)\, d\sigma(y) \Big(= \frac{1}{N\omega_N} \int_{\partial B(0,1)} u(x+Ry)\, d\sigma(y) \Big) \\ \forall R > 0 \textit{ such that } B(x,R) \subset \Omega, \forall x \in \Omega. \tag{2.1}$$

Proof. If $\Delta u \geq 0$ in Ω, then by the Gauss-Green theorem we have

$$0 \leq \int_{B(x,R)} \Delta u(y)\, dy = \int_{\partial B(x,R)} \frac{\partial u}{\partial \vec{\nu}}(y)\, d\sigma(y) = R^{N-1} \int_{\partial B(0,1)} \frac{\partial u}{\partial R}(x+Ry)\, d\sigma(y) \\ = R^{N-1} \frac{d}{dR} \int_{\partial B(0,1)} u(x+Ry)\, d\sigma(y).$$

Therefore the function

$$\varphi_x : R \mapsto \frac{1}{N\omega_N} \int_{\partial B(0,1)} u(x+Ry)\, d\sigma(y) = \frac{1}{N\omega_N R^{N-1}} \int_{\partial B(x,R)} u(y)\, d\sigma(y)$$

is nondecreasing. As $\varphi_x(R) \to u(x)$ as $R \to 0^+$, (1) follows.

We prove the converse statement by contradiction. If $\Delta u(x) < 0$ for some $x \in \Omega$, then by continuity $\Delta u < 0$ in an open subset of Ω. By the above argument, φ_x is then strictly decreasing in a neighbourhood of $R = 0$, and (2.1) fails. □

This theorem has several important consequences.

Corollary 2.2 *Let Ω be a domain of $\mathbf{R}^N$ $(N \ge 1)$ and $u \in C^2(\Omega)$. Then $\Delta u \ge 0$ in Ω iff*

$$u(x) \le \frac{1}{\omega_N R^N} \int_{B(x,R)} u(y)\, dy \qquad \forall R > 0 \textit{ such that } B(x,R) \subset \Omega, \forall x \in \Omega. \tag{2.2}$$

Proof. This is easily checked multiplying (2.1) by $N\omega_N R^{N-1}$ and integrating w.r.t. R. □

Corollary 2.3 *Let Ω be a domain of $\mathbf{R}^N$ $(N \ge 1)$, and $u \in C^2(\Omega)$. Then $\Delta u = 0$ in Ω iff either of the following properties holds*

$$u(x) = \frac{1}{N\omega_N R^{N-1}} \int_{\partial B(x,R)} u(y)\, d\sigma(y) \qquad \forall R > 0 \textit{ such that } B(x,R) \subset \Omega, \forall x \in \Omega, \tag{2.3}$$

$$u(x) = \frac{1}{\omega_N R^N} \int_{B(x,R)} u(y)\, dy \qquad \forall R > 0 \textit{ such that } B(x,R) \subset \Omega, \forall x \in \Omega. \tag{2.4}$$

Corollary 2.4 *(Strong Maximum Principle) Let Ω be a domain of $\mathbf{R}^N$ $(N \ge 1)$, and $u \in C^2(\Omega)$. If $\Delta u \ge 0$ in Ω, then either u is constant, or $u(x) < \sup_\Omega u$ for any $x \in \Omega$.*

Proof. We can assume that $S := \sup_\Omega u < +\infty$. By continuity of u, $A := \{x \in \Omega : u(x) = S\}$ is a closed subset of Ω, and by (2.4) it is open. Therefore either $A = \Omega$ or $A = \emptyset$. □

Let now Ω be a bounded domain of $\mathbf{R}^N$. For any subharmonic function $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, the strong maximum principle entails the weaker inequality $u(x) \le \sup_\Gamma u$. This is known as the *weak maximum principle,* and entails the following properties: [(2)]

(i) The solution depends monotonically on the interior and boundary data. That is, assuming that $u_1, u_2 \in C^2(\Omega) \cap C^0(\bar{\Omega})$,

$$\text{if } \; -\Delta u_1 \le -\Delta u_2 \; \text{ in } \; \Omega \; \text{ and } \; u_1 \le u_2 \; \text{ on } \; \Gamma, \quad \text{then } \; u_1 \le u_2 \; \text{ in } \; \Omega.$$

(ii) The solution depends continuously on the boundary datum w.r.t. to the uniform topology. That is, assuming that $u_1, u_2 \in C^2(\Omega) \cap C^0(\bar{\Omega})$,

$$\text{if } \; -\Delta u_1 = -\Delta u_2 \; \text{ in } \; \Omega \; \text{ then } \; \max\nolimits_\Omega |u_1 - u_2| \le \max\nolimits_\Gamma |u_1 - u_2|.$$

(iii) The solution of the Dirichlet problem for the Poisson equation $-\Delta u = f$ is unique.

The weak maximum principle is also at the basis of the following classical result (which can also be stated under weaker regularity hypotheses on Ω).

Theorem 2.5 *(Perron) Let Ω be a bounded domain of $\mathbf{R}^N$ of class C^2, $g \in C^0(\Gamma)$, and set*

$$S_g := \{v \in C^0(\Omega) : -\Delta v \le 0 \textit{ in } \mathcal{D}'(\Omega), v \le g \textit{ on } \Gamma\}, \qquad u(x) := \max_{v \in S_g} v(x) \quad \forall x \in \Omega. \tag{2.5}$$

Then $u \in C^0(\Omega)$ and

$$\Delta u = 0 \quad \textit{ in } \mathcal{D}'(\Omega), \qquad u = g \quad \textit{ on } \Gamma. \qquad [] \tag{2.6}$$

Analogous results can be stated for superharmonic functions; in this case one derives *minimum* principles.

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(2) Despite of the terminology, the strong and weak maximum principles have no relation with the concepts of strong and weak solution.

Linear Elliptic Operators in Nondivergence Form. Let

$$a_{ij}, b_i, c \in L^\infty(\Omega) \qquad \forall i,j \in \{1,...,N\}. \tag{2.7}$$

For any $p \in]1,+\infty[$, we then define the second order, linear operator in *nondivergence form*

$$L : W^{2,p}(\Omega) \to L^p(\Omega) : u \mapsto -\sum_{i,j=1}^{N} a_{ij} D_i D_j u + \sum_{i=1}^{N} b_i D_i u + cu. \tag{2.8}$$

With no loss of generality, we may assume that the matrix $\{a_{ij}\}$ is symmetric.

The operator L is said *elliptic* at a point $x \in \Omega$ iff $\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \neq 0$ for any $\xi \in \mathbf{R}^N \setminus \{0\}$. We shall assume the stronger condition $\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j > 0$ for any $\xi \neq 0$. L is then said *uniformly elliptic* in Ω iff

$$\exists \gamma > 0 : \forall x \in \Omega, \forall \xi \in \mathbf{R}^N \qquad \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \gamma|\xi|^2. \tag{2.9}$$

The strong maximum principle can also be proved for general elliptic operators of the form (2.8), if $c > 0$ a.e. in Ω.

Theorem X. *(Weak Maximum Principle) Let Ω be a bounded domain of $\mathbf{R}^N$. Assume that (2.7), (2.8) and (2.9) hold, $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, and $Lu \leq 0$ in Ω. Then:*
(i) If $c \equiv 0$, then $\sup_\Omega u \leq \sup_\Gamma u$.
(ii) If $c \geq 0$, then $\sup_\Omega u \leq \sup_\Gamma u^+$.

Proof. (i) Let $c \equiv 0$. We claim that if $Lu < 0$ in Ω then u cannot achieve an interior maximum in Ω. If x_0 were such a point, then the matrix $\{D_{ij}u(x_0)\}$ would be negative semi-definite, and $\nabla u(x_0) = 0$. This would yield $Lu(x_0) \geq 0$. [3]

Let us now just assume that $Lu \leq 0$ in Ω. For a sufficiently large constant $\gamma > 0$, we have $Le^{\gamma x_1} = \big(-a_{11}\gamma^2 + b_1\gamma + c\big)e^{\gamma x_1} < 0$. Setting $u_\varepsilon := u + \varepsilon e^{\gamma x_1}$ we then get $Lu_\varepsilon < 0$ for any $\varepsilon > 0$. The previous statement then yields $\sup_\Omega u_\varepsilon \leq \sup_\Gamma u_\varepsilon$. Passing to the limit as $\varepsilon \to 0$, we then conclude that $\sup_\Omega u \leq \sup_\Gamma u$.

(ii) Let $c \geq 0$, and assume that $\sup_\Omega u > 0$, as otherwise no proof is needed. As $Lu \leq 0$ in Ω, we have

$$(L - cI)u \leq -cu \leq 0 \qquad \text{in } \Omega_+ := \{x \in \Omega : u(x) \geq 0\}.$$

Applying part (i) to $L - cI$, we get that the maximum of u is then achieved on $\partial\Omega_+$. By the continuity of u, $u = 0$ on $(\partial\Omega_+) \cap \Omega$. The maximum of u is then attained on $(\partial\Omega_+) \cap \partial\Omega$. □

A *strong* maximum principle can also be proved without assuming the boundedness of Ω.

Strong and Classical Solutions. Henceforth we assume that Ω is a bounded domain of class $C^{0,1}$ and that (2.7) is fulfilled; we then fix any $p \in]1,+\infty[$, define the operator L as in (2.8), and assume that it is uniformly elliptic, cf. (2.9). For any prescribed $f \in L^p(\Omega)$ and any $g \in W^{2-1/p,p}(\Gamma)$, we then search for $u \in W^{2,p}(\Omega)$ such that

$$\begin{cases} Lu = f & \text{a.e. in } \Omega, \\ \gamma_0 u = g & \text{a.e. on } \Gamma. \end{cases} \tag{2.10}$$

[3] For whenever two matrices $A, B \in \mathbf{R}^N \times N$ are both symmetric and positive-definite, $\sum_{ij=1}^{N} A_{ij}B_{ij} \geq 0$.

A result analogous to Perron's Theorem 2.5 holds for problem (2.10), with L in place of $-\Delta$. (The elements of S_g are here named *subsolutions.*) []

The following two classical results may be derived either via potential theory, or via Banach space interpolation.

*** Theorem 2.6** *(Agmon-Douglis-Nirenberg) Let $k \in \mathbf{N}_0$, Ω be a bounded domain of class $C^{k+1,1}$, and $1 < p < +\infty$. Moreover, let $a_{ij}, b_i, c \in W^{k,\infty}(\Omega)$ for any i, j, $c \geq 0$ a.e., and L be as in (2.8). Then:*

(i) For any $f \in W^{k,p}(\Omega)$ and $g \in W^{k+2-1/p,p}(\Gamma)$, there exists a unique solution $u \in W^{k+2,p}(\Omega)$ of problem (2.10).

(ii) There exists a constant $C_k > 0$ such that

$$\|v\|_{W^{k+2,p}(\Omega)} \leq C_k\big(\|Lv\|_{W^{k,p}(\Omega)} + \|\gamma_0 v\|_{W^{k+2-1/p,p}(\Gamma)}\big) \qquad \forall v \in W^{k+2,p}(\Omega). \quad [] \tag{2.11}$$

*** Theorem 2.7** *(Schauder) Let $k \in \mathbf{N}_0$, $0 < \alpha < 1$, and Ω be a bounded domain of class $C^{k+2,\alpha}$. Moreover, let $a_{ij}, b_i, c \in C^{k,\alpha}(\bar\Omega)$ for any i, j, $c \geq 0$ a.e., and L be as in (2.8). Then:*

(i) For any $f \in C^{k,\alpha}(\bar\Omega)$ and $g \in C^{k+2,\alpha}(\bar\Omega)$, then there exists a unique solution $u \in C^{k+2,\alpha}(\bar\Omega)$ of problem (2.10).

(ii) There exists a constant $C_k > 0$ such that

$$\|v\|_{C^{k+2,\alpha}(\bar\Omega)} \leq C_k\big(\|Lv\|_{C^{k,\alpha}(\bar\Omega)} + \|v\|_{C^{k+2,\alpha}(\Gamma)}\big) \qquad \forall v \in C^{k+2,\alpha}(\Omega). \quad [] \tag{2.12}$$

If $N > 1$, for $\alpha = 0$ Theorem 2.7 fails. For either theorem, if the hypothesis $c \geq 0$ is dropped, then a Fredholm alternative applies, see ahead.

Finally, as a simple consequence of part (i) of the latter theorem, for any $f, g \in C^\infty(\bar\Omega)$, there exists a unique solution $u \in C^\infty(\bar\Omega)$ of problem (2.10).

3 The Green Formula and Related Trace Results

In view of the analysis of weak formulations, we need an extension of the formula of partial integration to several dimensions. This will also provide some further trace theorems. We still assume that Ω is a domain of $\mathbf{R}^N$ and denote its boundary by Γ.

Theorem 3.1 *(Classical Green's Formulae for the Laplace Operator) Let Ω be bounded domain of class $C^{0,1}$, and $\vec\nu$ be the exterior normal unit vector on Γ. Then*

$$-\int_\Omega (\Delta u)v\,dx = \int_\Omega \nabla u \cdot \nabla v\,dx - \int_\Gamma \frac{\partial u}{\partial \vec\nu} v\,d\sigma \qquad \forall u \in C^2(\bar\Omega), \forall v \in C^1(\bar\Omega), \tag{3.1}$$

$$\int_\Omega [(\Delta v)u - (\Delta u)v]\,dx = \int_\Gamma \Big(\frac{\partial v}{\partial \vec\nu}u - \frac{\partial u}{\partial \vec\nu}v\Big)\,d\sigma \qquad \forall u, v \in C^2(\bar\Omega). \tag{3.2}$$

Proof. The classical Gauss theorem reads

$$\int_\Omega \nabla \cdot \vec z\,dx = \int_\Gamma \vec z \cdot \vec\nu\,d\sigma \qquad \forall \vec z \in C^1(\bar\Omega)^N \;\; (\nabla\cdot := \mathrm{div}), \tag{3.3}$$

and taking $\vec z := v\vec w$ we get the following extension of the formula of partial integration to several dimensions

$$\int_\Omega (\nabla v) \cdot \vec w\,dx + \int_\Omega v\nabla \cdot \vec w\,dx = \int_\Gamma v\vec w \cdot \vec\nu\,d\sigma \qquad \forall \vec w \in C^1(\bar\Omega)^N, \forall v \in C^1(\bar\Omega). \tag{3.4}$$

The choice $\vec w := \nabla u$ then yields the *first Green formula* (3.1). By exchanging u and v and subtracting the two formulas, we then get the *second Green formula* (3.2). □

Green's Function. The second Green formula allows us to transform the Dirichlet problem as follows. Let $E \in \mathcal{D}'(\mathbf{R}^N)$ be a *fundamental solution* of the operator $-\Delta$ [4] For any $y \in \Omega$, let $w(x,y)$ (we do not specify its regularity) be such that

$$\begin{cases} -\Delta_x w(x,y) = 0 \qquad \forall x \in \Omega, \\ w(x,y) = E(x-y) \qquad \forall x \in \Gamma. \end{cases} \tag{3.5}$$

In general it is not easier to solve this problem rather than (2.10); however, in presence of special symmetries, this problem may be solved explicitly. The second Green formula (3.2) then yields

$$-\int_\Omega (\Delta_x u(x)) w(x,y)\,dx = \int_\Gamma \Big(\frac{\partial w}{\partial \vec{\nu}}(x,y) u(x) - \frac{\partial u}{\partial \vec{\nu}}(x) w(x,y) \Big)\, d\sigma(x). \tag{3.6}$$

(Here and below, the normal derivative $\frac{\partial}{\partial \vec{\nu}}$ acts on the variable x.) Setting

$$G(x,y) := E(x-y) - w(x,y) \qquad \text{for a.a. } (x,y) \in \Omega,$$

we get

$$\begin{cases} -\Delta_x G(x,y) = \delta_y \qquad \text{in } \mathcal{D}'(\Omega) \\ G(x,y) = 0 \qquad \forall x \in \Gamma \end{cases} \qquad \forall y \in \Omega. \tag{3.7}$$

Although $G \notin C^2(\bar{\Omega})$, taking $v(x) := G(x,y)$ a *formal* application of (3.2) yields

$$\begin{aligned} u(y) &= \langle \delta_y, u \rangle = \langle \Delta_x G(x,y), u(x) \rangle \\ &= \int_\Omega G(x,y) \Delta_x u(x)\,dx + \int_\Gamma \frac{\partial G}{\partial \vec{\nu}}(x,y) u(x)\, d\sigma(x) \qquad \forall u \in C^2(\bar{\Omega}). \end{aligned} \tag{3.8}$$

Therefore the solution of the (nonhomogeneous) Dirichlet problem (2.10) reads

$$u(y) = \int_\Omega G(x,y) f(x)\,dx + \int_\Gamma \frac{\partial G}{\partial \vec{\nu}}(x,y) g(x)\, d\sigma(x). \tag{3.9}$$

G is named the *Green function* relative to $-\Delta$ in the domain Ω.

This classical procedure allows one to exhibit explicit solutions for special geometries (e.g., a ball, a half-plane, and so on). However here we intend to make a different use of the Green formula.

Normal Traces. Let Ω be a domain of $\mathbf{R}^N$ of class $C^{0,1}$, and set

$$L^2_{\rm div}(\Omega)^N := \{ \vec{v} \in L^2(\Omega)^N : \nabla \cdot \vec{v} \in L^2(\Omega) \}, \tag{3.10}$$

which is a Banach space equipped with the graph norm

$$\| \vec{v} \|_{L^2_{\rm div}(\Omega)^N} := \big(\|\vec{v}\|^2_{L^2} + \|\nabla \cdot \vec{v}\|^2_{L^2} \big)^{1/2}. \tag{3.11}$$

Lemma 3.2 *If Ω be a domain of $\mathbf{R}^N$ of class $C^{0,1}$, then the space $\mathcal{D}(\bar{\Omega})^N$ is dense in $L^2_{\rm div}(\Omega)^N$.* []

Theorem 3.3 *(Normal Traces) Let Ω be a bounded domain of $\mathbf{R}^N$ of class $C^{0,1}$.*

(i) There exists a unique linear and continuous trace operator $\gamma_\nu : L^2_{\rm div}(\Omega)^N \to H^{-1/2}(\Gamma)$

[4] We remind the reader that $E \in \mathcal{D}'(\mathbf{R}^N)$ is called a fundamental solution the differential operator L whenever $LE = \delta_0$ (the Dirac measure at the origin) in $\mathcal{D}'(\mathbf{R}^N)$.

such that, denoting by $\langle\cdot,\cdot\rangle$ *the duality pairing between* $H^{1/2}(\Gamma)$ *and* $H^{-1/2}(\Gamma)$*, the following* generalized formula of partial integration *holds*

$$-\int_\Omega (\nabla\cdot\vec{u})v\,dx = \int_\Omega \vec{u}\cdot\nabla v\,dx - \langle\frac{\partial\vec{u}}{\partial\vec{\nu}}, v\rangle \qquad \forall\vec{u}\in L^2_{\rm div}(\Omega)^N, \forall v\in\mathcal{D}(\bar{\Omega}). \tag{3.12}$$

(ii) There exists a (nonunique) linear and continuous lift operator $\mathcal{R}_\nu : H^{-1/2}(\Gamma)\to L^2_{\rm div}(\Omega)^N$ *such that* $\gamma_\nu\mathcal{R}_\nu v = v$ *for any* $v\in H^{-1/2}(\Gamma)$*. []*

Proof. By (3.4) and by the continuity of the lift operator $\mathcal{R} : H^{1/2}(\Gamma)\to H^1(\Omega)$, there exist constants $C_1, C_2 > 0$ such that

$$\Big|\int_\Gamma v\vec{w}\cdot\vec{\nu}\,d\sigma\Big| \le C_1\|\vec{w}\|_{L^2_{\rm div}(\Omega)}\,\|v\|_{H^1(\Omega)} \le C_2\|\vec{w}\|_{L^2_{\rm div}(\Omega)}\,\|\gamma_0 v\|_{H^{1/2}(\Gamma)}$$
$$\forall\vec{w}\in\mathcal{D}(\bar{\Omega})^N, \forall v\in H^1(\Omega).$$

By the density lemma, the operator $\mathcal{D}(\bar{\Omega})^N\to\mathcal{D}(\Gamma) : \vec{w}\mapsto\vec{w}\cdot\vec{\nu}$ can then be extended by continuity to an operator $L^2_{\rm div}(\Omega)^N\to H^{1/2}(\Gamma)'\ (= H^{-1/2}(\Gamma))$. Therefore the generalized formula of partial integration (3.12) holds.

In order to prove the converse, let us now fix any $g\in H^{-1/2}(\Gamma)$, and notice that the mapping $F_g : v\mapsto\langle g, v\rangle$ is an element of $H^1(\Omega)'$. By Theorem 5.1 (see later on), for any large enough constant λ, there exists one and only one $u\in H^1(\Omega)$ such that

$$-\int_\Omega (\nabla u\cdot\nabla v + \lambda uv)\,dx = F_g(v) (= -\langle g, v\rangle) \qquad \forall v\in H^1(\Omega), \tag{3.13}$$

and the mapping $H^1(\Omega)'\to H^1(\Omega) : F_g\mapsto u$ is continuous. The same then holds for the mapping $H^{-1/2}(\Gamma)\to L^2_{\rm div}(\Omega)^N : g\mapsto\nabla u$.

By the Green formula (3.12) we conclude that $\gamma_\nu\mathcal{R}_\nu v = v$ for any $v\in H^{-1/2}(\Gamma)$. □

Second Order Elliptic Operators in Divergence Form.

Let

$$a_{ij}, b_j, c_i, d\in L^\infty(\Omega) \qquad \forall i,j\in\{1,...,N\}, \tag{3.14}$$

and define the second order, linear operator in *divergence form*

$$L : H^1(\Omega)\to H^{-1}(\Omega) : u\mapsto -\sum_{j=1}^N D_j\Big(\sum_{i=1}^N a_{ij}D_i u + b_j u\Big) + \sum_{i=1}^N c_i D_i u + du, \tag{3.15}$$

and the *formal adjoint operator*

$$L^* : H^1(\Omega)\to H^{-1}(\Omega) : v\mapsto -\sum_{i=1}^N D_i\Big(\sum_{j=1}^N a_{ij}D_j v + c_i v\Big) + \sum_{j=1}^N b_j D_j v + dv. \tag{3.16}$$

We shall see that

$$\int_\Omega vLu\,dx = \int_\Omega uL^*v\,dx \qquad \forall v\in H^1_0(\Omega), \tag{3.17}$$

which explains the denomination we used for L^*.

(Here we confine ourselves to $p = 2$, since the most important results are known for this setting.) Ellipticity and uniform ellipticity are here defined as above. If the coefficients are sufficiently regular, it is easy to see that any operator in divergence form can also be represented in nondivergence form, and conversely. Moreover, any operator in divergence form is elliptic (uniform elliptic, resp.) iff the same holds for the equivalent operator in nondivergence form.

The next result can be proved by an argument similar to that of Theorem 3.1.

Theorem 3.4 *(Classical Green's Formulae for Operators in Divergence Form) Let Ω be a bounded domain of $\mathbf{R}^N$ of class $C^{0,1}$, Γ be its boundary, and $\vec{\nu}$ be the exterior normal unit vector on Γ. Then*

$$\begin{aligned} -\int_\Omega vLu\,dx = &\int_\Omega \Big(\sum_{i,j=1}^N a_{ij} D_i u D_j v + \sum_{j=1}^N b_j u D_j v + \sum_{i=1}^N c_i (D_i u) v + duv \Big)\,dx \\ &- \int_\Gamma v \sum_{j=1}^N \Big(\sum_{i=1}^N a_{ij} D_i u + b_j u \Big) \nu_j \,d\sigma \qquad \forall u \in C^2(\bar{\Omega}), \forall v \in C^1(\bar{\Omega}), \end{aligned} \tag{3.18}$$

$$\begin{aligned} \int_\Omega [vLu - uL^* v]\,dx = &\int_\Gamma v \sum_{j=1}^N \Big(\sum_{i=1}^N a_{ij} D_i u + b_j u \Big) \nu_j \,d\sigma \\ &- \int_\Gamma u \sum_{i=1}^N \Big(\sum_{j=1}^N a_{ij} D_j v + c_i v \Big) \nu_i \,d\sigma \qquad \forall u, v \in C^2(\bar{\Omega}). \end{aligned} \tag{3.19}$$

(3.17) then follows, by a density argument.

Conormal Traces. Let L be defined as in (3.15), and set

$$H^1_L(\Omega)^N := \{ v \in H^1(\Omega) : Lv \in L^2(\Omega) \}. \tag{3.20}$$

This is a Banach space equipped with the graph norm

$$\|v\|_{H^1_L(\Omega)} := \Big(\|v\|^2_{H^1} + \|Lv\|^2_{L^2} \Big)^{1/2}. \tag{3.21}$$

Theorem 3.5 *(Conormal Traces) Let Ω be a bounded domain of $\mathbf{R}^N$ of class $C^{0,1}$, and the operator L be defined as in (3.15).*

(i) There exists a unique linear and continuous operator $\gamma_L : H^1_L(\Omega) \to H^{-1/2}(\Gamma)$ such that

$$\begin{aligned} -\int_\Omega vLu\,dx = &\int_\Omega \Big(\sum_{i,j=1}^N a_{ij} D_i u D_j v + \sum_{j=1}^N b_j u D_j v + \sum_{i=1}^N c_i (D_i u) v + duv \Big)\,dx \\ &- \langle \gamma_L u, v \rangle \qquad \forall u \in \mathcal{D}(\bar{\Omega}), \forall v \in H^1(\Omega), \end{aligned} \tag{3.22}$$

(ii) There exists a (nonunique) linear and continuous lift operator $\mathcal{R}_L : H^{-1/2}(\Gamma) \to H^1_L(\Omega)^N$ such that $\gamma_L \mathcal{R}_L \vec{v} = v$ for any $\vec{v} \in H^{-1/2}(\Gamma)$. []

By generalized partial integration $\gamma_L u = \sum_{i,j=1}^N (a_{ij} D_i u + b_j uv)\nu_j$.

Proof. This is analogous to that of Thereom 3.3. □

4 The Fredholm Alternative and the Lax-Milgram Theorem

NOTE: Theorems 4.1 and 4.2 have been skipped this year......

Theorem 4.1 *(Riesz-Schauder) Let B be a Banach space and $T \in \mathcal{L}(B)$ be a compact operator.* [5] *Then:*
(i) $R(I-T) = N(I-T^*)^\perp$;
(ii) $\dim N(I-T) = \dim N(I-T^*) < +\infty$. []

As for part (i), the point is that $R(I-T)$ is closed, as one can prove that $\overline{R(A)} = N(A^*)^\perp$ for any $A \in \mathcal{L}(B)$. [] (The inclusion $\overline{R(A)} \subset N(A^*)^\perp$ is straightforward, for one easily checks that $N(A^*) \subset R(A)^\perp$.)
For part (ii) the point is that $N(I-T)$ and $N(I-T^*)$ have the same dimension, as by the compactness of T it is clear that both are finite dimensional.
In terms of the Fredholm theory, Theorem 4.1 entails that $I-T$ has null *Fredholm index*

$$\mathrm{Ind}(I-T) := \dim N(I-T) - \mathrm{codim}\ R(I-T).$$

In fact

$$\mathrm{codim}\ R(I-T) = \dim R(I-T)^\perp = (\text{by (i)})\ \dim \overline{N(I-T^*)} = (\text{by (ii)})\ \dim N(I-T).$$

Notice that by (i)

$$R(I-T) = B \quad \Leftrightarrow \quad N(I-T) = \{0\};$$

namely, the operator $I-T$ is surjective iff it is injective. In this case $(I-T)^{-1} \in \mathcal{L}(B)$ (e.g., by the closed graph theorem). The following statement is more precise.

Corollary 4.2 *("Existence ⇔ Uniqueness") Let B be a Banach space and $T \in \mathcal{L}(B)$ be a compact operator.*
(i) If $Tw = w$ only for $w = 0$, then the equation $u - Tu = f$ has a (unique) solution u for any $f \in B$, and the mapping $B \to B : f \mapsto u$ is continuous.
(ii) If $Tw = w$ for some $w \neq 0$, then the equation $u - Tu = f$ has a (nonunique) solution u iff

$$_B\langle f, v\rangle_{B'} = 0 \qquad \forall v \in B' : T^*v = v. \tag{4.1}$$

Moreover, these functions v span a finite dimensional vector space. Furthermore, denoting by P the operator that selects the minimal norm element, the operator $P \circ (I-T)^{-1}$ is continuous in B; that is, there exists a constant $C > 0$ such that

$$\inf\{\|v\|_B : v = Tv\} \le C\|v\|_B \qquad \forall v \in B. \tag{4.2}$$

Thus there are two alternatives:
(i) either the equation $u - Tu = f$ has one and only one solution for any $f \in B$,
(ii) or it has a (nonunique) solution iff f fulfils a finite number of *orthogonality conditions,* cf. (4.1).
This extends what is already known for linear systems in $\mathbf{R}^N$, $Au = f$. Indeed, for any linear operator A in $\mathbf{R}^N$, $T := I - A$ is trivially compact.
Here the hypothesis of compactness is essential. E.g., in ℓ^2 the left (right, resp.) shift: s_ℓ : $(u_1, u_2, ...) \to (u_2, u_3, ...)$ ($s_r : (u_1, u_2, ...) \to (0, u_1, u_3, ...)$, resp.) are linear and continuous, but the operator $I - s_\ell$ ($I - s_r$, resp.) is noncompact. The mapping s_ℓ is surjective but noninjective, whereas s_r is injective but nonsurjective.

---

[5] This means that $\overline{T(S)}$ is compact for any bounded set $S \subset B$.

Theorem 4.3 *(Lax-Milgram) Let H be a Hilbert space, and $A \in \mathcal{L}(H)$ be such that, for some $\alpha > 0$,*

$$(Av, v) \geq \alpha \|v\|^2 \qquad \forall v \in H \text{ (coerciveness).} \tag{4.3}$$

Then A is bijective, and $\|A^{-1} f\| \leq \alpha^{-1} \|f\|$ for any $f \in H$.

Proof. The coerciveness yields $\alpha\|v\|^2 \leq (Av, v) \leq \|Av\|\|v\|$ for any $v \in H$, whence $\alpha\|v\| \leq \|Av\|$. This entails that A is injective, and, for any sequence $\{v_n\}$ in H, that $\{Av_n\}$ is a Cauchy sequence only if the same holds for $\{v_n\}$. By the continuity of A, $A(H)$ is then a closed vector subspace of H.

For any $v \in A(H)^\perp$, we have $\alpha\|v\|^2 \leq (Av, v) = 0$, whence $v = 0$. Therefore $A(H) = H$. The boundedness of A^{-1} then follows from the stated inequality $\alpha\|v\| \leq \|Av\|$ for any $v \in H$. □

This theorem generalizes the Riesz-Fréchet representation Theorem IV.1.9 to nonsymmetric operators. We check this assuming that H is a *real* Hilbert space, for the sake of simplicity. If $(Au, v) = (Av, u)$ for any $u, v \in H$, then $(u, v) \mapsto ((u, v)) := (Au, v)$ is a scalar product over H; moreover, by the continuity and coerciveness of A, the corresponding norm is equivalent to the original one. Then, by the representation theorem, for any $f \in H'$ setting $u := R^{-1} f \in H$ we have $((u, v)) = \langle f, v \rangle$ for any $v \in H$, i.e., $Au = f$.

5 Elliptic Equations in Divergence Form

Henceforth we assume that Ω is a bounded domain of class $C^{0,1}$ and that (3.14) is fulfilled; we then define L as in (3.15), and assume that it is uniformly elliptic, cf. (2.9). For any prescribed $f \in H^{-1}(\Omega)$, any $g \in H^{1/2}(\Gamma)$ and any parameter $\lambda \in \mathbf{R}$, we then search for $u \in H^1(\Omega)$ such that

$$\begin{cases} Lu + \lambda u = f & \text{in } H^{-1}(\Omega), \\ \gamma_0 u = g & \text{a.e. on } \Gamma. \end{cases} \tag{5.1}$$

For some purposes it is convenient to transform this problem to a homogeneous Dirichlet problem. Let $R \in \mathcal{L}\big(H^{1/2}(\Gamma); H^1(\Omega)\big)$ be a lift operator. By setting $\hat{f} := f - LRg - \lambda Rg$ and $\hat{u} := u - Rg$, (5.1) equivalently reads

$$\begin{cases} L\hat{u} + \lambda \hat{u} = \hat{f} & \text{in } H^{-1}(\Omega), \\ \gamma_0 \hat{u} = 0 & \text{a.e. on } \Gamma \ \ (\text{i.e., } \hat{u} \in H^1_0(\Omega)). \end{cases} \tag{5,2}$$

Theorem 5.1 *(Well-Posedness) Let Ω be a bounded domain of class $C^{0,1}$, (3.14) be fulfilled, L be as in (3.15), and uniformly elliptic. Then there exists $\tilde{\lambda} \in \mathbf{R}$ such that, for any $\lambda \geq \tilde{\lambda}$:*

(i) For any $f \in H^{-1}(\Omega)$ and any $g \in H^{1/2}(\Gamma)$, there exists a unique solution $u \in H^1(\Omega)$ of problem (5.1).

(ii) There exists a constant $C > 0$ such that

$$\|u\|_{H^1(\Omega)} \leq C\big(\|Lu\|_{H^{-1}(\Omega)} + \|\gamma_0 u\|_{H^{1/2}(\Gamma)}\big) \qquad \forall u \in H^1(\Omega). \tag{5.3}$$

Proof. Let us consider the homogeneous problem (5.2). If $\tilde{\lambda}$ is large enough, the linear and continuous operator $L + \tilde{\lambda} I : H^1_0(\Omega) \to H^{-1}(\Omega)$ is coercive. [Ex] By the Lax-Milgram Theorem 4.3, this operator is then bijective, and $\|\hat{u}\|_{H^1_0(\Omega)} \leq \alpha^{-1} \|\hat{f}\|_{H^{-1}(\Omega)}$. In terms of the solution u of the corresponding problem (5.1), this reads

$$\|u - Rg\|_{H^1_0(\Omega)} \leq \alpha^{-1} \|f - LRg - \lambda Rg\|_{H^{-1}(\Omega)}.$$

As

$$\|LRg - \lambda Rg\|_{H^{-1}(\Omega)} \leq \|Rg\|_{H^1(\Omega)} \leq \|g\|_{H^{1/2}(\Gamma)}$$

we then get (5.3). □

Theorem 5.2 *(Fredholm Alternative) Let Ω be a bounded domain of class $C^{0,1}$, (3.14) be fulfilled, L be as in (3.15), and uniformly elliptic. Let $\lambda \in \mathbf{R}$ and $f \in H^{-1}(\Omega)$. Then there exists a finite dimensional subspace F of $H_0^1(\Omega)$ such that problem (5.2) has a solution $\hat{u}$ iff $\langle f, v\rangle = 0$ for any $v \in F$. Moreover, $F \neq \{0\}$ only for a discrete family of λs (with $\lambda < \tilde{\lambda}$). Finally, there exists a constant $C > 0$ such that*

$$\begin{aligned}\inf\left\{\|u\|_{H_0^1(\Omega)} : (5.1) \text{ holds}\right\} \leq C\big(\|f\|_{H^{-1}(\Omega)} + \|g\|_{H^{1/2}(\Gamma)}\big)\\ \forall f \in H^{-1}(\Omega), \forall g \in H^{1/2}(\Gamma).\end{aligned} \tag{5.4}$$

Proof. Let us define $\tilde{\lambda}$ as in Theorem 5.1, and denote by J the canonic imbedding $H_0^1(\Omega) \to H^{-1}(\Omega)$. The operator $T := J \circ (L + \tilde{\lambda} I)^{-1}$ is then a linear and compact operator in $H^{-1}(\Omega)$. Henceforth we shall omit the operator J. The equation $(5.1)_1$ also reads $(L+\tilde{\lambda} I)u = f + (\tilde{\lambda} - \lambda)u =: w$, namely, $u = Tw$. In terms of w this equation is also equivalent to

$$w - (\tilde{\lambda} - \lambda)Tw = f \qquad \text{in } H^{-1}(\Omega).$$

By the Fredholm alternative, cf. Theorem 4.2, this problem has a solution iff

$$(f, z)_{H^{-1}(\Omega)} = 0 \qquad \forall z \in H^{-1}(\Omega) \text{ such that } z = (\tilde{\lambda} - \lambda)T^* z.$$

By the compactness of T, the latter equality just holds for an at most countable number of isolated $\lambda < \tilde{\lambda}$.

Notice also that $(f, z)_{H^{-1}(\Omega)} = \langle f, v\rangle$ if $v \in H_0^1(\Omega)$ is such that $\Delta v = z$ in $\mathcal{D}'(\Omega)$. Setting $F := \{v \in H_0^1(\Omega) : \Delta v = (\lambda - \tilde{\lambda})T^* \Delta v\}$, we thus get

$$\langle f, v\rangle = 0 \qquad \forall v \in F.$$

□

Theorem 5.3 *(Regularity) Let $k \in \mathbf{N}_0$, Ω be a bounded domain of class C^{k+2}, L be as in (3.15) and uniformly elliptic, $a_{ij}, b_j \in W^{k+1,\infty}(\Omega)$ and $c_i, d \in W^{k,\infty}(\Omega)$ for any i, j. Then:*

(i) For any $u \in H^1(\Omega)$, if $Lu \in H^k(\Omega)$ and $\gamma_0 u \in H^{k+3/2}(\Gamma)$, then $u \in H^{k+2}(\Omega)$.

(ii) There exists a constant $C_k > 0$ such that

$$\|u\|_{H^{k+2}(\Omega)} \leq C_k\big(\|Lu\|_{H^k(\Omega)} + \|\gamma_0 u\|_{H^{k+3/2}(\Gamma)}\big) \qquad \forall u \in H^{k+2}(\Omega).$$

[]

As a simple consequence of part (i) of the latter theorem, for any $u \in H^1(\Omega)$, if $Lu \in C^\infty(\bar{\Omega})$ and $\gamma_0 u \in C^\infty(\bar{\Omega})$, then $u \in C^\infty(\bar{\Omega})$.

Theorem 4 *(Weak Maximum Principle) Let $k \in \mathbf{N}_0$, Ω be a bounded domain of class C^{k+2}, L be as in (3.15) and uniformly elliptic, with $b_j = c_i = 0$ for any i, j and $d > 0$. Then, for any $u \in H^1(\Omega)$ such that $Lu \in L^1_{\mathrm{loc}}(\Omega)$:*

(i) If $Lu \leq 0$ a.e. in Ω, then

$$u \leq \max\left\{\operatorname*{ess\,sup}_{\Omega} \frac{Lu}{d},\ \operatorname*{ess\,sup}_{\Gamma} \gamma_0 u\right\} =: M \qquad a.e.\ in\ \Omega.$$

(ii) If $Lu \geq 0$ a.e. in Ω, then

$$u \geq \min\left\{\operatorname*{ess\,inf}_{\Omega} \frac{Lu}{d},\ \operatorname*{ess\,inf}_{\Gamma} \gamma_0 u\right\} =: m \qquad a.e.\ in\ \Omega.$$

(Both $m = -\infty$ and $M = +\infty$ are not excluded.)

Proof. Obviously, it suffice to prove part (i). With no loss of generality, we may assume that M is finite. Let us multiply Lu by $(u-M)^+$ $(\in H_0^1(\Omega))$. We have

$$D_j u D_i (u-M)^+ = D_j(u-M) D_i(u-M)^+ = D_j(u-M)^+ D_i(u-M)^+ \qquad \text{a.e. in } \Omega, \forall i,j,$$

and $du(u-M)^+ \geq 0$ a.e. in Ω. Hence

$$a \int_\Omega |\nabla (u-M)^+|^2 \, dx \leq \sum_{i,j=1}^N \int_\Omega a_{ij} D_j (u-M)^+ D_i (u-M)^+ \, dx$$
$$\leq \langle Lu - du, (u-M)^+ \rangle \leq 0.$$

We then conclude that $(u-M)^+ = 0$ a.e. in Ω. □

The hypotheses of the latter theorem might be weakened. Under these hypotheses, one might also prove a *strong* maximum principle:

if $u \in H^1(\Omega)$ and, for some ball $B \subset\subset \Omega$, $\operatorname{ess\,sup}_B v = \operatorname{ess\,sup}_\Omega v < +\infty$, then u is a.e. constant in Ω. []

*** Theorem 5.** *(L^∞-Bound and De Giorgi-Nash) Let L be as above, $u \in H^1(\Omega)$ be such that $Lu = f_1 + \nabla \cdot f_2$ with $f_1 \in L^p(\Omega)$, $f_2 \in L^{2p}(\Omega)$ for some $p > N$. [Hence $f_1 + \nabla \cdot f_2 \in H^{-1}(\Omega)$.] Then $u \in L^\infty(\Omega) \cap C^{0,\alpha}(\Omega)$. Moreover, there exist constants $C > 0$ and $C_{\Omega'} > 0$ (for any $\Omega' \subset\subset \Omega$) such that, for any $f_1 \in L^p(\Omega)$ and any $f_2 \in L^{2p}(\Omega)$,*

$$\|u\|_{L^\infty(\Omega)} \leq C\big(\|u\|_{L^2(\Omega)} + \|f_1\|_{L^p(\Omega)} + \|f_2\|_{L^{2p}(\Omega)}\big),$$

$$\|u\|_{C^{0,\alpha}(\Omega')} \leq \tilde{C}_{\Omega'}\big(\|u\|_{L^2(\Omega)} + \|f_1\|_{L^p(\Omega)} + \|f_2\|_{L^{2p}(\Omega)}\big).$$

[]

The Galerkin Method.

Lemma X. *Let Ω be a bounded domain of class $C^{0,1}$, L be as in (3.15), symmetric and uniformly elliptic. Then there a exists a countable orthonormal basis $\{w_n\}$ of $L^2(\Omega)$ which consists of eigenvectors of L. The associated sequence of eigenvalues $\{\lambda_n\} \subset \mathbf{R}^+$ diverges.*

For any $m \in \mathbf{N}$, let V_m be the subspace of $L^2(\Omega)$ spanned by $\{\lambda_n : n = 1, ..., m\}$. Let us set $a_m := (f, w_m)$, $u_m := \sum_{n=1}^m \lambda_n^{-1} a_n w_n$, and $f_m := \sum_{n=1}^m a_n w_n$. Then $Lu_m = f_m$ in V_m', and $u_m \to u$ in V.

6 Variational Techniques of $\mathbf{L}^2$-Type

In this section we outline the variational approach to two simple initial- and boundary-value problems for the heat equation and for the wave equation, resp., in the framework of Sobolev spaces of Hilbert type.

Let Ω be a bounded domain of $\mathbf{R}^N$ $(N \geq 1)$ of class $C^{0,1}$. We denote its boundary by Γ, fix any $T > 0$ and set $Q := \Omega \times]0,T[$, $\Sigma := \Gamma \times]0,T[$. As above, we set $\Delta := \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$. We assume that $f_1 : Q \to \mathbf{R}$, $u^0, w^0 : \Omega \to \mathbf{R}$ are prescribed functions, and deal with the following Cauchy-Dirichlet problems

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f_1 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(\cdot, 0) = u^0 & \text{in } \Omega, \end{cases} \tag{6.1}$$

$$
\begin{cases}
\dfrac{\partial^2 u}{\partial t^2} - \Delta u = f_1 & \text{in } Q, \\
u = 0 \qquad \text{on } \Sigma, \\
u(\cdot, 0) = u^0, \quad \dfrac{\partial u}{\partial t}(\cdot, 0) = w^0 & \text{in } \Omega.
\end{cases} \tag{6.2}
$$

We set $V := H_0^1(\Omega)$ and $H := L^2(\Omega)$, and identify H with its topological dual H'. As the injection $V \to H$ is continuous and dense, in turn H' can be identified with a subspace of V'. This yields the *Hilbert triplet* $V \subset H = H' \subset V'$, with dense injections. We denote by ${}_{V'}\langle \cdot, \cdot \rangle_V$ the duality pairing between V' and V, and define the operator

$$
A \in \mathcal{L}(V; V'), \qquad {}_{V'}\langle Au, v \rangle_V := \int_\Omega \nabla u \cdot \nabla v \, dx \qquad \forall u, v \in V.
$$

Hence $Av = -\Delta v$ in $\mathcal{D}'(\Omega)$, and, after an obvious identification, $A : L^2(0, T; V) \to L^2(0, T; V')$ is linear and continuous.

Weak Formulation of the Heat Equation. We assume that

$$
u^0 \in H, \qquad f \in L^2(0, T; V'), \tag{6.3}
$$

and introduce a *weak formulation* of (6.1).

Problem 6.1 *To find $u \in L^2(0, T; V)$ such that*

$$
\begin{aligned}
\iint_Q \Big(- u \frac{\partial v}{\partial t} + \nabla u \cdot \nabla v \Big) \, dx dt = \int_0^T {}_{V'}\langle f, v \rangle_V \, dt + \int_\Omega u^0 v(\cdot, 0) \, dx \\
\forall v \in H^1(0, T; H) \cap L^2(0, T; V), v(\cdot, T) = 0.
\end{aligned} \tag{6.4}
$$

Interpretation. (6.4) yields

$$
\frac{\partial u}{\partial t} + Au = f \qquad \text{in } L^2(0, T; V'), \tag{6.5}
$$

whence $\partial u / \partial t = f - Au \in L^2(0, T; V')$. Therefore $u \in H^1(0, T; V')$, and (6.5) is satisfied in the sense of $L^2(0, T; V')$ $(= L^2(0, T; V)')$. Integrating (6.4) by parts in time and using (6.5), we then get

$$
u|_{t=0} = u^0 \qquad \text{in } V' \text{ (in the sense of traces of } H^1(0, T; V')). \tag{6.6}
$$

In turn (6.5) and (6.6) yield (6.4).

Theorem 6.1 *(Existence) If (6.3) holds, then Problem 1.1 has a solution such that $u \in L^\infty(0, T; H)$.*

Proof. The following argument is based on approximation – derivation of a priori estimates – passage to the limit. This classical may also be applied to a great number of nonlinear parabolic equations.

(i) *Approximation.* This problem may be approximated in many ways: by finite dimensional approximation *Faedo-Galerkin method,* by regularization, and so on.

We approximate our problem by (implicit) time discretization. One might also use other procedures, e.g.,

– complete space- and time-discretization.

– hyperbolic or elliptic regularization (this corresponds to adding a term $\varepsilon \frac{\partial^2 u}{\partial t^2}$ or $-\varepsilon \frac{\partial^2 u}{\partial t^2}$, and an initial or a final condition, resp.; eventually, one will then pass to the limit as $\varepsilon \to 0$);

– finite-dimensional approximation (see Sect. 8), or the *Faedo-Galerkin method.*

Let us fix any $m \in \mathbf{N}$, set $k := T/m$, $u_m^0 := u^0$ and

$$f_m^n := \frac{1}{k} \int_{(n-1)k}^{nk} f(\tau) d\tau \qquad \text{in } V', \text{ for } n = 1, \dots, m. \tag{6.7}$$

Problem 6.1$_\mathbf{m}$ *To find $u_m^n \in V$ for $n = 1, \dots, m$, such that*

$$\frac{u_m^n - u_m^{n-1}}{k} + A u_m^n = f_m^n \qquad \text{in } V', \text{ for } n = 1, \dots, m, \tag{6.8}$$

By the results of Sect. 5, this problem can be solved step by step in time; that is, u_m^n can be solved in terms of u_m^{n-1} for any n.

(ii) *A Priori Estimates.* Let us multiply (6.8) by $k u_m^n$ and sum for $n = 1, \dots, \ell$, for any $\ell \in \{1, \dots, m\}$. By the elementary inequality $2(a-b)a \geq a^2 - b^2$, valid for any $a, b \in \mathbf{R}$, we have

$$(u_m^n - u_m^{n-1}) u_m^n \geq \frac{1}{2} (u_m^\ell)^2 - \frac{1}{2} (u^0)^2 \qquad \text{a.e. in } Q. \tag{6.8$'$}$$

For any ℓ and any $\varepsilon > 0$, we then get

$$\begin{aligned} &\frac{1}{2} \int_\Omega [(u_m^\ell)^2 - (u^0)^2] \, dx + k \sum_{n=1}^{\ell} \int_\Omega |\nabla u_m^n|^2 \, dx \leq k \sum_{n=1}^{\ell} \|f_m^n\|_{V'} \|u_m^n\|_V \\ &\leq \frac{k}{2} \sum_{n=1}^{\ell} \Big(\frac{1}{\varepsilon} \|f_m^n\|_{V'}^2 + \varepsilon \|u_m^n\|_V^2 \Big) \leq \frac{k}{2\varepsilon} \sum_{n=1}^{\ell} \|f_m^n\|_{V'}^2 + \frac{\varepsilon C k}{2} \sum_{n=1}^{\ell} \int_\Omega |\nabla u_m^n|^2 \, dx. \end{aligned} \tag{6.9}$$

The Schwartz-Hölder inequality yields $k \sum_{n=1}^{\ell} \|f_m^n\|_{V'}^2 \leq \int_0^T \|f(t)\|_{V'}^2 \, dt$. Choosing $\varepsilon = 1/C$ in (6.9), we then have

$$\int_\Omega (u_m^\ell)^2 \, dx + k \sum_{n=1}^{\ell} \int_\Omega |\nabla u_m^n|^2 \, dx \leq \int_\Omega (u^0)^2 \, dx + C \|f\|_{L^2(0,T;V')}^2 \,, \tag{6.9$'$}$$

whence

$$\max_{n=1,\dots,m} \|u_m^n\|_H, k \sum_{n=1}^{m} \|u_m^n\|_V^2 \leq \text{Constant}. \tag{6.10}$$

In order to formulate a *continuous-time* approximate equation, let us set

$$u_m(x, \cdot) := \text{ linear time interpolate of } \{u_m(x, nk) := u_m^n(x)\}_{n=0,\dots,m}, \tag{6.10$'$}$$

$$\bar{u}_m(x,t) := u_m^n(x) \qquad \text{if } (n-1)k < t \leq nk, \text{ for } n = 1, \dots, m, \tag{6.11}$$

for almost any $x \in \Omega$, and define $\bar{f}_m$ similarly. The equation (6.8) and the uniform estimate (6.10) then read

$$\frac{\partial u_m}{\partial t} + A \bar{u}_m = \bar{f}_m \qquad \text{in } V', \text{ a.e. in }]0, T[, \tag{6.12}$$

$$\|u_m\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \leq \text{Constant (independent of } m). \tag{6.13}$$

Hence $A \bar{u}_m$ is uniformly bounded in $L^2(0, T; V')$, and comparing the terms of (6.12) we get

$$\|u_m\|_{H^1(0,T;V')} \leq \text{Constant}. \tag{6.14}$$

(iii) *Limit Procedure.* By the estimates (6.13), (6.14) and by classical compactness results, there exist $u, \bar{u}$ such that, possibly taking $m \to \infty$ along a subsequence, [5]

$$\bar{u}_m \to \bar{u} \qquad \text{weakly in } L^2(0,T;V), \tag{6.15}$$

$$u_m \to u \qquad \text{weakly star in } L^\infty(0,T;H) \cap H^1(0,T;V'). \tag{6.16}$$

As the injection $H^1(0,T) \to C^0([0,T])$ is compact, (16) entails that

$$\langle u_m - u, v\rangle \to 0 \qquad \text{uniformly in } [0,T], \forall v \in V. \tag{6.17}$$

By the Ascoli-Arzelà Theorem, the sequence $\{\langle u_m, v\rangle\}$ is equicontinuous; hence $\langle \bar{u}_m - u, v\rangle \to 0$ a.e. in $]0,T[$. Therefore $\langle \bar{u} - u, v\rangle = 0$ a.e. in $]0,T[$ for any $v \in V$, whence $\bar{u} = u$ a.e. in Q. Taking the limit in (6.12) we then get (6.5).

As $\langle u_m(0) - u^0, v\rangle \to 0$ for any $v \in V$; by (6.17) we also derive the initial condition (6.6). □

Theorem 6.2 *(Regularity) If*

$$u^0 \in V, \qquad f \in L^2(Q) \cap W^{1,1}(0,T;V'), \tag{6.18}$$

then Problem 1.1 has a solution such that

$$u \in H^1(0,T;H) \cap L^\infty(0,T;V). \tag{6.19}$$

If moreover $f \in L^2(Q)$ and Ω is of class C^2, then $u \in L^2\big(0,T;H^2(\Omega)\big)$.

Outline of the Proof. We still use the time-discretized Problem 1.1_m. Multiplying (6.8) by $u^n_m - u^{n-1}_m$ and summing for $n = 1, \dots, \ell$, for any $\ell \in \{1, \dots, m\}$, we get a uniform bound for u_m in $H^1(0,T;H) \cap L^\infty(0,T;V)$. This yields (6.18).

If moreover $f \in L^2(Q)$, then by comparing the terms of (6.12) we see that Δu_m is uniformly bounded in $L^2(Q)$. On account of the regularity of Ω, u_m is then uniformly bounded in $L^2\big(0,T;H^2(\Omega)\big)$. Therefore this regularity is preserved in the limit. □

Remark. One can prove that

$$\begin{aligned} &L^2(0,T;V) \cap H^1(0,T;V') \subset C^0([0,T];H),\\ &L^2\big(0,T;V \cap H^2(\Omega)\big) \cap H^1(0,T;H) \subset C^0([0,T];V). \qquad [] \end{aligned} \tag{6.20}$$

For any solution Problem 1.1 we then have $u \in C^0([0,T];H)$, and the final statement of Theorem 2 entails $u \in C^0([0,T];V)$.

Theorem 6.3 *(Continuous and Monotone Dependence on the Data) (i) For $i = 1, 2$, let $u^0_i \in H$ and $f_i \in L^2(0,T;V')$, and u_i be a corresponding solution of Problem 6.1. Then, defining the constant $C > 0$ as in the proof of Theorem 6.1,*

$$\begin{aligned} &\int_\Omega [u_1(x,t) - u_2(x,t)]^2\, dx + \int_0^t d\tau \int_\Omega |\nabla(u_1 - u_2)|^2\, dx\\ &\le \int_\Omega (u^0_1 - u^0_2)^2\, dx + C\int_0^t \|f_1(\tau) - f_2(\tau)\|^2_{V'}\, d\tau \end{aligned} \tag{6.22}$$

for a.a. $t \in]0,T[$.

---

[5] By this statement we mean that there exists a sequence $\{m_\ell\}_{\ell \in \mathbf{N}}$ such that, as $\ell \to \infty$, $\bar{u}_{m_\ell} \to \bar{u}$ and $u_{m_\ell} \to u$ as it is indicated in (6.15) and (6.16). For the sake of simplicity, however we assume that these subsequences have been relabelled so that we may indicate a single index, which we still denote by m. Formally, this is tantamount to being in the conditions of extracting no subsequence.

(ii) The solution of Problem 6.1 is unique.
(iii) If $f_1 \le f_2$ in $\mathcal{D}'(Q)$ and $u_1^0 \le u_2^0$ a.e. in Ω, then $u_1 \le u_2$ a.e. in Q.

Outline of the Proof. (i) Setting $\tilde u := u_1 - u_2$, $\tilde f := f_1 - f_2$, $\tilde u^0 := u_1^0 - u_2^0$, (5) yields

$$\begin{cases} \dfrac{\partial \tilde u}{\partial t} - \Delta \tilde u = \tilde f & \text{in } V', \text{ a.e. in }]0,T[, \\ \gamma_0 \tilde u = 0 & \text{on } \Sigma, \\ \tilde u(\cdot,0) = \tilde u^0 & \text{in } \Omega. \end{cases} \tag{6.23}$$

Multiplying $(6.23)_1$ by $\tilde u$, integrating in time, and proceding as in the proof of Theorem 1 for the derivation of the a priori estimates, we then get (6.22).

In particular, the following formula is here applied: for any $v \in L^2(0,T;V) \cap H^1(0,T;V')$,

$$\int_0^{\hat t} \langle \frac{\partial v}{\partial t}, v \rangle \, dt = \frac{1}{2} \|v(\cdot,\hat t)\|^2_{L^2(\Omega)} - \frac{1}{2} \|v(\cdot,0)\|^2_H \qquad \forall \hat t \in]0,T[. \tag{6.24}$$

This equality is easily checked for any $v \in H^1(0,T;H) \cap L^2(0,T;V)$, and is then extended by density.

(ii) Uniqueness directly follows from (6.22).

(iii) Finallly, multiplying $(6.23)_1$ by $\tilde u^+$ ($\in L^2(0,T;V)$, by Stampacchia's Proposition XXX) and using the above procedure, one can prove that $\tilde u^+ = 0$ a.e. in Q. $\square$

The above developments can be extended

— to other boundary conditions, of the sort introduced in Sect. 1;

— to more general parabolic operators with the elliptic part in divergence form;

— to unbounded domains Ω. In this case the Friedrichs inequality (Corollary X.4.6) does not hold, and in place of (6.9) we get

$$\begin{aligned} &\frac{1}{2} \int_\Omega [(u_m^\ell)^2 - (u^0)^2] \, dx + k \sum_{n=1}^{\ell} \int_\Omega |\nabla u_m^n|^2 \, dx \\ &\le \frac{k}{2} \sum_{n=1}^{\ell} \|f_m^n\|^2_{V'} + \frac{k}{2} \sum_{n=1}^{\ell} \int_\Omega [(u_m^n)^2 + |\nabla u_m^n|^2] \, dx, \end{aligned} \tag{6.25}$$

or equivalently, using the notation (6.10)' and (6.11),

$$\begin{aligned} &\frac{1}{2} \int_\Omega [u_m(\cdot,t)^2 - (u^0)^2] \, dx + \int_0^t d\tau \int_\Omega |\nabla u_m|^2 \, dx \\ &\le \frac{1}{2} \int_0^t \|f_m\|^2_{V'} \, d\tau + \frac{1}{2} \int_0^t d\tau \int_\Omega [(u_m)^2 + |\nabla u_m|^2] \, dx. \end{aligned} \tag{6.26}$$

An a priori estimate of the form (6.10) then follows from the following classical result.

Lemma 6.4 *(Gronwall's Lemma) Let $0 < T \le +\infty$, and φ, α, $\beta : [0,T[\to \mathbf{R}$ be continuous functions, with α nondecreasing and $\beta \ge 0$. If*

$$\varphi(t) \le \alpha(t) + \int_0^t \beta(\tau)\varphi(\tau) d\tau \qquad \forall t \in [0,T[, \tag{6.27}$$

then

$$\varphi(t) \le \alpha(t) \exp\left(\int_0^t \beta(\tau) d\tau \right) \qquad \forall t \in [0,T[. \tag{6.28}$$

[]

(It is easy to see that the assumption (6.27) may be replaced by the weaker condition $\varphi(t) \leq \alpha(t) + \int_0^t \beta(\tau) \max_{[0,\tau]} |\varphi| d\tau$ for any $t \in [0, T[$.)

7 Variational Techniques.........
Weak Formulation of the Wave Equation. Let us assume that

$$u^0, w^0 \in H, \qquad f \in L^2(0,T;V'). \tag{7.1}$$

Problem 7.2 *To find $u \in H^1(0,T;H) \cap L^2(0,T;V)$ such that*

$$\begin{aligned}
&\iint_Q \Big(-\frac{\partial u}{\partial t}\frac{\partial v}{\partial t} + \nabla u \cdot \nabla v \Big)\, dxdt = \int_0^T {}_{V'}\langle f, v\rangle_V \, dt + \int_\Omega w^0 v(\cdot,0)\, dx \\
&\forall v \in H^1(0,T;H) \cap L^2(0,T;V), v(\cdot,T) = 0, \\
&u(\cdot,0) = u^0 \qquad \textit{a.e. in } \Omega.
\end{aligned} \tag{7.2}$$

Interpretation. (7.2) yields

$$\frac{\partial^2 u}{\partial t^2} + Au = f \qquad \text{in } \mathcal{D}'(Q), \tag{7.3}$$

whence $\frac{\partial^2 u}{\partial t^2} = f - Au \in L^2(0,T;V')$. Therefore $u \in H^2(0,T;V')$, and (7.3) is satisfied in the sense of $L^2(0,T;V')$. Integrating (7.2) in time by parts, we then get

$$\frac{\partial u}{\partial t}\Big|_{t=0} = w^0 \qquad \text{in } V' \text{ (in the sense of traces of } H^1(0,T;V')). \tag{7.4}$$

In turn (7.3) and (7.4) yield (7.2). (The initial condition $u(\cdot,0) = u^0$ is obviously meaningful.)

Theorem 7.3 *(Existence) Assume that*

$$u^0, w^0 \in H, \qquad f = f_1 + f_2, \quad f_1 \in L^1(0,T;H), \quad f_2 \in L^2(0,T;V'). \tag{7.1}$$

Then Problem 1.2 has a solution such that

$$u \in W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V). \tag{7.5}$$

Proof. (i) *Approximation.* Let us fix any $m \in \mathbf{N}$, set $k := T/m$, $u_m^0 := u^0$, $u_m^1 := w^0$, and define f_m^n as in (6.7). We approximate our problem by *implicit time-discretization.*

Problem 7.2$_\mathbf{m}$ *To find $u_m^n \in V$ for $n = 1, \dots, m$, such that, setting $w_m^n := (u_m^n - u_m^{n-1})/k$,*

$$\frac{w_m^n - w_m^{n-1}}{k} + Au_m^n = f_m^n \qquad \textit{in } V', \textit{ for } n = 1, \dots, m. \tag{7.6}$$

This problem can be solved step by step in time.

(ii) *A Priori Estimates.* Let us multiply (7.6) by kw_m^n and sum for $n = 1, \dots, \ell$, for any $\ell \in \{1, \dots, m\}$. Similarly to (6.8)' here we have

$$\left.\begin{aligned}
&(w_m^n - w_m^{n-1})w_m^n \geq \frac{1}{2}(w_m^n)^2 - \frac{1}{2}(w_m^{n-1})^2 \\
&\nabla(u_m^n - u_m^{n-1}) \cdot \nabla u_m^n \geq \frac{1}{2}|\nabla u_m^n|^2 - \frac{1}{2}|\nabla u_m^{n-1}|^2
\end{aligned}\right\} \quad \text{a.e. in } Q, \forall n. \tag{7.7}$$

We then get

$$\begin{aligned} &\frac{1}{2}\int_{\Omega}[(w_m^{\ell})^2-(w_m^0)^2]\,dx+\frac{1}{2}\int_{\Omega}\left(|\nabla u_m^n|^2-|\nabla u_m^0|^2\right)dx \\ &\le \sum_{n=1}^{\ell}\int_{\Omega} f_{1m}^n w_m^n\,dx+\sum_{n=1}^{\ell}\langle f_{2m}^n, w_m^n\rangle. \end{aligned} \tag{7.8}$$

Moreover, defining $\bar{f}_{1m}$ and f_{2m} similarly to () and (), we have

$$\sum_{n=1}^{\ell}\int_{\Omega} f_{1m}^n w_m^n\,dx \le C_1\|\bar{f}_{1m}\|_{L^1(0,T;H)}\max_{n=0,\dots,\ell}\|w_m^n\|_H\ ,$$

$$\begin{aligned} \sum_{n=1}^{\ell}\langle f_{2m}^n, u_m^n-u_m^{n-1}\rangle &\le \langle f_{2m}^{\ell}, u_m^{\ell}\rangle-\langle f_{2m}^1, u^0\rangle-\sum_{n=1}^{\ell}\langle f_{2m}^n-f_{2m}^{n-1}, u_m^{n-1}\rangle \\ &\le C_2\|f_{2m}\|_{W^{1,1}(0,T;V')}\max_{n=0,\dots,\ell}\|u_m^n\|_V. \end{aligned}$$

We then get

$$\max_{n=1,\dots,m}\|w_m^n\|_H, k\sum_{n=1}^{m}\|u_m^n\|_V^2 \le \text{Constant (independent of } m). \tag{7.9}$$

Let us define u_m, $\bar{u}_m$, $\bar{f}_m$ ... as above. The equation (7.3) and the uniform estimate (7.9) then read

$$\frac{\partial^2 u_m}{\partial t^2}+A\bar{u}_m=\bar{f}_m \qquad \text{in } V',\ \text{a.e. in }]0,T[, \tag{7.10}$$

$$\|u_m\|_{W^{1,\infty}(0,T;H)\cap L^{\infty}(0,T;V)}\le \text{Constant}. \tag{7.11}$$

Hence $A\bar{u}_m$ is uniformly bounded in $L^{\infty}(0,T;V')$, and by comparing the terms of (7.10) we get

$$\|u_m\|_{W^{2,\infty}(0,T;V')}\le \text{Constant}. \tag{7.12}$$

(iii) *Limit Procedure.* By (7.11), (7.12) and by classical compactness results, there exist $\bar{u}, u$ such that, possibly taking $m\to\infty$ along a subsequence,

$$\bar{u}_m \to \bar{u} \qquad \text{weakly in } L^{\infty}(0,T;V), \tag{7.13}$$

$$u_m \to u \qquad \text{weakly star in } L^{\infty}(0,T;H)\cap H^1(0,T;V'). \tag{7.14}$$

As we saw in the proof of Theorem 1, $\bar{u}=u$ a.e. in Q. Hence by taking the limit in (7.10) we get (7.3).

By (7.14) we also have

$$u_m \to u \qquad \text{strongly in } C^0([0,T];H); \tag{7.15}$$

the initial condition $u_m(\cdot,0)=u^0$ is thus preserved in the limit. □

For any reflexive Banach space B, let us denote by $C^0_w([0,T];B)$ the space of weakly measurable functions $u:[0,T]\to B$; by this we mean that $\langle u,v\rangle\in C^0([0,T])$ for any $v\in B'$. $C^0_w([0,T];B)$ is a Banach subspace of $L^{\infty}(0,T;B)$.

Proposition 7.2 *Let B_0, B_1 be Banach space, and B_1 be reflexive. Then*

$$L^{\infty}(0,T;B_1)\cap C^0([0,T];B_0)=C^0_w([0,T];B_1). \tag{7.16}$$

Proof. .....

By the latter Proposition, as $u \in H^2(0,T;H)$, (7.5) entails that

$$\frac{\partial u}{\partial t} \in C^0_w\big([0,T];H\big), \qquad u \in C^0_w([0,T];V). \tag{7.17}$$

Theorem 7.3 *(Continuous Dependence on the Data) For $i = 1,2$, let $u_i^0 \in H$, $w_i^0 \in H$ and $f_i \in L^2(0,T;V')$, and u_i be a corresponding solution of Problem 7.1. Then*

$$\begin{aligned}&\int_\Omega \Big[\Big|\frac{\partial(w_1 - w_2)}{\partial t}\Big|^2(x,t) + |\nabla(u_1 - u_2)|^2(x,t)\Big]\, dx \\ &\le \int_\Omega \big[(u_1^0 - u_2^0)^2 + (w_1^0 - w_2^0)^2\big]\, dx + C\int_0^t \|f_1(\tau) - f_2(\tau)\|^2_{V'}\, d\tau\end{aligned} \tag{7.18}$$

for a.a. $t \in]0,T[$.

Therefore the solution of Problem 7.1 is unique.

Proof. Setting $\tilde u := u_1 - u_2$, $\tilde f := f_1 - f_2$, $\tilde u^0 := u_1^0 - u_2^0$, () yields

$$\begin{cases} \dfrac{\partial^2 \tilde u}{\partial t^2} - \Delta \tilde u = \tilde f \qquad \text{in } V', \text{ a.e. in }]0,T[, \\ \gamma_0 \tilde u = 0 \qquad \text{on } \Sigma, \\ \tilde u(\cdot,0) = \tilde u^0, \quad \dfrac{\partial \tilde u}{\partial t}(\cdot,0) = w^0 \qquad \text{in } V'. \end{cases} \tag{7.19}$$

Multiplying $(7.19)_1$ by $\tilde u$ and integrating in time, we then get (7.18). □

The above developments can be extended to other boundary conditions, and to more general hyperbolic operators in divergence form.

The above developments can be extended

— to other boundary conditions, of the sort introduced in Sect. 1;

— to more general hyperbolic operators with the elliptic part in divergence form (the symmetry of the matrix A being essential);

— to unbounded domains Ω.