

Elements of Convex Analysis

1 The Legendre Transform. omissis

2 Convex Lower Semicontinuous Functions.

We deal with a real Banach space B , although most of the results of this chapter also hold for any real separated locally convex space. By $\langle \cdot, \cdot \rangle$ we denote the duality pairing between B and its topological dual B^* , as well as that between B^* and B . If not otherwise specified, we refer to the *strong* topology.

Proposition 2.0 (i) Any set $K \subset B$ is closed and convex iff I_K is closed and convex.

(ii) Any function $f : B \rightarrow]-\infty, +\infty]$ is lower semicontinuous and convex iff $\text{epi}(f)$ is closed and convex.

(iii) If $\{K_i\}_{i \in I}$ is a family of closed convex subsets of B , then $\bigcap_i K_i$ is closed and convex.

(iv) If $\{f_i\}_{i \in I}$ is a family of lower semicontinuous convex functions $B \rightarrow]-\infty, +\infty]$, then their upper hull $f(\cdot) := \sup_i f_i(\cdot)$ is lower semicontinuous and convex.

• **Proposition 2.1** (i) Any convex set $K \subset B$ is strongly closed iff it is weakly closed.

(ii) Any convex function $F : B \rightarrow]-\infty, +\infty]$ is strongly lower semicontinuous iff it is weakly lower semicontinuous.

Proof. Let K be a strongly closed subset of B . If $u \notin K$, then by a corollary of the Hahn-Banach theorem the compact set $\{u\}$ can be strongly separated from K by means of a closed hyperplane. Hence u does not belong to the weak closure of K , and this proves the first statement.

The second statement then follows from part (ii) of Proposition 2.0. □

Remark. Of course the same result also holds for B^* . However, if B is not reflexive, a subset of B^* may be convex and weakly (equivalently, strongly) closed without being weakly star closed. For instance, this occurs for the half-space $\{u^* \in B^* : \langle u^{**}, u^* \rangle \geq 0\}$ for any $u^{**} \in B^{**} \setminus B$.⁽¹⁾

In order to avoid any reference to B^{**} , we deal with the duality between B^* and B , rather than that between B^* and B^{**} . This can be expressed by saying that B and B^* are regarded as *spaces in duality*. We shall then often use the weak star topology instead of the weak one.

The argument of Proposition 2.1 also yields the following result.

• **Proposition 2.2** (i) Any set $K \subset B$ ($K \neq B$) is convex and closed iff it is the intersection of a (nonempty) family of closed half-spaces.

(ii) Any function $F : B \rightarrow]-\infty, +\infty]$ is lower semicontinuous and convex iff it is the upper hull of a (nonempty) family of continuous affine functions.⁽²⁾ []

(Part (i) easily follows from the Hahn-Banach theorem.)

We denote by $\Gamma(B)$ the class of convex lower semicontinuous functions $B \rightarrow]-\infty, +\infty]$, and by $\Gamma_0(B)$ the subclass of functions not identically equal to $+\infty$.

For any set $K \subset B$, the smallest closed convex subset of B which contains K is called the *closed convex hull* of K . It coincides with the intersection of all the closed convex subsets of B which contain K . By Proposition 2.2(i), it also coincides with the intersection of the closed half-spaces which contain K . As the closure of any convex set is convex, that hull also coincides with $\overline{\text{co}}(K)$, namely, the closure of the convex hull of K . (But it may not coincide with the convex hull of the

⁽¹⁾ Here $\langle \cdot, \cdot \rangle$ represents the duality pairing between B^{**} and B^* .

⁽²⁾ In passing note that the convex set B is the intersection of the empty family, and $F \equiv -\infty$ is the upper hull of the empty family of functions.

closure of K , for this may not be closed. The set $K := \{(0, 0)\} \cup \{(x, y) \in (\mathbf{R}^+)^2 : xy \geq 1\}$ is a counterexample.)

Similarly, let us consider any function $F : B \rightarrow]-\infty, +\infty]$ which has a convex and lower semi-continuous lower bound. By Proposition 2.2(ii), F has then a continuous affine lower bound, and the upper hull of these lower bounds is the largest lower bound of F in $\Gamma(B)$. It is called the Γ -regularized function of F , and its epigraph coincides with the closed convex hull of the epigraph of F . \square

Regularity Properties of Convex Functions. Convexity is a source of regularity, as it is shown by the following classical result.

* **Theorem 2.2'.** (Alexandrov) *If $F : \mathbf{R}^N \rightarrow \mathbf{R}$ is convex, then it is twice differentiable a.e. in \mathbf{R}^N .* \square

* **Theorem 2.3** *Let $F : B \rightarrow]-\infty, +\infty]$ be convex and upperly bounded in a (nonempty) open set $A \subset B$. Then F is locally Lipschitz continuous in $\text{int}(\text{Dom}(F))$.* ⁽²⁾ \square

Whenever F is convex and continuous at a point, the latter result applies.

Corollary 2.4 *If $F : \mathbf{R}^N \rightarrow]-\infty, +\infty]$ is convex, then it is locally Lipschitz continuous in $\text{int}(\text{Dom}(F))$.*

Proof. $\text{int}(\text{Dom}(F))$ contains an N -simplex S , i.e., a set of convex combinations of $N + 1$ affinely independent points of $\text{int}(\text{Dom}(F))$. By the convexity, F is bounded in the interior of S , which is nonempty. It then suffices to apply the latter theorem. \square

• **Corollary 2.5** *If $F \in \Gamma_0(B)$, then it is locally Lipschitz continuous in $\text{int}(\text{Dom}(F))$.* \square

Exercises. (i) Prove that the interior and the closure of any convex subset of B are convex.

(ii) Show that $\text{co}(\bar{K}) \subset \bar{\text{co}}(K)$. Check that the opposite inclusion fails for $K := (\mathbf{R} \times \{1\}) \cup \{(0, 0)\}$.

(iii) Let $F : B \rightarrow]-\infty, +\infty]$ be convex. Show that F is continuous at $u_0 \in \text{Dom}(B)$ whenever it is upper semicontinuous at the same point u_0 .

Hint: Apply Theorem 2.3.

(iv) Let $F : B \rightarrow]-\infty, +\infty]$ be convex. Show that any point of relative minimum for F is also of absolute minimum. \square

3 The Fenchel Transform.

Let $F : B \rightarrow]-\infty, +\infty]$ be proper (i.e., with nonempty domain). We define the *conjugate function* ⁽²⁾

$$F^*(u^*) := \sup_{u \in B} \{\langle u^*, u \rangle - F(u)\} \quad (= \sup\{\langle u^*, u \rangle - r : (u, r) \in \text{epi}(F)\}) \quad \forall u^* \in B^*. \quad (3.1)$$

As B^* is a Banach space, for any proper function $G : B^* \rightarrow]-\infty, +\infty]$, the conjugate function $G^* : B^{**} \rightarrow]-\infty, +\infty]$ might be defined by using the duality pairing between B^* and its dual B^{**} . However, it seems more convenient to deal with the duality pairing between B^* and B , as we did above. We then restrict G^* to B : ⁽³⁾

$$G^*(u) := \sup_{u^* \in B^*} \{\langle u^*, u \rangle - G(u^*)\} \quad \forall u \in B. \quad (3.2)$$

⁽²⁾ By $\text{int}(A)$ we denote the interior of the set A .

⁽²⁾ This has nothing to do with the transposed L^* of a linear operator L .

⁽³⁾ No confusion should arise by denoting this restriction by G^* .

If F is as above and F^* is also proper, we introduce the *biconjugate function* of F :

$$F^{**}(u) := \sup_{u^* \in B^*} \{\langle u^*, u \rangle - F^*(u^*)\} \quad (= \sup\{\langle u^*, u \rangle - r : (u^*, r) \in \text{epi}(F^*)\}) \quad \forall u \in B. \quad (3.3)$$

Similarly, if G^* is proper, we define the biconjugate function of a proper function $G : B^* \rightarrow]-\infty, +\infty]$:

$$G^{**}(u^*) := \sup_{u \in B} \{\langle u^*, u \rangle - G^*(u)\} \quad \forall u^* \in B^*. \quad (3.4)$$

Note that

$$F^*(0) = - \inf_{u \in B} F(u), \quad F^{**}(0) = - \inf_{u^* \in B^*} F^*(u^*). \quad (3.5)$$

For any proper function $F : B \rightarrow]-\infty, +\infty]$, F^* is proper iff F has a continuous affine lower bound.

We may then summarize the above definitions as follows:

- (i) If $F : B \rightarrow]-\infty, +\infty]$ is proper, then $F^* : B^* \rightarrow]-\infty, +\infty]$ is defined as in (3.1).
- (ii) If $G : B^* \rightarrow]-\infty, +\infty]$ is proper, then $G^* : B^{**} \rightarrow]-\infty, +\infty]$ is defined. Most often one restricts G^* to B .
- (iii) If $F^* : B^* \rightarrow]-\infty, +\infty]$ is proper (i.e., if F has a continuous affine lower bound), then $(F^*)^* : B^{**} \rightarrow]-\infty, +\infty]$ is defined. Most often one restricts $(F^*)^*$ to B ; we set

$$F^{**} := (F^*)^*|_B.$$

- (iv) If $G^*|_B : B \rightarrow]-\infty, +\infty]$ is proper (i.e., if G has a continuous affine lower bound), then $(G^*|_B)^* : B^* \rightarrow]-\infty, +\infty]$ is defined.

Here is an application of the above setting to economics.

“If we interpret the vector space B as a *commodity space* and accordingly its dual B^* as a *price space*, and if we interpret $F : B \rightarrow]-\infty, +\infty]$ as a *cost function* that associates to every commodity $u \in B$ its cost $F(u) \in]-\infty, +\infty]$, then the conjugate function F^* can be interpreted as a *profit function* that associates to every price $u^* \in B^*$ the maximum profit $F^*(u^*) = \sup_{u \in B} \{\langle u^*, u \rangle - F(u)\}$ (since $\langle u^*, u \rangle$ is the value of u when the price system u^* prevails).”⁽⁴⁾

• **Theorem 3.1** *Let $F : B \rightarrow]-\infty, +\infty]$ be proper and admit a continuous affine lower bound. Then:*

- (i) $F^* \in \Gamma_0(B^*)$ and $F^{**} \in \Gamma_0(B)$. F^* is even weakly star lower semicontinuous.
- (ii) (Fenchel-Moreau theorem) F^{**} is the Γ -regularized function of F ; that is,

$$F^{**}(u) = \sup\{\langle u^*, u \rangle + \alpha : u^* \in B^*, \alpha \in \mathbf{R}, \langle u^*, u \rangle + \alpha \leq F(u)\} \quad \forall u \in B, \quad (3.5)'$$

or also $\text{epi}(F^{**}) = \overline{\text{co}}(\text{epi}(F))$. [Therefore $F^{**} = F$ whenever $F \in \Gamma_0(B)$.]

- (iii) (Fenchel's inequality and dual Fenchel's inequality)

$$\begin{aligned} \langle u^*, u \rangle &\leq F(u) + F^*(u^*) & \forall u \in \text{Dom}(F), \forall u^* \in \text{Dom}(F^*), \\ \langle u^*, u \rangle &\leq F^{**}(u) + F^*(u^*) & \forall u \in \text{Dom}(F^{**}), \forall u^* \in \text{Dom}(F^*). \end{aligned} \quad (3.6)$$

Proof. (i) is a consequence of part (ii) of Proposition 2.2.

Let us now come to the proof of (3.5'). Although in this formula u is kept fixed and u^*, α are varied, here we fix any $u^* \in \text{Dom}(F^*)$ and $\alpha \in \mathbf{R}$, and let u vary. By definition of $F^*(u^*)$ we have

$$\langle u^*, u \rangle + \alpha \leq F(u) \quad \forall u \in B \quad \Leftrightarrow \quad \alpha \leq -F^*(u^*).$$

⁽⁴⁾ From J.-P. Aubin: Applied Functional Analysis, 1979, p. 211.

That is, the function $L_{u^*} : u \mapsto \langle u^*, u \rangle - F^*(u^*)$ is a continuous and affine lower bound of F , and its constant term, $-F^*(u^*)$, is maximal in the family of these lower bounds. This family of functions is parameterized by u^* ; its upper hull, F^{**} , is the Γ -regularized function of F , by definition of Γ -regularization. (ii) thus holds.

The inequalities (3.6) directly follows from the definitions of F^* and F^{**} . \square

*** Proposition 3.2** *Let X_1, X_2 be vector spaces over \mathbf{R} and $f : X_1 \times X_2 \rightarrow]-\infty, +\infty]$. Let us define the infimal value function $g(u) := \inf_{v \in X_2} f(u, v)$ for any $u \in X_1$. ⁽⁵⁾ If f is convex (quasi-convex, resp.) then g is also convex (quasi-convex, resp.).*

Proof. For $i = 1, 2$, let us fix any $u_i \in \text{Dom}(g)$, any $v_i \in \text{Dom}(f(u_i, \cdot))$, and any $\lambda \in]0, 1[$. If f is convex we have

$$\begin{aligned} g(\lambda u_1 + (1 - \lambda)u_2) &\leq f(\lambda u_1 + (1 - \lambda)u_2, \lambda v_1 + (1 - \lambda)v_2) \\ &= f(\lambda(u_1, v_1) + (1 - \lambda)(u_2, v_2)) \leq \lambda f(u_1, v_1) + (1 - \lambda)f(u_2, v_2). \end{aligned}$$

By taking the infimum with respect to (v_1, v_2) we then get $g(\lambda u_1 + (1 - \lambda)u_2) \leq \lambda g(u_1) + (1 - \lambda)g(u_2)$.

Similarly, if f is quasi-convex we have

$$\begin{aligned} g(\lambda u_1 + (1 - \lambda)u_2) &\leq f(\lambda u_1 + (1 - \lambda)u_2, \lambda v_1 + (1 - \lambda)v_2) \\ &= f(\lambda(u_1, v_1) + (1 - \lambda)(u_2, v_2)) \leq \max\{f(u_1, v_1), f(u_2, v_2)\}. \end{aligned}$$

By taking the infimum with respect to (v_1, v_2) we get $g(\lambda u_1 + (1 - \lambda)u_2) \leq \max\{g(u_1), g(u_2)\}$. \square

*** The Infimal Convolution.** Let X be a vector space over \mathbf{R} . For any $F, G : X \rightarrow]-\infty, +\infty]$, we define the *infimal convolution* of F and G by

$$(F \nabla G)(u) := \inf_{v, w \in X: v+w=u} \{F(v) + G(w)\} \quad (= \inf_{v \in X} \{F(v) + G(u - v)\}) \quad \forall u \in X. \quad (3.7)$$

(The analogy with the usual convolution product is obvious.) This operation is commutative and associative. [Ex] We have [Ex]

$$\text{Dom}(F \nabla G) = \text{Dom}(F) + \text{Dom}(G), \quad F \nabla I_{\{0\}} = F \quad \forall F, G : X \rightarrow]-\infty, +\infty],$$

$$\text{Dom}(F \nabla G) = \text{Dom}(F) + \text{Dom}(G), \quad I_A \nabla I_B = I_{A+B} \quad \forall A, B \subset X,$$

$$\text{if } F_1 \leq F_2 \text{ in } X, \text{ then } F_1 \nabla G \leq F_2 \nabla G \text{ in } X \quad \forall F_1, F_2, G : X \rightarrow]-\infty, +\infty],$$

$$\text{epi}(F) + \text{epi}(G) \subset \text{epi}(F \nabla G) \quad \forall F, G : X \rightarrow]-\infty, +\infty],$$

$$F \nabla G \text{ is convex} \quad \forall \text{convex } F, G : X \rightarrow]-\infty, +\infty].$$

*** Proposition 3.3** (i) *If $F, G : B \rightarrow]-\infty, +\infty]$ are proper, then $(F \nabla G)^* = F^* + G^*$.*

(ii) *If $F, G \in \Gamma_0(B)$ and either F or G is continuous at some point $u_0 \in \text{Dom}(F) \cap \text{Dom}(G)$, then $(F + G)^* = F^* \nabla G^*$. []*

Part (i) is easily proved. For any $u^* \in B^*$ we have

$$\begin{aligned} (F \nabla G)^*(u^*) &= \sup_{u \in B} \{\langle u^*, u \rangle - \inf_{w \in B} \{F(w) + G(u - w)\}\} \\ &= \sup_{u, w \in B} \{\langle u^*, w \rangle + \langle u^*, u - w \rangle - F(w) - G(u - w)\} \\ &= \sup_{w \in B} \{\langle u^*, w \rangle - F(w)\} + \sup_{v \in B} \{\langle u^*, v \rangle - G(v)\} = F^*(u^*) + G^*(u^*). \end{aligned}$$

⁽⁵⁾ Notice that g may attain the value $-\infty$.

If $F, G \in \Gamma_0(B)$, then $(F^* \nabla G^*)^{**} = (F^{**} + G^{**})^* = (F + G)^*$ by part (i). The further assumption of part (ii) allows one to show that $(F^* \nabla G^*)^{**} = F^* \nabla G^*$. \square

Exercises. (i) Show that $I_{\text{co}(K)} = \text{co}(I_K)$.

(ii) Exhibit a function $f : \mathbf{R} \rightarrow]-\infty, +\infty]$ which has no convex lower bound.

(iii) Set $f(u) := u^2$ for any $u \in \mathbf{R} \setminus \{0\}$, $f(0) := 1$. Check that $\text{epi}(\text{co}(f)) \neq \text{co}(\text{epi}(f))$. (Notice that there exists no function $g : \mathbf{R} \rightarrow]-\infty, +\infty]$ such that $\text{epi}(g) = \text{co}(\text{epi}(f))$).

(iv) Check that $f : \mathbf{R} \rightarrow]-\infty, +\infty]$ is convex iff $\text{Dom}(f)$ is convex and $f|_{\text{Dom}(f)}$ is also convex.

(v) Check that the pointwise limit of a sequence of convex functions is convex.

(vi) Let K be a convex subset of a Banach space, equipped with a norm $\|\cdot\|$, and define the distance function, $d_K(u) := \inf\{\|u - v\| : v \in K\}$ for any $u \in B$. Check that $I_K \nabla \|\cdot\| = d_K$.

(vii) Prove the characterization (4.3) of quasi-convex functions.

(viii) The sum of two quasi-convex functions is necessarily quasi-convex?

(ix) Let K be a subset of a real vector space. Does $\text{co}(K)$ coincide with the set of convex combinations of *pairs* of elements of K ?

Exercises. (i) Let $p, q \in]1, +\infty[$, $1/p + 1/q = 1$. Check that the function $\mathbf{R} \rightarrow \mathbf{R} : v^* \mapsto |v^*|^q/q$ is the conjugate of the function $v \mapsto |v|^p/p$. The Fenchel inequality (3.6)₁ then generalizes the classical *Young inequality*: $uu^* \leq |u|^p/p + |u^*|^{p'}/p'$ for any $u, u^* \in \mathbf{R}$.

Let Ω be an open subset of \mathbf{R}^N . Check that the functional $L^q(\Omega) \rightarrow \mathbf{R} : v^* \mapsto (1/q) \int_{\Omega} |v^*|^q dx$ is the conjugate of the functional $L^p(\Omega) \rightarrow \mathbf{R} : v \mapsto (1/p) \int_{\Omega} |v|^p dx$.

(ii) Check that (3.6) is equivalent to

$$\begin{aligned} \langle u^*, u \rangle &\leq r + s & \forall (u, r) \in \text{epi}(F), \forall (u^*, s) \in \text{epi}(F^*), \\ \langle u^*, u \rangle &\leq r + s & \forall (u, r) \in \text{epi}(F^{**}), \forall (u^*, s) \in \text{epi}(F^*); \end{aligned}$$

moreover, for instance,

$$\text{epi}(F^*) = \{(u^*, s) \in B^* \times \mathbf{R} : \langle u^*, u \rangle \leq r + s, \forall (u, r) \in \text{epi}(F)\}.$$

(iii) Under the assumptions of Theorem 3.1, prove that:

(a) $F \leq G$ entails $G^* \leq F^*$ (whenever G fulfils the conditions which we assumed for F);

(b) $F^{***} := (F^{**})^* = (F^*)^{**} = F^*$;

(iv) Let B be a Banach space equipped with the norm $\|\cdot\|$, and let B^* be equipped with the dual norm $\|\cdot\|_*$. Let $\varphi \in \Gamma_0(\mathbf{R})$ be even, set $F(u) := \varphi(\|u\|)$ for any $u \in B$, and $G(u^*) := \varphi^*(\|u^*\|_*)$ for any $u^* \in B^*$. Show that $F^* = G$.

* (v) Let F be convex. Show that if F is lower semicontinuous at $u \in B$, then $F(u) = F^{**}(u)$.

* (vi) Let B be a Banach space, $u_0 \in B$, set $F(u) := \|u - u_0\|$ for any $u \in B$ and denote the unit ball of B^* by K^* . Check that $F^*(u^*) = I_{K^*}(u^*) + \langle u^*, u_0 \rangle$ for any $u^* \in B^*$.

Let us then set $F_c(u) := F(cu)$ for any $u \in B$ and any $c \in \mathbf{R}$. How can F_c^* be represented? \square

4 The Subdifferential.

Let $F : B \rightarrow]-\infty, +\infty]$ be proper. We define its *subdifferential*, ∂F , as follows:

$$\partial F(u) := \{u^* \in B^* : \langle u^*, u - v \rangle \geq F(u) - F(v), \forall v \in B\} \quad \forall u \in \text{Dom}(F). \quad (4.1)$$

$\partial F(u) = \emptyset$ is not excluded, and we set $\partial F(u) = \emptyset$ for any $u \in B \setminus \text{Dom}(F)$. We also define the (*effective*) *domain* of ∂F by $\text{Dom}(\partial F) := \{u \in B : \partial F(u) \neq \emptyset\}$, and say that F is subdifferentiable at u iff $\partial F(u) \neq \emptyset$. The elements of $\partial F(u)$ are called *subgradients* of F at u .

The condition (4.1) means that:

(i) the continuous and affine function $L : v \mapsto \langle u^*, v - u \rangle + F(u)$ is a lower bound of F , and

(ii) L is *exact* at u , that is, $L(u) = F(u)$.

If $F^* : B^* \rightarrow]-\infty, +\infty]$ is proper, we set

$$\partial F^*(u^*) := \{u \in B : \langle u, u^* - v^* \rangle \geq F^*(u^*) - F^*(v^*), \forall v^* \in B^*\} \quad \forall u^* \in \text{Dom}(F^*), \quad (4.2)$$

and $\partial F^*(u^*) = \emptyset$ for any $u^* \in B^* \setminus \text{Dom}(F^*)$. As above, the duality pairing between B^* and B is here exploited. This is tantamount to defining $\partial F^*(u^*)$ as a subset of B^{**} , and then to restrict it to B .

Examples. (i) Let H be a (real) Hilbert space, $1 \leq p < +\infty$ and set $F_p(u) := \|u\|^p/p$ for any $u \in H$. It is convenient to identify H with its dual space, and then to define the subdifferential as a subset of H , simply by replacing the duality pairing by the scalar product in the definition.

If $p > 1$, then $\partial F_p(u) = \{\|u\|^{p-2}u\}$ for any $u \in H$; in particular, $\partial F_1(u) = \{\|u\|^{-1}u\}$ for any $u \in H \setminus \{0\}$, and $\partial F_1(0) = \{v \in H : \|v\| \leq 1\}$.

In particular, if $H := \mathbf{R}$ then $\partial F_1 = \text{sign}$, where we set

$$\text{sign}(x) := \{-1\} \text{ if } x < 0, \quad \text{sign}(0) := [-1, 1], \quad \text{sign}(x) := \{1\} \text{ if } x > 0. \quad (4.3)$$

(ii) Let (A, \mathcal{A}, μ) be a measure space, $1 \leq p < +\infty$, $F_p : \mathbf{R} \rightarrow \mathbf{R} : v \mapsto |v|^p/p$, and set $\Phi_p(u) := \int_A F_p(u) d\mu$ for any $u \in B := L^p(A, \mathcal{A}, \mu)$. Then

$$\partial \Phi_p(u) = \{u^* \in L^{p'}(A, \mathcal{A}, \mu) : u^* \in \partial F_p(u), \mu\text{-a.e. in } A\} \quad \forall u \in B.$$

(Here $p' := p/(p-1)$ if $p \neq 1$, $1' := +\infty$, as usual.)

(iii) Let us set $B := \mathbf{R}$ and $F_+(u) = F_-(u) := |u|$ for any $u \in \mathbf{R} \setminus \{0\}$, $F_+(0) := 1$, $F_-(0) := -1$. Then $\partial F_+(u) = \text{sign}(u)$ for any $\mathbf{R} \setminus \{0\}$ and $\partial F_+(0) = \emptyset$. On the other hand, $\partial F_-(u) = \emptyset$ for any $\mathbf{R} \setminus \{0\}$ and $\partial F_-(0) = [-1, 1]$.

(iv) Let Ω be an open subset of \mathbf{R}^N ($N \geq 1$), $1 < p < +\infty$, either $B := W_0^{1,p}(\Omega)$ or $B := W^{1,p}(\Omega)$, and set

$$F(u) := \frac{1}{p} \int_{\Omega} |\nabla u|^p dx \quad \forall u \in B. \quad (4.4)$$

This functional is convex and continuous on the whole B . By examples (i) and (ii), for any $u \in B$, $\xi := |\nabla u|^{p-2} \nabla u$ is the only element of $L^{p'}(\Omega)^N$ such that

$$\int_{\Omega} \xi \cdot \nabla(u-v) dx \geq \frac{1}{p} \int_{\Omega} (|\nabla u|^p - |\nabla v|^p) dx \quad \forall v \in B. \quad (4.5)$$

Thus, setting $L_u : v \mapsto \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx$, $L_u \in B^*$ and $L_u \in \partial F(u)$. If $B := W_0^{1,p}(\Omega)$ then $\partial F(u) = -\nabla \cdot (|\nabla u|^{p-2} \nabla u)$ in $\mathcal{D}'(\Omega)$. ∂F is thus single-valued.

This holds also for $p = 1$, provided that we replace $|\nabla u|^{p-2} \nabla u$ by any element of $\partial g(\nabla u)$, where $g(\xi) := |\xi|$ for any $\xi \in \mathbf{R}^N$. In this case ∂F is multi-valued.

(v) Let Ω be as above, $1 < p < +\infty$, set $B := L^p(\Omega)$ and

$$\tilde{F}(u) := \begin{cases} \frac{1}{p} \int_{\Omega} |\nabla u|^p dx & \forall u \in W_0^{1,p}(\Omega), \\ +\infty & \forall u \in L^p(\Omega) \setminus W_0^{1,p}(\Omega). \end{cases} \quad (4.6)$$

By (4.5), $\text{Dom}(\partial \tilde{F}) = \{u \in W_0^{1,p}(\Omega) : \nabla \cdot (|\nabla u|^{p-2} \nabla u) \in L^{p'}(\Omega)\}$ (otherwise the first integral of (4.5) is meaningless) and $\partial \tilde{F}(u) = -\nabla \cdot (|\nabla u|^{p-2} \nabla u)$ for any $u \in \text{Dom}(\partial \tilde{F})$. For instance, for $p = 2$, $\text{Dom}(\partial \tilde{F}) = H_0^1(\Omega) \cap H^2(\Omega)$ and $\partial \tilde{F}(u) = -\Delta u$ for any $u \in \text{Dom}(\partial \tilde{F})$.

If in (4.6) we replace $W_0^{1,p}(\Omega)$ by $\tilde{B} := W^{1,p}(\Omega)$, the representation of $\partial \tilde{F}$ is more delicate. Actually,

$$\langle \xi, v \rangle = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx \quad \forall \xi \in \partial \tilde{F}(u), \forall u, v \in W^{1,p}(\Omega), \quad (4.6')$$

(so that $\langle \xi, u - v \rangle \geq \tilde{F}(u) - \tilde{F}(v) \geq 0$). By the Hahn-Banach theorem, ξ may then be extended to an element of $W^{1,p}(\Omega)'$.

(vi) These examples can easily be extended if $|\nabla u|^p$ is replaced by $\varphi(\nabla u)$, where $\varphi : \mathbf{R}^N \rightarrow \mathbf{R}$ is convex and $\varphi(\xi)$ grows at infinity at most like $|\xi|^p$ (that is, there exist $C, M > 0$ such that $\varphi(\xi) \leq C|\xi|^p + M$ for any $\xi \in \mathbf{R}^N$). \square

• **Theorem 4.1** *Let $F : B \rightarrow]-\infty, +\infty]$ and assume that any function of which here we consider either the conjugate or the subdifferential is proper. Then for any $u \in B$ and any $u^* \in B^*$ we have:*

(i) $u^* \in \partial F(u) \Leftrightarrow F(u) + F^*(u^*) = \langle u^*, u \rangle$ (Fenchel's equality), or equivalently:

$u^* \in \partial F(u) \Leftrightarrow$ there exist $(u, r) \in \text{epi}(F)$ and $(u^*, s) \in \text{epi}(F^*)$ such that $r + s = \langle u^*, u \rangle$.

(ii) $u \in \partial F^*(u^*) \Leftrightarrow F^{**}(u) + F^*(u^*) = \langle u^*, u \rangle$ (dual Fenchel's equality), or equivalently:

$u \in \partial F^*(u^*) \Leftrightarrow$ there exist $(u, r) \in \text{epi}(F^{**})$ and $(u^*, s) \in \text{epi}(F^*)$ such that $r + s = \langle u^*, u \rangle$.

(iii) $u^* \in \partial F(u) \Rightarrow u \in \partial F^*(u^*)$. The converse holds if $F(u) = F^{**}(u)$.

(iv) $\partial F(u)$ is convex and weakly star closed (hence strongly and weakly closed); $\partial F^*(u^*)$ is convex and closed.

(v) The operator ∂F is monotone, that is,

$$\langle u_1^* - u_2^*, u_1 - u_2 \rangle \geq 0 \quad \forall u_i \in \text{Dom}(\partial F), \forall u_i^* \in \partial F(u_i) (i = 1, 2). \quad (4.7)$$

(vi) $F(u) = \inf F \Leftrightarrow \partial F(u) \ni 0 \Leftrightarrow [F(u) = F^{**}(u), \partial F^{**}(u) \ni 0] \Rightarrow u \in \partial F^*(0)$. If $F \in \Gamma_0(B)$ then the latter implication can be inverted.

Proof. By (4.1), $u^* \in \partial F(u)$ iff $\langle u^*, v \rangle - F(v) \leq \langle u^*, u \rangle - F(u)$ for any $v \in B$, that is,

$$F^*(u^*) := \sup_{v \in B} \{\langle u^*, v \rangle - F(v)\} = \langle u^*, u \rangle - F(u).$$

(i) thus holds. (ii) can be derived similarly.

As $F^{**} \leq F$, (iii) follows from (i), (ii) and (3.6).

By (4.1) we have $\partial F(u) = \bigcap_{v \in B} \{u^* \in B^* : \langle u^*, u - v \rangle \geq F(u) - F(v)\}$; thus $\partial F(u)$ is the intersection of a (nonempty) family of weakly star closed half-spaces. This yields the first part of (iv). The remainder can be proved similarly.

(v) and (vi) easily follow from the definition of subdifferential (cf. Proposition 4.2 below). \square

By the Fenchel's inequality, $F(u) + F^*(u^*) \geq \langle u^*, u \rangle$ (which holds for any $(u, u^*) \in B \times B^*$), the Fenchel's equality is equivalent to the opposite Fenchel's inequality, $F(u) + F^*(u^*) \leq \langle u^*, u \rangle$. By the same token, the dual Fenchel's equality is equivalent to the opposite dual Fenchel's inequality: $F^{**}(u) + F^*(u^*) \geq \langle u^*, u \rangle$.

By part (iii) of the latter result, for any $F \in \Gamma_0(B)$, $\partial F^* = (\partial F)^{-1}$.⁽⁵⁾

Corollary 4.1' *Let $F : B \rightarrow]-\infty, +\infty]$ and F^* be proper, and set*

$$\Phi(u, u^*) := F(u) + F^*(u^*) - \langle u^*, u \rangle \quad \forall u \in \text{Dom}(F), \forall u^* \in \text{Dom}(F^*).$$

Then

$$u^* \in \partial F(u) \quad \text{iff} \quad \Phi(u, u^*) = \inf_{\text{Dom}(F) \times \text{Dom}(F^*)} \Phi. \quad (4.7)'$$

Proof. The Fenchel equality and inequality respectively read

$$u^* \in \partial F(u) \quad \text{iff} \quad \Phi(u, u^*) = 0; \quad 0 \leq \inf_{\text{Dom}(F) \times \text{Dom}(F^*)} \Phi.$$

⁽⁵⁾ The inverse of any multi-valued function $f : A \rightarrow 2^B$ is defined as follows: for any $(a, b) \in A \times B$, $a \in f^{-1}(b)$ iff $b \in f(a)$. For multi-valued functions, this is not always equivalent to the property $f^{-1} \circ f = f \circ f^{-1} = \text{Id}$.

This yields the “only if” part. Conversely, if $\Phi(u, u^*) = \inf_{\text{Dom}(F) \times \text{Dom}(F^*)} \Phi$, then $\Phi(u, u^*) \leq \Phi(v, u^*)$ for any $v \in B$, whence $u^* \in \partial F(u)$. The “if” part thus holds. ⁽⁶⁾ \square

Note that (4.7)’ yields

$$u^* \in \partial F(u) \quad \text{iff} \quad \partial_u \Phi(u, u^*) \ni 0, \quad \partial_{u^*} \Phi(u, u^*) \ni 0. \quad (4.7)'$$

The following result can easily be proved via the Fenchel equality and inequality.

Proposition 4.2 *Under the assumptions of Theorem 4.1 we have:*

- (i) *If $\partial F(u) \neq \emptyset$, then $F(u) = F^{**}(u)$.*
- (ii) *If $F(u) = F^{**}(u)$, then $\partial F(u) = \partial F^{**}(u)$ (possibly $= \emptyset$). [Ex]*

The two latter implications cannot be inverted in general. As counterexamples take, e.g., $B = \mathbf{R}$ and respectively $F_1(x) := +\infty$ for any $x < 0$, $F_1(x) := -\sqrt{x}$ for any $x \geq 0$, $F_2(x) := F_1(x)$ for any $x \neq 0$, $F_2(0) := 1$. In either case consider the point $x = 0$.

• **Theorem 4.3** (Rockafellar) *Let $F_1, F_2 : B \rightarrow]-\infty, +\infty]$. Then*

$$\partial F_1(u) + \partial F_2(u) \subset \partial(F_1 + F_2)(u) \quad \forall u \in \text{Dom}(F_1) \cap \text{Dom}(F_2). \quad (4.8)$$

The opposite inclusion holds if $F_1, F_2 \in \Gamma_0(B)$, and either F_1 or F_2 is continuous at some point $u_0 \in \text{Dom}(F_1) \cap \text{Dom}(F_2)$. ⁽⁷⁾

Proof. To check (4.8) it suffices to write the definition of subdifferential for F_1 and F_2 , and sum the two inequalities.

The opposite inclusion is less easy to be proved. \square

The continuity assumption cannot be dropped. As a counterexample take $B = \mathbf{R}$, $F_1(x) := +\infty$ for any $x < 0$, $F_1(x) := -\sqrt{x}$ for any $x \geq 0$, $F_2 := I_{]_{-\infty, 0]}}$. Hence $(F_1 + F_2)(0) = 0$, $(F_1 + F_2)(x) = +\infty$ for any $x \neq 0$. Therefore $\partial(F_1 + F_2)(0) = \mathbf{R}$, whereas $\partial F_1(0) + \partial F_2(0) = \emptyset + [0, +\infty[= \emptyset$.

Proposition 4.4 *Let B_1, B_2 be Banach spaces over \mathbf{R} , $L : B_1 \rightarrow B_2$ be linear and continuous, and $F \in \Gamma(B_2)$. Then $F \circ L \in \Gamma(B_1)$.*

Moreover, if F is continuous at some point $\bar{p} \in B_2$, then

$$\partial(F \circ L)(u) = (L^* \circ \partial F)(Lu) \quad \forall u \in B_1. \quad \square \quad (4.9)$$

• **Proposition 4.5** *Let $F : B \rightarrow]-\infty, +\infty]$ be lower semicontinuous at some $u \in B$, and $\{(u_n, u_n^*)\}$ be a sequence in $B \times B^*$. If*

$$u_n^* \in \partial F(u_n) \quad \forall n \in \mathbf{N}, \quad u_n \rightarrow u \text{ weakly in } B, \quad u_n^* \rightarrow u^* \text{ weakly star in } B^*, \quad (4.10)$$

$$\liminf_{n \rightarrow \infty} \langle u_n^*, u_n \rangle \leq \langle u^*, u \rangle, \quad (4.11)$$

then $u^ \in \partial F(u)$.*

Proof. For any n , $u_n^* \in \partial F(u_n)$ iff

$$\langle u_n^*, u_n - v \rangle \geq F(u_n) - F(v) \quad \forall v \in \text{Dom}(F).$$

⁽⁶⁾ This is just another way of proving the Fenchel equality.

⁽⁷⁾ By Corollary 2.5, this hypothesis is equivalent to $[\text{int}(\text{Dom}(F_1)) \cap \text{Dom}(F_2)] \cup [\text{Dom}(F_1) \cap \text{int}(\text{Dom}(F_2))] \neq \emptyset$.

It then suffices to pass to the inferior limit as $n \rightarrow \infty$. \square

The latter result entails that the operator ∂F is strongly-weakly star and weakly-strongly sequentially closed in $B \times B^*$.

Proposition 4.6 *Let $F : B \rightarrow]-\infty, +\infty]$ be convex. If F has a point of continuity, then $\text{int}(\text{Dom}(F)) \subset \text{Dom}(\partial F) (\subset \text{Dom}(F))$.*

Proof. If F is continuous at some point, then it is continuous at any interior point of $\text{Dom}(F)$, by Theorem 2.3. Let us fix any $u_0 \in \text{int}(\text{Dom}(F))$. As $\text{int}(\text{epi}(F))$ is nonempty and $(u_0, F(u_0)) \in \text{int}(\text{epi}(F))$, by Theorem II.3.10 there exists a closed hyperplane through $(u_0, F(u_0))$ which is tangent to $\text{epi}(F)$. Thus there exists $(u^*, r) \in (B^* \times \mathbf{R}) \setminus \{(0, 0)\}$ such that

$$\begin{aligned} {}_{B^* \times \mathbf{R}} \langle (u^*, r), (u - u_0, a - F(u_0)) \rangle_{B \times \mathbf{R}} = {}_{B^*} \langle u^*, u - u_0 \rangle_B + r[a - F(u_0)] \geq 0 \\ \forall (u, a) \in \text{int}(\text{epi}(F)). \end{aligned}$$

We have $r \neq 0$, since otherwise ${}_{B^*} \langle u^*, u - u_0 \rangle_B \geq 0$ for any $u \in \text{int}(\text{Dom}(F))$, whence $u^* = 0$ as $u_0 \in \text{int}(\text{Dom}(F))$. Moreover $r > 0$, as a may be arbitrarily large. Taking $a := F(u)$, we get ${}_{B^*} \langle -u^*/r, u_0 - u \rangle_B \geq F(u_0) - F(u)$ for any $u \in \text{Dom}(F)$, that is, $-u^*/r \in \partial F(u_0)$. \square

Proposition 4.7 *If $F \in \Gamma_0(B)$ then $\text{Dom}(\partial F)$ is dense in $\text{Dom}(F)$.*

This result will be proved in Sect. XIII.4.

Inverse Fenchel equality and inequality: for any $u^* \in B^*$ and any function $F : B \rightarrow \mathbf{R} \cup \{+\infty\}$,

$$\exists \xi \in \mathbf{R} : \begin{cases} \exists u \in B : \langle u^*, u \rangle = F(u) + \xi \\ \forall v \in B, \langle u^*, v \rangle \leq F(v) + \xi \end{cases} \Rightarrow u^* \in \partial F(u), \xi = F^*(u^*). \quad (4.12)$$

(The opposite implication coincides with the usual Fenchel equality and inequality.)

In particular this entails that, for any functions $F : B \rightarrow \mathbf{R} \cup \{+\infty\}$ and $G : B^* \rightarrow \mathbf{R} \cup \{+\infty\}$ and for any $(u, u^*) \in B \times B^*$,

$$\begin{cases} \langle u^*, u \rangle = F(u) + G(u^*) \\ \langle v^*, v \rangle \leq F(v) + G(v^*) \quad \forall (v, v^*) \in B \times B^* \end{cases} \Rightarrow \begin{cases} u^* \in \partial F(u), \quad G(u^*) = F^*(u^*) \\ u \in \partial G(u^*), \quad F(u) = G^*(u). \end{cases} \quad (4.13)$$

\square

Exercises. (i) Let $F \in \Gamma_0(B)$ and $u^* \in B^*$. Check that $\partial F^*(u^*)$ coincides with the set of points which minimize the function $u \mapsto F(u) - \langle u^*, u \rangle$.

(ii) Show by a counterexample that in general $u \in \partial F^*(u^*)$ does not entail $u^* \in \partial F(u)$.

(iii) Evaluate F_i^* , F_i^{**} , ∂F_i , ∂F_i^* , ∂F_i^{**} , for any F_i defined as follows:

- $F_1(x) := |x|^2$ for any $x \in \mathbf{R} \setminus \{0\}$, $F_1(0) := -1$.
- $F_2(x) := |x|^2$ for any $x \in \mathbf{R} \setminus \{0\}$, $F_2(0) := 1$.
- $F_3(x) := -|x|^2$ for any $x \in \mathbf{R}$.
- $F_4(x) := \arctan x$ for any $x \in \mathbf{R}$.
- $F_5(x) := +\infty$ for any $x < 0$, $F_5(x) := -\sqrt{x}$ for any $x \geq 0$.

Hint: In order to evaluate the conjugate, one may use part (iii) of Theorem 4.1.

(iv) Let $F : B \rightarrow]-\infty, +\infty]$ ($F \not\equiv +\infty$) and $u \in K \subset B$. Check that the two following properties are equivalent:

- $F(u) = \inf_K F$,
- $0 \in \partial(F + I_K)(u)$.

Show that, if $F \in \Gamma_0(B)$, K is closed and convex, and $[\text{int}(\text{Dom}(F)) \cap K] \cup [\text{Dom}(F) \cap \text{int}(K)] \neq \emptyset$, then (a) and (b) are also equivalent to

(c) $0 \in \partial F(u) + \partial I_K(u)$, i.e., $-\partial I_K(u) \cap \partial F(u) \neq \emptyset$.

(v) Consider the elementary statements

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 \geq xy \quad \forall x, y \in \mathbf{R},$$

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 = xy \quad \text{iff} \quad x = y \quad \forall x, y \in \mathbf{R}.$$

Check that they respectively express the Fenchel inequality and the Fenchel equality for the function $x \mapsto \frac{1}{2}x^2$ in \mathbf{R} . Apply then the Fenchel inequality and the Fenchel equality to the function $x \mapsto \frac{1}{2}x^p$ in \mathbf{R} for any $p \in]1, +\infty[$.

(vi) Show that the Fenchel inequality and equality entail the monotonicity of the subdifferential.

(vii) May the implication (4.13) be inverted? □