Mathematical methods for engineering
The boundary element method

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Introduction and notation

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = f & \text{in } \Gamma_D \\ \frac{\partial u}{\partial n} = g & \text{on } \Gamma_N \end{cases}$$

Given $\mathbf{z} \in \mathbb{R}^d$

- $E(\cdot, \mathbf{z})$ denotes the fundamental solution of the Laplace equation:
$$-\Delta E = \delta_{\mathbf{z}} \, .$$
- $T(\cdot, \mathbf{z})$ denotes the normal derivative of $E(\cdot, \mathbf{z})$:
$$T(\cdot, \mathbf{z}) := \nabla E(\cdot, \mathbf{z}) \cdot \mathbf{n} \, .$$

  It is defined in $\partial\Omega$.

If $\Omega \subset \mathbb{R}^2$

$$E(\mathbf{x},\mathbf{z}) = \frac{1}{2\pi}\log\frac{1}{|\mathbf{x}-\mathbf{z}|}\,,$$

$$T(\mathbf{x},\mathbf{z}) = -\frac{1}{2\pi}\frac{1}{|\mathbf{x}-\mathbf{z}|^2}(\mathbf{x}-\mathbf{z})\cdot\mathbf{n}\,.$$

$|\mathbf{x}-\mathbf{z}|^2 = (x_1 - z_1)^2 + (x_2 - z_2)^2.$

Basic integral equations

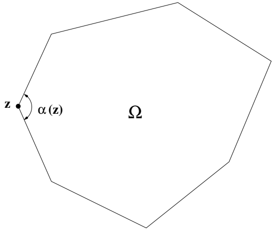
- Internal points: $\mathbf{z} \in \Omega$

$$u(\mathbf{z}) = \int_{\partial\Omega} E(\mathbf{x},\mathbf{z})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x})\, ds(\mathbf{x}) - \int_{\partial\Omega} T(\mathbf{x},\mathbf{z})u(\mathbf{x})\, ds(\mathbf{x})$$

- Boundary points: $\mathbf{z} \in \partial\Omega$

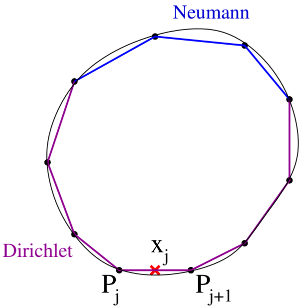
$$C(\mathbf{z})u(\mathbf{z}) = \int_{\partial\Omega} E(\mathbf{x},\mathbf{z})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x})\, ds(\mathbf{x}) - \int_{\partial\Omega} T(\mathbf{x},\mathbf{z})u(\mathbf{x})\, ds(\mathbf{x})$$

$$C(\mathbf{z}) \equiv \text{free term} \equiv \frac{\alpha(\mathbf{z})}{2\pi}.$$



Constant elements - 1

- The boundary of Ω is assumed to be polygonal divided in N elements.
- Both u and $q := \frac{\partial u}{\partial n}$ are approximated using piecewise constant functions: u_N, q_N .
- The unknowns are the values at the mid-element node, $\mathbf{x}_j$, $j = 1, \ldots, N$.
- Notice that $\alpha(\mathbf{x}_i) = \frac{1}{2}$ for $i = 1, \ldots, N$.



Constant elements - 2

Boundary points: $\mathbf{z} \in \partial\Omega$

$$C(\mathbf{z})u(\mathbf{z}) = \int_{\partial\Omega} E(\mathbf{x},\mathbf{z})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x})\, ds(\mathbf{x}) - \int_{\partial\Omega} T(\mathbf{x},\mathbf{z})u(\mathbf{x})\, ds(\mathbf{x})$$

For $i = 1, \ldots, N$

$$\frac{1}{2}u_N(\mathbf{x}_i) = \sum_{j=1}^{N} q_N(\mathbf{x}_j)\int_{\Gamma_j} E(\mathbf{x},\mathbf{x}_i)\, ds(\mathbf{x}) - \sum_{j=1}^{N} u_N(\mathbf{x}_j)\int_{\Gamma_j} T(\mathbf{x},\mathbf{x}_i)\, ds(\mathbf{x})$$

$$G_{i,j} := \int_{\Gamma_j} E(\mathbf{x},\mathbf{x}_i)\, ds(\mathbf{x}) \quad \hat{H}_{i,j} := \int_{\Gamma_j} T(\mathbf{x},\mathbf{x}_i)\, ds(\mathbf{x})$$

$$\frac{1}{2}u_i = \sum_{j=1}^{N} G_{i,j}q_j - \sum_{j=1}^{N} \hat{H}_{i,j}u_j$$

Constant elements - 3

$$\frac{1}{2}u_i + \sum_{j=1}^{N} \hat{H}_{i,j} u_j = \sum_{j=1}^{N} G_{i,j} q_j \,.$$

Setting $H_{i,j} = \begin{cases} \hat{H}_{i,j} & \text{when } i \neq j \\ \hat{H}_{i,j} + \frac{1}{2} & \text{when } i = j \end{cases}$

$$\sum_{j=1}^{N} H_{i,j} u_j = \sum_{j=1}^{N} G_{i,j} q_j \,.$$

Evaluation of integrals - 1

$$G_{i,j} := \frac{1}{2\pi}\int_{\Gamma_j} \log\frac{1}{|\mathbf{x}-\mathbf{x}_i|}\,ds(\mathbf{x}) \qquad \hat{H}_{i,j} := -\frac{1}{2\pi}\int_{\Gamma_j} \frac{(\mathbf{x}-\mathbf{x}_i)\cdot\mathbf{n}_j}{|\mathbf{x}-\mathbf{x}_i|^2}\,ds(\mathbf{x})$$

- For the case $i \neq j$ integrals $G_{i,j}$ and $\hat{H}_{i,j}$ can be calculted using, for instance, Gauss quadrature rules.
- For $i = j$ the presence on the singularity requires a more accurate integration.
  For the particular case of constant elements $\hat{H}_{i,i}$ and $G_{i,i}$ can be computed analitically.
  Notice that if $\mathbf{x} \in \Gamma_i$ then $(\mathbf{x}-\mathbf{x}_i)\cdot\mathbf{n}_i = 0$ hence $\hat{H}_{i,i} = 0$.

Evaluation of integrals - 2

$$G_{i,i} = \frac{1}{2\pi}\int_{\Gamma_i} \log\frac{1}{|\mathbf{x}-\mathbf{x}_i|}ds(\mathbf{x}) = \frac{1}{2\pi}\frac{L_i}{2}\int_{-1}^{1}\log\frac{1}{|r\,L_i/2|}dr$$

$$\mathbf{x} = \mathbf{P}_i\frac{1-r}{2} + \mathbf{P}_{i+1}\frac{1+r}{2} \qquad \mathbf{x}_i = \frac{\mathbf{P}_i + \mathbf{P}_{i+1}}{2}$$

where L_i is the length of Γ_i.

By integration by parts $\displaystyle\int_0^1 \log\left(r\frac{L_i}{2}\right)dr = \log\frac{L_i}{2} - 1\,.$

$$G_{i,i} = \frac{L_i}{2\pi}\left(1 - \log\frac{L_i}{2}\right)\,.$$

Gauss quadrature rules - 1

$$\int_{-1}^{1} f(r)\,dr \approx \sum_{k=1}^{K} \omega_k f(r_k)$$

It is a quadrature rule constructed to yield an exact result for polynomials of degree $2K - 1$ or less.
For instance, in the four points rule the point are

$$\pm\sqrt{\frac{1}{7}(3 - 2\sqrt{6/5})} \text{ with weight } \frac{18+\sqrt{30}}{36}$$

$$\pm\sqrt{\frac{1}{7}(3 + 2\sqrt{6/5})} \text{ with weight } \frac{18-\sqrt{30}}{36}.$$

$$\begin{aligned} \int_{\Gamma_j} F(\mathbf{x})\,ds(\mathbf{x}) &= \tfrac{L_j}{2} \int_{-1}^{1} F\left(\mathbf{P}_j \tfrac{1-r}{2} + \mathbf{P}_{j+1} \tfrac{1+r}{2}\right)\,dr \\ &\approx \tfrac{L_j}{2} \sum_{k=1}^{4} \omega_k F\left(\mathbf{P}_j \tfrac{1-r_k}{2} + \mathbf{P}_{j+1} \tfrac{1+r_k}{2}\right). \end{aligned}$$

Gauss quadrature rules - 2

```
function [h,g]=GQcon(P1,P2,Q)
   r=[-0.8611363116 -0.3399810436 0.3399810436 0.8611363116];
   w=[0.3478548451 0.6521451549 0.6521451549 0.3478548451];
   L=norm(P2-P1);
   n=[0 1; -1 0]*(P2-P1)/L;
   for i=1:4
     x=P1*(1-r(i))/2+P2*(1+r(i))/2;
     H(i)=(x-Q)'*n/norm(x-Q)^2;
     G(i)=log(norm(x-Q));
   end
   h=-L/(4*pi)*w*H;
   g=-L/(4*pi)*w*G;
```

Constant elements - Boundary data

$$\sum_{j=1}^{N} H_{i,j} u_j = \sum_{j=1}^{N} G_{i,j} q_j \qquad \begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = f & \text{in } \Gamma_D \\ \frac{\partial u}{\partial n} = g & \text{on } \Gamma_N \end{cases}$$

- If $\Gamma_D = \partial\Omega$ the vector $\mathbf{u} = (u_j)_{j=1}^N$ is given and the unknowns are the componente of the vector $\mathbf{q} = (q_j)_{j=1}^N$.
- On the other hand if $\Gamma_N = \partial\Omega$ then the vector $\mathbf{q} = (u_j)_{j=1}^N$ is given and the unknowns are the componente of the vector $\mathbf{u} = (q_j)_{j=1}^N$ (but matrix H is singular.)
- In general N_1 values of u corresponding to Γ_D and N_2 values of q corresponding to Γ_N are known with $N_1 + N_2 = N$.

Constant elements - Boundary data

We rearrange the unknowns to obtain a system:

$$A\mathbf{x} = \mathbf{b}$$

where $\mathbf{x}$ is the vector of unknowns (u on Γ_N, q on Γ_D)and $\mathbf{b}$ is a data vector obtained multiplying the corresponding coefficient by the known values of u or q:

```
for i=1:N
  for j=1:N
    if u(j) is unknown
      A(i,j)=H(i,j)
      b(i)=b(i)+G(i,j) *q(j)
    else
      A(i,j)=-G(i,j)
      b(i)=b(i)-H(i,j)*u(j)
    end
  end
end
```

The input data

The geometry and the boundary conditions.

- The geometry is given by a $2 \times N$ matrix with the coordinates of the points $\mathbf{P}_j$ that define the elements.
  Notice that:

$$\mathbf{x}_j = \frac{\mathbf{P}_j + \mathbf{P}_{j+1}}{2} \quad \text{if } j \neq N\,.$$

$$\mathbf{x}_N = \frac{\mathbf{P}_N + \mathbf{P}_1}{2} \quad \text{if } j \neq N\,.$$

- The boundary conditions are given in a $2 \times N$ matrix. The first line indicates the type of boundary condition (0 for Dirichlet boundary condition and 1 for Neumann boundary condition), the second one contains the boundary data.

The output

The solution is saved in a $2 \times N$ matrix that contains both the boundary data and the computed solution.

- The first row contains the Dirichlet data;
- the second row the Neumann one.

The program

```
function sol=Bemcon(coor,bc)
N=length(coor);
b=zeros(N,1);
coor=[coor,coor(:,1)];
L=sqrt(sum((coor(:,2:N+1)-coor(:,1:N)).^2))
gdiag=L/(2*pi).*(1-log(L/2))
for i=1:N
  Q=0.5*(coor(:,i)+coor(:,i+1));
  for j=1:N
    if j==i
      h=1/2;
      g=gdiag(j);
    else
      [h,g]=GQcon(coor(:,j),coor(:,j+1),Q);
    end
    if bc(1,j)==0
      A(i,j)=-g;
      b(i)=b(i)-h*bc(2,j);
    else
      A(i,j)=h;
      b(i)=b(i)+g*bc(2,j);
    end
  end
end
```

The program - 2

```
x=A\b;
for i=1:N
  if bc(1,i)==0
    sol(1,i)=bc(2,i);
    sol(2,i)=x(i);
  else
    sol(1,i)=x(i);
    sol(2,i)=bc(2,i);
  end
end
```

The solution at internal points - 1

$$u(\mathbf{z}) \approx \sum_{j=1}^{N} \left[q_j \int_{\Gamma_j} E(\mathbf{x}, \mathbf{z})\, ds(\mathbf{x}) - \sum_{j=1}^{N} u_j \int_{\Gamma_j} T(\mathbf{x}, \mathbf{z})\, ds(\mathbf{x}) \right]$$

The integrals are calculated using Gauss quadrature.

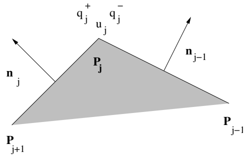
```
function s=SolIntcon(coor,sol,data)
K=length(data);
N=length(coor);
coor=[coor,coor(:,1)];
s=zeros(1,K);
for i=1:K
  z=data(:,i);
  for j=1:N
    [h,g]=GQcon(coor(:,j),coor(:,j+1),z);
    s(i)=s(i)-sol(1,j)*h+sol(2,j)*g;
  end
end
```

The solution at internal points - 2

- `coor` is the matrix with the vertices in the discrete boundary;
- `sol` is the matrix with both the Dirichlet and the Neumann data at the meddle point of the edges in the discrete boundary;
- `data` is a matrix with the coordinates of the internal points where the solution would be computed.

Linear elements - 1

- The functions u and q are assumed to be liner over each element.
- We will assume that u is continuous but q can be discontinuous in corner points



$$u_{|\Gamma_j} = u(P_j)N_1(r) + u(P_{j+1})N_2(r)$$

$$q_{|\Gamma_j} = q^+(P_j)N_1(r) + q^-(P_{j+1})N_2(r)$$

$N_1(r) = \frac{1-r}{2}$, $N_2(r) = \frac{1+r}{2}$, with $r \in [-1, 1]$.

Linear elements - 2

$$C(\mathbf{P}_i)u(\mathbf{P}_i) + \int_{\partial\Omega} u(\mathbf{x})T(\mathbf{x},\mathbf{P}_i)\,ds(\mathbf{x}) = \int_{\partial\Omega} q(\mathbf{x})E(\mathbf{x},\mathbf{P}_i)\,ds(\mathbf{x})$$

$\int_{\Gamma_j} u(\mathbf{x})T(\mathbf{x},\mathbf{P}_i)\,ds(\mathbf{x}) =$

$$u_j \tfrac{L_j}{2} \textstyle\int_{-1}^{1} N_1(r)T(\mathbf{P}_j N_1(r) + \mathbf{P}_{j+1}N_2(r), \mathbf{P}_i)\,ds(\mathbf{x})$$

$$+u_{j+1} \tfrac{L_j}{2} \textstyle\int_{-1}^{1} N_2(r)T(\mathbf{P}_j N_1(r) + \mathbf{P}_{j+1}N_2(r), \mathbf{P}_i)\,ds(\mathbf{x})$$

For $k = 1, 2$

$$A_k(i,j) := \frac{L_j}{2}\int_{-1}^{1} N_k(r)T(\mathbf{P}_j N_1(r) + \mathbf{P}_{j+1}N_2(r), \mathbf{P}_i)\,ds(\mathbf{x})$$

$$\int_{\Gamma_j} u(\mathbf{x})T(\mathbf{x},\mathbf{P}_i)\,ds(\mathbf{x}) = A_1(i,j)u_j + A_2(i,j)u_{j+1}$$

Linear elements - 3

Analogously

$\int_{\Gamma_j} q(\mathbf{x}) E(\mathbf{x}, \mathbf{P}_i) =$

$$q_j^+ \frac{l_j}{2} \int_{-1}^1 N_1(r) E(\mathbf{P}_j N_1(r) + \mathbf{P}_{j+1} N_2(r), \mathbf{P}_i)\, ds(\mathbf{x})$$

$$+ q_{j+1}^- \frac{l_j}{2} \int_{-1}^1 N_2(r) E(\mathbf{P}_j N_1(r) + \mathbf{P}_{j+1} N_2(r), \mathbf{P}_i)\, ds(\mathbf{x})$$

For $k = 1, 2$

$$B_k(i,j) := \frac{L_j}{2} \int_{-1}^1 N_k(r) E(\mathbf{P}_j N_1(r) + \mathbf{P}_{j+1} N_2(r), \mathbf{P}_i)\, ds(\mathbf{x})$$

$$\int_{\Gamma_j} q(\mathbf{x}) E(\mathbf{x}, \mathbf{P}_i) = q_j^+ B_1(i,j) + q_{j+1}^- B_2(i,j)$$

Linear elements - 4

$$C(\mathbf{P}_i)u(\mathbf{P}_i) + \int_{\partial\Omega} u(\mathbf{x})T(\mathbf{x},\mathbf{P}_i)\,ds(\mathbf{x}) = \int_{\partial\Omega} q(\mathbf{x})E(\mathbf{x},\mathbf{P}_i)\,ds(\mathbf{x})$$

$$C(\mathbf{P}_i)u_i + \sum_{j=1}^{N}\left[A_1(i,j)u_j + A_2(i,j)u_s\right] = \sum_{j=1}^{N}\left[B_1(i,j)q_j^+ + B_2(i,j)q_s^-\right]$$

$$s = s(j) = \begin{cases} j+1 & \text{if } j = 1,\dots,N-1 \\ 1 & \text{if } j = N \end{cases}$$

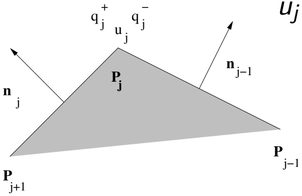
for i=1:N
  for j=1:N
    compute $A_1(i,j)$, $A_2(i,j)$, $B_1(i,j)$, $B_2(i,j)$.
    For each one of this four coefficients
      if it multiplies an unknown then the coefficient is added to the corresponding entry of the matrix ($a_{i,j}$ or $a_{i,s}$),
      if it multiplies a data the product is added to the right hand term.

Boundary conditions

1. Neumann - Neumann condition
   u_j unknown, q_j^- given, q_j^+ given.
2. Dirichlet - Neumann condition
   u_j given, q_j^- unknown, q_j^+ given.
3. Neumann - Dirichlet condition
   u_j given, q_j^- given, q_j^+ unknown.
4. Dirichlet - Dirichlet condition (regular point)
   u_j given, q_j^- unknown, q_j^+ unknown.
   but $q_j^- = q_j^+$
5. Dirichlet - Dirichlet condition (non regular point)
   u_j given, q_j^- unknown, q_j^+ unknown.
   and $q_j^- \neq q_j^+$ $\rightsquigarrow$ The gradient approach

The gradient approach

$$\begin{array}{rcl} q_j^- & = & \nabla u(\mathbf{P}_j)\cdot \mathbf{n}_{j-1} \\ q_j^+ & = & \nabla u(\mathbf{P}_j)\cdot \mathbf{n}_j \end{array} \qquad \nabla u(\mathbf{P}_j) = |\nabla u(\mathbf{P}_j)|\mathbf{e}(\mathbf{P}_j)$$



Assuming linear evolution along the elements adjacent to the corner in $\mathbf{P}_j$ a linear $\hat{u}_j(\mathbf{P})$ can be calculated.

- $\mathbf{e}(\mathbf{P}_j)$ is approximated using $\nabla \hat{u}_j$ (that is a constant vector).

$$\mathbf{e}(\mathbf{P}_j) \rightsquigarrow \frac{\nabla \hat{u}_j}{|\nabla \hat{u}_j|}$$

- $|\nabla u(\mathbf{P}_j)|$ is the unknown $\rightsquigarrow v_j$.

$$q_j^- \rightsquigarrow v_j \frac{\nabla \hat{u}_j}{|\nabla \hat{u}_j|}\cdot \mathbf{n}_{j-1} \quad q_j^+ \rightsquigarrow v_j \frac{\nabla \hat{u}_j}{|\nabla \hat{u}_j|}\cdot \mathbf{n}_j \,.$$

Evaluation of the coefficients - 1

- If $\mathbf{P}_i \notin \Gamma_j$ Gauss quadrature
- If $\mathbf{P}_i \in \Gamma_j$
  - $A_k(i,j) = 0$ for $k = 1,2$ because $\forall\, \mathbf{P} \in \Gamma_j \quad (\mathbf{P} - \mathbf{P}_i) \cdot \mathbf{n} = 0\,.$
  - $B_k(i,j)$ are improper integrals that can be calculated exactly.

$$
\begin{aligned}
B_1(j,j) &= -\frac{L_j}{2} \int_{-1}^{1} \frac{1}{2\pi} \frac{1-r}{2} \log \left| \mathbf{P}_j \frac{1-r}{2} + \mathbf{P}_{j+1} \frac{1+r}{2} - \mathbf{P}_j \right| dr \\
&= -\frac{L_j}{2} \int_{-1}^{1} \frac{1}{2\pi} \frac{1-r}{2} \log \left| (\mathbf{P}_{j+1} - \mathbf{P}_j) \frac{1+r}{2} \right| dr \\
&\qquad \left(\frac{1+r}{2} = s\,, \quad dr = 2\,ds\,, \quad \frac{1-r}{2} = 1 - s \right) \\
&= -\frac{L_j}{2\pi} \int_0^1 (1-s) \log(L_j s)\, ds = \frac{L_j}{2\pi} \left(\frac{3}{4} - \frac{1}{2} \log L_j \right)
\end{aligned}
$$

Evaluation of the coefficients - 2

$$
\begin{aligned}
B_1(j,j) &= \frac{L_j}{2\pi}\left(\frac{3}{4} - \frac{1}{2}\log L_j\right) \\
B_2(j,j) &= \frac{L_j}{2\pi}\left(\frac{1}{4} - \frac{1}{2}\log L_j\right) \\
B_1(j+1,j) &= \frac{L_j}{2\pi}\left(\frac{1}{4} - \frac{1}{2}\log L_j\right) \\
B_2(j+1,j) &= \frac{L_j}{2\pi}\left(\frac{3}{4} - \frac{1}{2}\log L_j\right)
\end{aligned}
$$

- Free terms: for a constant solution $u = M$

$$C(P_i)M + \sum_{j=1}^{N}\left[A_1(i,j)M + A_2(i,j)M\right] = 0$$

$$C(P_i) = -\sum_{j=1}^{N}\left[A_1(i,j) + A_2(i,j)\right]$$

Input data

The geometry and the boundary conditions.

- The geometry is given by a $2 \times N$ matrix with the coordinates of the points $\mathbf{P}_j$ that define the elements (as in the case of constant elements).
- The type of boundary conditions is given in a vector with N components. Each entry is an integer from 1 to 5.
- The boundary data are given in a matrix $3 \times N$ matrix. The first line contains the values of u, the second one q^- and the last one q^+.
  Notice that in this matrix there are unknowns coefficients.