Topological Quantum Computation

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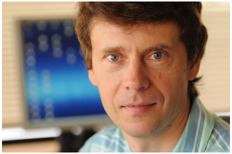
Topological Quantum Computation

Topological Quantum Computation (TQC) is a computational model built upon systems of topological phases.

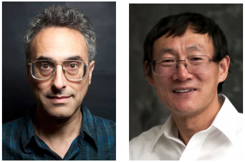
Co-creators:
Freedman and Kitaev



- M. Freedman, *P/NP, and the quantum field computer.* Proc. Natl. Acad. Sci. USA 1998.
- A. Kitaev, *Fault-tolerant quantum computation by anyons*. Ann. Physics 2003. (preprint 1997).

Some Milestones

- ∼1998: Freedman “Quantum field computer” and Kitaev “Anyonic quantum computation”
- 2002: Freedman, Kitaev, Larsen & Wang: quantum circuit model and topological model polynomially equivalent

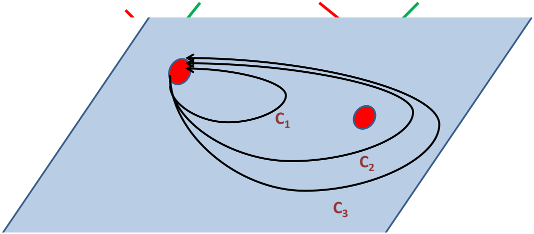


- 2005: Microsoft Station Q (Santa Barbara): Freedman, Wang, Walker,...
- 2011-2017: Many more Stations Q–Delft (Kouwenhoven), Copenhagen (Marcus), Sydney (Reilly), Purdue (Manfra)...

Anyons

For Point-like particles:

- In $\mathbb{R}^3$: bosons or fermions: $\psi(z_1, z_2) = \pm\psi(z_2, z_1)$
- Particle exchange $\rightsquigarrow$ reps. of symmetric group S_n
- In $\mathbb{R}^2$: (abelian) anyons: $\psi(z_1, z_2) = e^{i\theta}\psi(z_2, z_1)$
- or, if state space has dimension > 1,
  $\psi_1(z_2, z_1) = \sum_j a_j \psi_j(z_1, z_2)$ non-abelian anyons.
- Particle exchange $\rightsquigarrow$ reps. of braid group $\mathcal{B}_n$
- Why? $\pi_1(\mathbb{R}^3 \setminus \{z_i\}) = 1$ but $\pi_1(\mathbb{R}^2 \setminus \{z_i\}) = F_n$ Free group.



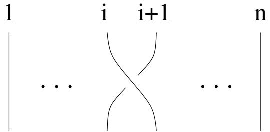
$C_1 \not\approx C_2 \approx C_3$

The hero of 2D topological materials is the **Braid Group** $\mathcal{B}_n$:
generators σ_i, $i = 1, \ldots, n-1$ satisfying:

(R1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

(R2) $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| > 1$

$\sigma_i \quad \mapsto$

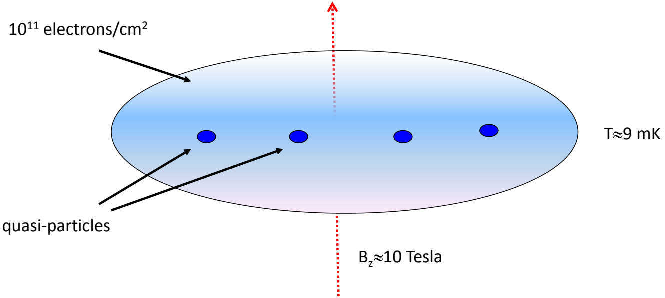


Motions of n points in a disk.

$$\mathcal{B}_n \hookrightarrow \operatorname{Aut} \pi_1(\mathbb{R}^2 \setminus \{z_1, \ldots, z_n\}) = \operatorname{Aut}(F_n)$$

Topological Phases of Matter Exist?

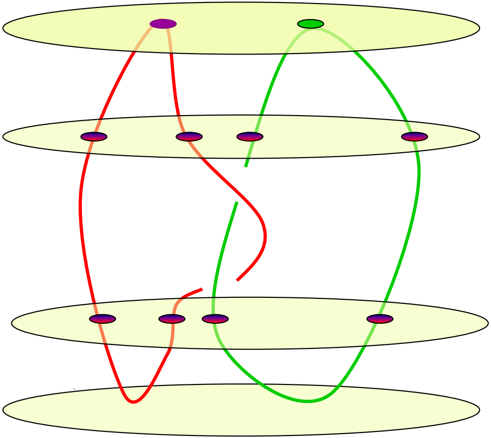
Fractional Quantum Hall Liquid



See also: 2016 Physics Nobel Prize...

Topological Model

Computation	**Physics**
output	**measure (fusion)**
apply gates	**braid anyons**
initialize	**create anyons**



vacuum

Foundational (Math) Questions

1. How to model Anyons on Surfaces?
2. What are the state spaces?
3. What are the quantum gates?

Modeling Anyons on Surfaces

Definition (Nayak, et al '08)

a (bosonic) system is in a topological phase if its low-energy effective field theory is a topological quantum field theory (TQFT).

A (2+1)D **TQFT** assigns to any (surface, boundary data) (M, ℓ) a Hilbert space:

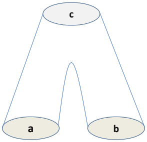
$$(M, \ell) \to \mathcal{H}(M, \ell).$$

Boundary $\bigcirc$ labeled by $i \in \mathcal{L}$: finite set of colors$\leftrightarrow$ (anyons).
$0 \in \mathcal{L}$ is **neutral**$\leftrightarrow$ vacuum. Orientation-reversing map: $x \to x^*$.

Basic pieces

Any surface can be built from the following basic pieces:

- disk: $\mathcal{H}($  $; i) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & else \end{cases}$
- annulus: $\mathcal{H}($  $; a, b) = \begin{cases} \mathbb{C} & a = b^* \\ 0 & else \end{cases}$
- pants:

$P :=$ 

$$\mathcal{H}(P; a, b, c) = \mathbb{C}^{N(a,b,c)}$$ ↖ choices!

Two more axioms

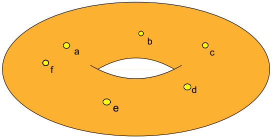
Axiom (Disjoint Union)

$\mathcal{H}[(M_1,\ell_1)\coprod(M_2,\ell_2)] = \mathcal{H}(M_1,\ell_1)\otimes\mathcal{H}(M_2,\ell_2)$

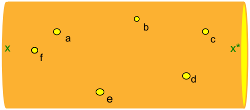
Axiom (Gluing)

If M is obtained from gluing two boundary circles of M_g together then

$$\mathcal{H}(M,\ell) = \bigoplus_{x\in\mathcal{L}} \mathcal{H}(M_g,\ell,x,x^*)$$



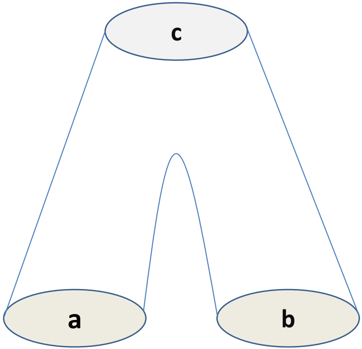
(M,ℓ)



(M_g,ℓ,x,x^*)

Fusion Channels

The state-space dimension $N(a, b, c)$ of:



represents the number of ways a and b may fuse to c

Fusion Matrix: $a \to (N_a)_{b,c} = N(a, b, c)$

Computational State Spaces/Quantum Dimensions

Principle

The Computational Spaces $\mathcal{H}_n := \mathcal{H}(D^2; a, \ldots, a)$: the state space of n identical type a anyons in a disk.

Definition

Let $\dim(a)$ be the maximal eigenvalue of N_a.

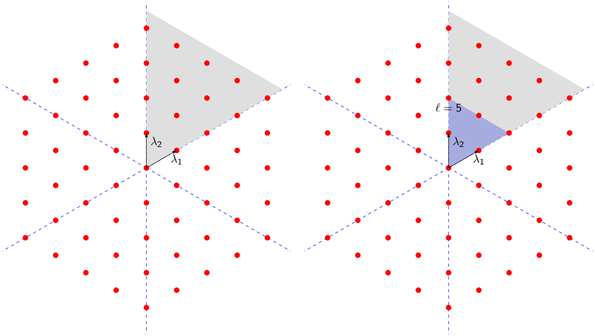
Fact

1. $\dim(a) \in \mathbb{R}$
2. $\dim(a) \geq 1$
3. *Physically: Loop amplitudes* $\bigcirc_a$
4. $\dim \mathcal{H}_n \approx \dim(a)^n$ *highly non-local Herein Lies Fault-Tolerance: Errors are local.*

Are There Any Non-trivial Examples?!

Quantum Groups at roots of unity:

$$\mathfrak{g} \rightsquigarrow U\mathfrak{g} \rightsquigarrow U_q\mathfrak{g} \overset{q=e^{\pi i/\ell}}{\rightsquigarrow} \mathrm{Rep}(U_q\mathfrak{g}) \overset{/\langle Ann(Tr)\rangle}{\rightsquigarrow} \mathcal{C}(\mathfrak{g},\ell)$$



Anyons$\longleftrightarrow$Objects (representations)

Computational space $\mathcal{H}_n \longleftrightarrow \mathrm{End}_{\mathcal{C}(\mathfrak{g},\ell)}(X_a^{\otimes n})$

Example (Fibonacci)

- $\mathcal{L} = \{0, 1\}$: $N(a, b, c) = \begin{cases} 1 & a = b = c \text{ or } a + b + c \in 2\mathbb{Z} \\ 0 & \text{else} \end{cases}$
- $N_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $\dim(X_1) = \frac{1+\sqrt{5}}{2}$ G_2 at $\ell = 15$

Example (Ising)

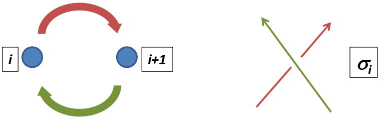
- $\mathcal{L} = \{0, 1, 2\}$: $N(a, b, c) = \begin{cases} 1 & a + b + c \in 2\mathbb{Z} \\ 0 & \text{else} \end{cases}$
- $N_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\dim X_1 = \sqrt{2}$ $SU(2)$ at $\ell = 4$.

Most generally: Modular Tensor Categories

Braiding Gates

Fix anyon a $\mathcal{B}_n$ acts on **state spaces**:

- Braid group representation ρ_a on $\mathcal{H}(D^2 \setminus \{z_i\}; a, \ldots, a) = \mathrm{End}(X_a^{\otimes n})$ by particle exchange



- (Topological) Quantum Gates: $\rho_a(\sigma_i)$,circuits: $\rho_a(\beta)$, $\beta \in \mathcal{B}_n$

More Foundational Questions

- Simulate TQC on QCM?
- Simulate (universal) QCM on TQC?
- Computes What? Complexity?

Simulating TQCs on QCM

[Freedman, Kitaev, Wang] showed that TQCs have hidden locality:
Let $U(\beta) \in \mathbf{U}(\mathcal{H}_n)$ be a unitary braiding matrix.
Goal: simulate U on $V^{\otimes k(n)}$ for some v.s. V.

- Set $V = \bigoplus_{(a,b,c)\in\mathcal{L}^3} \mathcal{H}(P; a, b, c)$ and $W_n = V^{\otimes(n-1)}$
- TQFT axioms (gluing, disjoint union) imply:

$$\mathcal{H}_n \hookrightarrow W_n$$

Remark

V can be quite large and $U(\beta)$ only acts on the subspace $\mathcal{H}_n$.
Forced to project, etc...

Local $\mathcal{B}_n$ representations: Yang-Baxter eqn.

Definition

(R, V) is a **braided vector space** if $R \in Aut(V \otimes V)$ satisfies

$$(R \otimes I_V)(I_V \otimes R)(R \otimes I_V) = (I_V \otimes R)(R \otimes I_V)(I_V \otimes R)$$

Induces a sequence of local $\mathcal{B}_n$-reps $(\rho^R, V^{\otimes n})$ by

$$\rho^R(\sigma_i) = I_V^{\otimes i-1} \otimes R \otimes I_V^{\otimes n-i-1}$$

$$v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \xrightarrow{\rho^R(\sigma_i)} v_1 \otimes \cdots \otimes R(v_i \otimes v_{i+1}) \otimes \cdots \otimes v_n$$

Idea: braided QCM gate set $\{R\}$

Square Peg, Round Hole?

Definition (R,Wang '12)

A **localization** of a sequence of $\mathcal{B}_n$-reps. (ρ_n, V_n) is a braided vector space (R, W) and injective algebra maps $\tau_n : \mathbb{C}\rho_n(\mathcal{B}_n) \to \mathrm{End}(W^{\otimes n})$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}\mathcal{B}_n & & \\ \downarrow{\scriptstyle \rho_n} & \searrow^{\rho^R} & \\ \mathbb{C}\rho_n(\mathcal{B}_n) & \xrightarrow{\tau_n} & \mathrm{End}(W^{\otimes n}) \end{array}$$

Idea: Push braiding gates inside a braided QCM.

Example $\mathcal{C}(\mathfrak{sl}_2, 4)$

Let $R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$

Theorem (Franko,R,Wang '06)

$(R, \mathbb{C}^2)$ *localizes* $(\rho_n^X, \mathcal{H}_n)$ *for* $X = X_1 \in \mathcal{C}(\mathfrak{sl}_2, 4)$

Remark

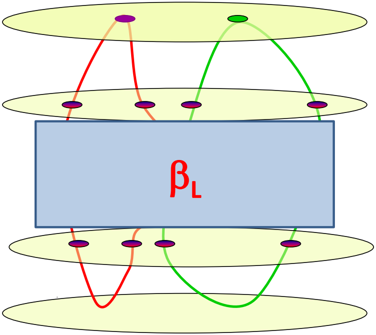
Notice: object X is not a vector space! $(\dim(X) = \sqrt{2})$ hidden locality has $\dim(V) = 10$, $\dim(W) = 10^{n-1}$ while $\dim(\mathcal{H}_n) \in O(2^n)$.

What do TQCs compute?

Answer

(Approximations to) Link invariants!

Associated to $X \in \mathcal{C}$ is a link invariant $Inv_L(X)$ approximated by the corresponding Topological Model efficiently.



Prob(●) ~ $x^{-t}|Inv_L($●$)|$

Complexity of Jones Polynomial Evaluations

For $\mathcal{C}(\mathfrak{sl}_2, \ell)$, $Inv_L(X) = V_L(q)$ Jones polynomial at $q = e^{2\pi i/\ell}$

Theorem (Vertigan,Freedman-Larsen-Wang)

- *(Classical) exact computation of $V_L(q)$ at $q = e^{\pi i/\ell}$ is:*
  $$\begin{cases} FP & \ell = 3, 4, 6 \\ FP^{\sharp P} - complete & else \end{cases}$$
- *(Quantum) approximation of $|V_L(q)|$ at $q = e^{\pi i/\ell}$ is BQP*

Universal Anyons

Question (Quantum Information)

When does an anyon x provide universal computation models?
Informally: when can any unitary gate be (approximately) realized by particle exchange?

Example

Fibonacci $\dim(X) = \frac{1+\sqrt{5}}{2}$ is universal.

Example

Ising $\dim(X) = \sqrt{2}$ is not universal: particle exchange generates a finite group.

Anyon a is

- Abelian/non-abelian if $\rho_a(\mathcal{B}_n)$ is abelian/non-abelian
- Universal if $\overline{\rho_a(\mathcal{B}_n)} \supset \Pi_i SU(n_i)$ for $n >> 0$ where $\rho_a = \oplus \rho_a^i$ irreps.
- Localizable if $\rho_a(\mathcal{B}_n)$ simulated on QCM via Yang-Baxter operator gate R
- Classical if $Inv_a(L)$ is in FP

Principle

All (conjecturally) determined by $\dim(a)$:

- Abelian anyons: $\dim(a) = 1$
- Non-abelian anyons: $\dim(a) > 1$ (PRA 2016, with Wang)
- Universal anyons: $\dim(a)^2 \notin \mathbb{Z}$ (conj. 2007)
- Localizable and Classical anyons: $\dim(a)^2 \in \mathbb{Z}$ (conj. 2010, with Wang)

Other questions...

- For fixed $n = |\mathcal{L}|$ classify TQFTs by $|\mathcal{L}|$. (Recently: finitely many for fixed $|\mathcal{L}|$ Bruillard-Ng-R-Wang JAMS 2016)
- Measurement assisted?
- Gapped boundaries/defects?
- Fermions?
- 3D Materials/loop excitations?

THANK YOU!

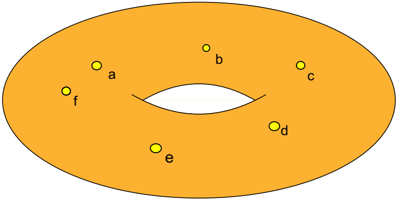
Modeling Anyons on Surfaces

Topology of marked surfaces+quantum mechanics

Marks$\leftrightarrow$ anyons$\leftrightarrow$boundary components.

Principle

Superposition: a state is a vector in a Hilbert space $|\psi\rangle \in \mathcal{H}$.

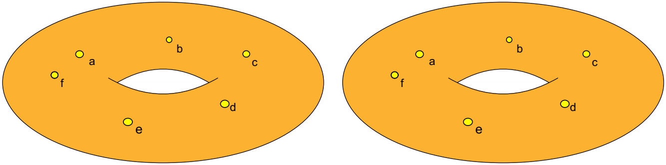


$\rightarrow \mathcal{H}(T^2, a, b, c, d, e, f)$

Principle

The **composite state space** of two physically separate systems A and B is the **tensor product** $\mathcal{H}_A \otimes \mathcal{H}_B$ of their state spaces.

Interpretation



$\to \mathcal{H} \otimes \mathcal{H}$

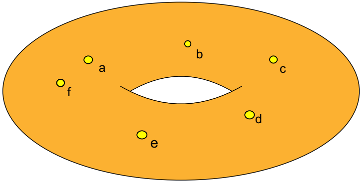
Key to: Entanglement

Principle

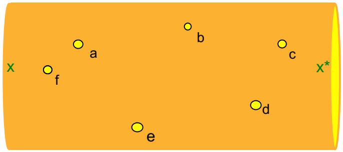
Locality: the global state is determined from pieces.

Interpretation

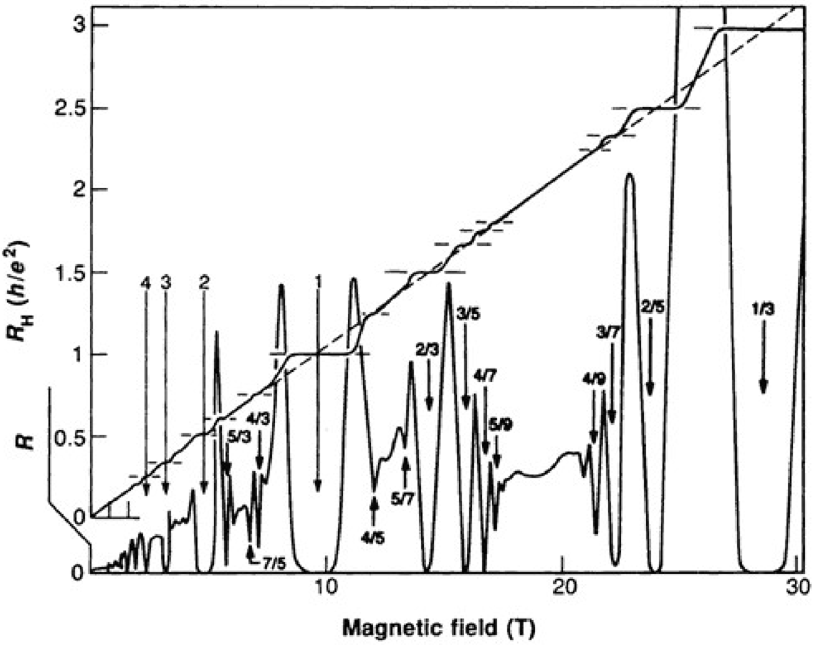
The Hilbert space of a marked surface M is a direct sum over all boundary labelings of a surface M_g obtained by cutting M along a circle.



$$\mathcal{H}(T^2; a, b, c, d, e) =$$



$\bigoplus_x \mathcal{H}(A; a, b, c, d, e, x, x^*)$
(x^* is anti-particle to x)
.



R_H (h/e^2)
R
3
2.5
2
1.5
1
0.5
0
0
4
3
2
1
5/3
4/3
7/5
4/5
5/7
2/3
3/5
4/7
5/9
4/9
3/7
2/5
1/3
10
20
30
Magnetic field (T)